Smallest area surface evolving with unit areal speed

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Abstract. The theory of smallest area surfaces evolving with unit areal speed is a particular case of the theory of surfaces of minimum area subject to various constraints. Based on our recent results, such problems can be solved using the two-time maximum principle in a controlled evolution.

Section 1 studies a controlled dynamics problem (smallest area surface evolving with unit areal speed) via the two-time maximum principle. The evolution PDE is of 2-flow type and the adjoint PDE is of divergence type. Section 2 analyzes the smallest area surfaces evolving with unit areal speed, avoiding an obstacle. Section 3 reconsiders the same problem for touching an obstacle, detailing the results for the cylinder and the sphere.

M.S.C. 2010: 49J20, 49K20, 49N90, 90C29.
Key words: multitime maximum principle, smallest area surface, minimal surfaces, controlled dynamics with obstacle.

1 Smallest area surface evolving with unit areal speed, passing through two points

The minimal surfaces are characterized by zero mean curvature. They include, but are not limited to, surfaces of minimum area subject to various constraints. Minimal surfaces have become an area of intense mathematical and scientific study over the past 15 years, specifically in the areas of molecular engineering and materials sciences due to their anticipated nanotechnology applications (see [1]-[5]).

Let $\Omega_{0\tau}$ be a bidimensional interval fixed by the diagonal opposite points $0, \tau \in R^2_+$. Looking for surfaces $x^i(t) = x^i(t^1, t^2)$, $(t^1, t^2) \in \Omega_{0\tau}$, $i = 1, 2, 3$, that evolve with unit areal speed and relies transversally on two curves Γ_0 and Γ_1, let us show that a minimum area surface (2-sheet) is a solution of a special PDE system, via the optimal control theory (multitime maximum principle, see [11]-[22]). An example is a planar quadrilateral (totally geodesic surface in R^3) fixed by the origin $x^i(0) = x^i_0$ on Γ_0 and passing through the diagonal terminal point $x^i(\tau) = x^i_1$ on Γ_1.

Balkan Journal of Geometry and Its Applications, Vol.16, No.1, 2011, pp. 155-169.

In R^3 we introduce the *two-time controlled dynamics*

$$(PDE) \qquad \frac{\partial x^i}{\partial t^\alpha}(t) = u^i_\alpha(t),$$

$$t = (t^1, t^2) \in \Omega_{0\tau},\ i = 1,2,3;\ \alpha = 1,2,\ x^i(0) = x^i_0,\ x^i(\tau) = x^i_1,$$

where $u^i_\alpha(t)$ represents two open-loop C^1 control vectors, non-collinear, eventually fixed on the boundary $\partial\Omega_{0\tau}$. The complete integrability conditions of the (PDE) system, restrict the set of controls to

$$\mathcal{U} = \left\{ u = (u_\alpha) = (u^i_\alpha) \,\Big|\, \frac{\partial u^i_1}{\partial t^2}(t) = \frac{\partial u^i_2}{\partial t^1}(t) \right\}.$$

A solution of (PDE) system is a surface (2-sheet) $\sigma : x^i = x^i(t^1, t^2)$. Suppose $x(0) = x_0$ belongs to the image Γ_0 of a curve in R^3 and $\tau = (\tau^1, \tau^2)$ is the two-time when the 2-sheet $x(t^1, t^2)$ reaches the curve Γ_1 in R^3, at $x(\tau) = x_1$, with Γ_0 and Γ_1 transversal to σ. On the other hand, we remark that the area of the 2-sheet σ is

$$\int\!\!\int_\sigma d\sigma = \int\!\!\int_{\Omega_{0\tau}} \left(||u_1||^2 ||u_2||^2 - < u_1, u_2 >^2\right)^{\frac{1}{2}} dt^1 dt^2.$$

This surface integral generates the cost functional

$$(J) \qquad J(u(\cdot)) = -\int\!\!\int_\sigma d\sigma.$$

Of course, the maximization of $J(u(\cdot))$ is equivalent to minimization of the area, under the constraint (PDE).

Suppose the control set U is restricted to the hypersurface in $R^3 \times R^3$ corresponding to unit areal speed produced by two linearly independent vectors $u_\alpha = (u^1_\alpha, u^2_\alpha, u^3_\alpha)$, $\alpha = 1, 2$, in R^3, i.e.,

$$U : \ \delta_{ij} u^i_1 u^j_1 \delta_{k\ell} u^k_2 u^\ell_2 - (\delta_{ij} u^i_1 u^j_2)^2 = 1$$

or in short

$$U : q(u) = ||u_1||^2 ||u_2||^2 - < u_1, u_2 >^2 - 1 = 0$$

(see the Gram determinant). Then the surface integral

$$\int\!\!\int_\sigma d\sigma = \int\!\!\int_{\Omega_{0\tau}} dt^1 dt^2 = \tau^1 \tau^2$$

represents the area of the bidimensional interval $\Omega_{0\tau}$ whose image as (PDE) solution is the surface (2-sheet) $\sigma : x^i = x^i(t^1, t^2)$. We need to find the point $\tau(\tau^1, \tau^2)$ such that $x(\tau) = x_1$ and $\tau^1 \tau^2 = \min$.

Two-time optimal control problem of smallest area surface evolving with unit areal speed: *Find*

$$\max_{u(\cdot)} J(u(\cdot))$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t) = u^i_\alpha(t),\ i = 1,2,3;\ \alpha = 1,2;\ q(u) = 0,$$

$$u(t) \in \mathcal{U},\ t \in \Omega_{0\tau};\ x(0) = x_0,\ x(\tau) = x_1.$$

To solve the previous problem we apply the multitime maximum principle [11]-[22]. In general notations, we have

$$x = (x^i),\ u = (u_\alpha),\ u_\alpha = (u^i_\alpha),\ p = (p^\alpha),\ p^\alpha = (p^\alpha_i),\ \alpha = 1,2;\ i = 1,2,3$$
$$u_1 = (u^1_1, u^2_1, u^3_1),\ u_2 = (u^1_2, u^2_2, u^3_2),\ p^1 = (p^1_1, p^1_2, p^1_3),\ p^2 = (p^2_1, p^2_2, p^2_3)$$
$$X_\alpha(x(t), u(t)) = u_\alpha(t),\ X^0(x(t), u(t)) = -1,\ q(u) = 0$$

and the control Hamiltonian is

$$H(x, p, u) = p^\alpha_i X^i_\alpha(x, u) + p_0 X^0(x, u) - \mu q(u),$$

where $\mu(t)$ is a Lagrange multiplier. Taking $p_0 = 1$, we have $H(x,p,u) = p^\alpha_i u^i_\alpha - 1 - \mu q(u)$. The adjoint dynamics says

$$(ADJ) \qquad \frac{\partial p^\alpha_i}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i} = 0.$$

The general solution of the adjoint PDE system is

$$p^1_i = \frac{\partial F_i}{\partial t^2},\ p^2_i = -\frac{\partial F_i}{\partial t^1},\ i = 1,2,3,$$

where F_i are arbitrary C^2 functions.

We have to maximize the Hamiltonian $H(x,p,u) = p^\alpha_i u^i_\alpha - 1 - \mu q(u)$ with respect to the control u. The critical points u of H are solutions of the algebraic system

$$(1) \qquad \begin{aligned} &p^1_i - 2\mu(||u_2||^2 u^i_1 - <u_1, u_2> u^i_2) = 0 \\ &p^2_i - 2\mu(||u_1||^2 u^i_2 - <u_1, u_2> u^i_1) = 0 \\ &||u_1||^2 ||u_2||^2 - <u_1, u_2>^2 - 1 = 0. \end{aligned}$$

The system (1) is equivalent to the system

$$(2) \qquad \begin{aligned} &u^i_1 p^1_i = 2\mu, u^i_1 p^2_i = 0, u^i_2 p^1_i = 0, u^i_2 p^2_i = 2\mu \\ &||u_1||^2 ||u_2||^2 - <u_1, u_2>^2 - 1 = 0. \end{aligned}$$

In this way u_1 is orthogonal to p^2, and u_2 is orthogonal to p^1. Also

$$(3) \qquad \delta^{ij} p^1_i p^1_j = 4\mu^2 ||u_2||^2,\ \delta^{ij} p^1_i p^2_j = -4\mu^2 <u_1, u_2>,\ \delta^{ij} p^2_i p^2_j = 4\mu^2 ||u_1||^2.$$

The first two relations of (1) are equivalent to

$$2\mu u^i_1 = ||u_1||^2 p^1_i + <u_1, u_2> p^2_i,\ 2\mu u^i_2 = ||u_2||^2 p^2_i + <u_1, u_2> p^1_i$$

or, via the relations (3), we obtain the unique solution

$$u^i_1 = \frac{||p^2||^2}{8\mu^3} p^1_i - \frac{<p^1, p^2>}{8\mu^3} p^2_i,\ u^i_2 = \frac{||p^1||^2}{8\mu^3} p^2_i - \frac{<p^1, p^2>}{8\mu^3} p^1_i,$$

depending on the parameter $\mu(t)$. The functions F_i are constrained by the complete integrability conditions $\dfrac{\partial u^i_1}{\partial t^2} = \dfrac{\partial u^i_2}{\partial t^1}$.

Lemma 1.1. *The Lagrange multiplier $\mu(t)$ is a constant.*

Proof. The relations (2) can be written in the form $p_i^\alpha u_\beta^i = 2\mu\delta_\beta^\alpha$. By differentiation with respect to $\dfrac{\partial}{\partial t^\alpha}$, we find $\dfrac{\partial p_i^\alpha}{\partial t^\alpha}u_\beta^i + p_i^\alpha u_{\beta\alpha}^i = 2\dfrac{\partial \mu}{\partial t^\beta}$ and, via (ADJ), we can write $\dfrac{\partial \mu}{\partial t^\beta} = \dfrac{1}{2}p_i^\alpha u_{\beta\alpha}^i$. On the other hand, the relations (2), the condition $q(u) = 0$ and techniques from differential geometry give $p_i^\alpha u_{\beta\alpha}^i = 0$, i.e., $\mu(t) = constant$. □

Corollary 1.2. *The smallest area surface evolving with unit areal speed is a minimal surface.*

Introducing u_1^i and u_2^i in the restriction $U : q(u) = 0$, we find the areal speed in the dual variables $||p^1||^2||p^2||^2 - < p^1, p^2 >^2 = 16\mu^4(t) = constant$.

Consequently, we have the following

Theorem 1.3. *The smallest area surface evolving with unit areal speed is a minimal surface, solution of the PDE system*

$$(PDE)\qquad \begin{gathered} \frac{\partial x^i}{\partial t^1} = \frac{||p^2||^2}{8\mu^3}p_i^1 - \frac{< p^1, p^2 >}{8\mu^3}p_i^2, \\ \frac{\partial x^i}{\partial t^2} = \frac{||p^1||^2}{8\mu^3}p_i^2 - \frac{< p^1, p^2 >}{8\mu^3}p_i^1, \\ x(0) = x_0,\ x(\tau) = x_1;\ \mu = constant; \end{gathered}$$

$$(ADJ)\qquad \frac{\partial p_i^\alpha}{\partial t^\alpha} = 0,\ B(p(t))|_{\partial\Omega_{0\tau}} = 0,$$

where B means boundary condition.

In fact the strongest restriction on the surface is the existence of a global holonomic frame $\{u_1, u_2\}$.

1.1 Constant dual variables

A very special solution of the adjoint (divergence) PDE is the constant solution

$$p_i^\beta(t) = c_i^\beta(\text{constants}) \neq 0.$$

Particularly, if $p^1(0), p^2(0)$ are linearly independent, then p^1, p^2 rest linearly independent during the evolution. Also, by the initial condition

$$||p^1(0)||^2||p^2(0)||^2 - < p^1(0), p^2(0) >^2 = 1,$$

the constants vectors p^1, p^2 satisfy $||p^1||^2||p^2||^2 - < p^1, p^2 >^2 = 1$ throughout, where $||p^1|| = \delta^{ij}p_i^1p_j^1$, $||p^2|| = \delta^{ij}p_i^2p_j^2$, $< p^1, p^2 > = \delta^{ij}p_i^1p_j^2$. If we select p^1, p^2 with $\Delta = ||p^1||^2||p^2||^2 - < p^1, p^2 >^2 = 1$, then $\mu = \pm\frac{1}{2}$.

We select $\mu = \frac{1}{2}$. The vectors $u_\alpha(\cdot) = u_{\alpha 0} = (u_{\alpha 0}^i)$ are constant in time. Cauchy problem for (PDE) system has the solution

$$x^i(t) = u_{10}^i t^1 + u_{20}^i t^2 + x_0^i,\ i = 1, 2, 3.$$

depending on six arbitrary constants. We fix these constants by the following conditions: $x^i(\tau) = x_1^i$ which implies $\det[u_1, u_2, x_0 - x_1] = 0$, $q(u) = 1$, and finally, the transversality conditions

$$(\tau) \qquad p(0) \perp T_{x_0}\Gamma_0, \quad p(\tau) \perp T_{x_1}\Gamma_1,$$

which show that the optimal plane is orthogonal to Γ_0 and Γ_1, and so the tangent lines $T_{x_0}\Gamma_0$ and $T_{x_1}\Gamma_1$ are parallel.

That is why, the optimum 2-sheet transversal to the curves Γ_0, Γ_1 is a planar quadrilateral fixed by the starting point $x^i(0) = x_0^i$ on Γ_0 and the terminal point $x^i(\tau) = x_1^i$ on Γ_1.

Remark 1.4. *In the previous hypothesis, the surfaces evolving with unit areal speed and having a minimum area are planar 2-sheets.*

Remark 1.5. *In case of $i = 1, 2, 3$ and $\alpha = 1, 2, 3$ we obtain a problem of controlled diffeomorphisms, having as result a set which include the affine unimodular group.*

Remark 1.6. *The (PDE) system can be replaced by the Pfaff system $dx^i = u_\alpha^i dt^\alpha$. If this system is completely integrable, then we have the previous theory. If this system is not completely integrable, then we apply the theory in [13].*

1.2 Nonconstant dual variables

We start from the minimal revolution surface of Cartesian equation

$$z = \frac{1}{c}\sqrt{\mathrm{ch}^2(cx + c_1) - c^2y^2},$$

where

$$1 + z_x^2 + z_y^2 = \frac{\mathrm{ch}^4(cx + c_1)}{\mathrm{ch}^2(cx + c_1) - c^2y^2}.$$

Let us find a parametrization

$$x = x(t^1, t^2),\ y = y(t^1, t^2),\ z = \frac{1}{c}\sqrt{\mathrm{ch}^2(cx(t^1, t^2) + c_1) - c^2y^2(t^1, t^2)}$$

satisfying (unit areal speed)

$$(1 + z_x^2 + z_y^2)(x_{t^1}y_{t^2} - x_{t^2}y_{t^1})^2 = 1.$$

Taking $x_{t^2} = 0$, we find

$$x_{t^1}y_{t^2} = \frac{1}{1 + z_x^2 + z_y^2},$$

which can be realized for

$$x_{t^1} = \frac{1}{\mathrm{ch}^2(cx + c_1)},\ y_{t^2} = \sqrt{\mathrm{ch}^2(cx + c_1) - c^2y^2}.$$

We find

$$2cx(t^1) + \mathrm{sh}\,2(cx(t^1) + c_1) = 4ct^1 + c_2$$

$$y(t^1,t^2) = \mathrm{ch}\,(cx(t^1)+c_1)\sin(ct^2+\varphi(t^1)).$$

In particular, for $c=1$, $c_1=c_2=0$, $\varphi(t^1)=0$, the implicit equation define the function $x(t^1)$, and we obtain the parametrization

$$x = x(t^1),\ y = \mathrm{ch}\,x(t^1)\sin t^2,\ z = \mathrm{ch}\,x(t^1)\cos t^2.$$

To show that this non-planar minimal surface is a solution to the optimal problem in Section 1, we evidentiate the optimal control

$$u_1 = \left(\frac{1}{\mathrm{ch}^2 x}, \frac{\mathrm{sh}\,x\sin t^2}{\mathrm{ch}^2 x}, \frac{\mathrm{sh}\,x\cos t^2}{\mathrm{ch}^2 x}\right)$$

$$u_2 = \left(0, \mathrm{ch}\,x\cos t^2, -\mathrm{ch}\,x\sin t^2\right).$$

and the adjoint vectors

$$p^1 = 2\mu\ \mathrm{ch}^2\,x\ u_1,\ p^2 = \frac{2\mu}{\mathrm{ch}^2\,x}\ u_2.$$

Moreover, the adjoint PDEs give $\mu=$ constant.

2 Two time evolution with unit areal speed, passing through two points

2.1 Two time evolution avoiding an obstacle

Let us apply the multitime maximum principle to search the smallest area surface satisfying the following conditions: it evolves above the rectangle $\Omega_{0\tau}$ with unit areal speed, it contains two diagonal points $x(0)$ and $x(\tau)$, and it avoids an obstacle A whose boundary is ∂A. For that we start with the controlled dynamics problem and its solution in Section 1. Suppose $x(t)\notin\partial A$ for $t\in\Omega_{0\tau}$. In this hypothesis, the multi-time maximum principle applies, and hence the initial dynamics (PDE) and the adjoint dynamics (ADJ) are those in Section 1. Particularly, if the dual variables are constants, then the evolution (PDE) shows that the "surface" of evolution is a planar quadrilateral (starting from origin $x^i(0)=x^i_0$ and ending at a terminal point $x^i(\tau)=x_1$.

We accept that the boundary of the obstacle is a surface (manifold). Also, to simplify, we accept as obstacle a 2-dimensional cylinder (that supports a global tangent frame $\{u_1,u_2\}$) or a 2-dimensional sphere (that does not support a global tangent frame because any continuous vector field on such sphere vanishes somewhere).

Remark 2.1. *Of all the solids having a given volume, the sphere is the one with the smallest surface area; of all solids having a given surface area, the sphere is the one having the greatest volume. These properties define the sphere uniquely and can be seen by observing soap bubbles. A soap bubble will enclose a fixed volume and due to surface tension it will try to minimize its surface area. This is why a free floating soap bubble approximates a sphere (though external forces such as gravity will distort the bubble's shape slightly). The sphere has the smallest total mean curvature among all convex solids with a given surface area. The sphere has constant positive mean curvature. The sphere is the only imbedded surface without boundary or singularities with constant positive mean curvature.*

2.2 Two time evolution touching an obstacle

The points $0 \leq s_0 \leq s_1 \leq \tau$ generates a decomposition Ω_{0s_0}, $\Omega_{0s_1} \setminus \Omega_{0s_0}$, $\Omega_{0\tau} \setminus \Omega_{0s_1}$ of the bidimensional interval $\Omega_{0\tau}$. To simplify the problem, suppose the sheet $x(t) \notin \partial A$ for $t \in \Omega_{0s_0} \cup (\Omega_{0\tau} \setminus \Omega_{0s_1})$ is a union of two planar quadrilaterals (one starting from $x(0)$ and ending in $x(s_0)$ and the other starting from $x(s_1)$ and ending at $x(\tau)$). If $x(t) \in \partial A$ for $t \in \Omega_{0s_1} \setminus \Omega_{0s_0}$, then we need the study in Section 3 and Section 4, which evidentiates the controls and the dual variables capable to keep the evolution on the obstacle. Furthemore, suitable smoothness conditions on boundaries are necessary.

3 Touching, approaching and leaving a cylinder

3.1 Touching a cylinder

Let us take the cylinder $C : (x^1)^2 + (x^2)^2 \leq r^2$ as obstacle. Suppose $x(t) \in \partial C$ for $t \in \Omega_{0s_1} \setminus \Omega_{0s_0}$. In this case we use the modified version of two-time maximum principle.

We introduce the set $N = R^3 \setminus C : \; f(x) = r^2 - ((x^1)^2 + (x^2)^2) \leq 0$, where $x = (x^1, x^2, x^3)$ and we build the functions

$$c_\alpha(x,u) = \frac{\partial f}{\partial x^i}(x) X^i_\alpha(x,u), \; \alpha = 1,2,$$

i.e., $c_\alpha(x,u) = -2(x^1 u^1_\alpha + x^2 u^2_\alpha)$. Let us use the two-time maximum principle using the constraints

$$c_1(x,u) = -2(x^1 u^1_1 + x^2 u^2_1) = 0, \; c_2(x,u) = -2(x^1 u^1_2 + x^2 u^2_2) = 0$$

$$q(u) = ||u_1||^2 ||u_2||^2 - < u_1, u_2 >^2 - 1 = 0.$$

Then the adjoint equations

$$\frac{\partial p^\alpha_i}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i} + \lambda^\gamma(t) \frac{\partial c_\gamma}{\partial x^i}$$

are reduced to

$$(ADJ') \qquad \frac{\partial p^\alpha_1}{\partial t^\alpha}(t) = \lambda^\gamma(t)(-2u^1_\gamma), \; \frac{\partial p^\alpha_2}{\partial t^\alpha}(t) = \lambda^\gamma(t)(-2u^2_\gamma), \; \frac{\partial p^\alpha_3}{\partial t^\alpha}(t) = 0.$$

The critical point condition with respect to the control u is

$$\frac{\partial H}{\partial u} = \lambda^\gamma \frac{\partial c_\gamma}{\partial u},$$

i.e.,

$$\frac{\partial H}{\partial u^i_1} = \lambda^1 \frac{\partial c_1}{\partial u^i_1}, \; \frac{\partial H}{\partial u^i_2} = \lambda^2 \frac{\partial c_2}{\partial u^i_2},$$

or

$$(4) \qquad \begin{gathered} p^1_i = \lambda^1(-2x^i) + 2\mu(||u_2||^2 u^i_1 - < u_1, u_2 > u^i_2), \; i = 1,2 \\ p^1_3 = 2\mu(||u_2||^2 u^3_1 - < u_1, u_2 > u^3_2), \\ p^2_i = \lambda^2(-2x^i) + 2\mu(||u_1||^2 u^i_2 - < u_1, u_2 > u^i_1), \; i = 1,2, \\ p^2_3 = 2\mu(||u_1||^2 u^3_2 - < u_1, u_2 > u^3_1). \end{gathered}$$

We recall that $x(t) \in \partial C$ means $(x^1)^2 + (x^2)^2 = r^2$. Consequently,

$$x^i p_i^1 = \lambda^1(-2r^2),\ x^i p_i^2 = \lambda^2(-2r^2),\ i = 1, 2.$$

To develop further our ideas, we accept that the cylinder C is represented by the parametrization $x^1 = r\cos\theta^1$, $x^2 = r\sin\theta^1$, $x^3 = \theta^2$. We use the partial velocities (orthogonal vectors)

$$\frac{\partial x^1}{\partial \theta^1} = -r\sin\theta^1,\ \frac{\partial x^2}{\partial \theta^1} = r\cos\theta^1,\ \frac{\partial x^3}{\partial \theta^1} = 0$$

$$\frac{\partial x^1}{\partial \theta^2} = 0,\ \frac{\partial x^2}{\partial \theta^2} = 0,\ \frac{\partial x^3}{\partial \theta^2} = 1.$$

Then the area formula

$$\iint_{\sigma\subset\partial C} d\sigma = r\int_0^{\theta^1}\int_0^{\theta^2} d\theta^1 d\theta^2 = r\theta^1\theta^2 = t^1 t^2$$

suggests to take $t^1 = r\,\theta^1$, $t^2 = \theta^2$. On the other hand, the evolution PDEs are transformed in

$$\frac{\partial x^i}{\partial \theta^1} = \frac{\partial x^i}{\partial t^1}\frac{\partial t^1}{\partial \theta^1} + \frac{\partial x^i}{\partial t^2}\frac{\partial t^2}{\partial \theta^1} = r\frac{\partial x^i}{\partial t^1} = r u_1^i$$

$$\frac{\partial x^i}{\partial \theta^2} = \frac{\partial x^i}{\partial t^1}\frac{\partial t^1}{\partial \theta^2} + \frac{\partial x^i}{\partial t^2}\frac{\partial t^2}{\partial \theta^2} = \frac{\partial x^i}{\partial t^2} = u_2^i,$$

evidentiating the controls

$$u_1:\ u_1^1 = -\sin\theta^1,\ u_1^2 = \cos\theta^1,\ u_1^3 = 0$$

$$u_2:\ u_2^1 = 0,\ u_2^2 = 0,\ u_2^3 = 1,$$

with $< u_1, u_2 >= 0$, $||u_1|| = 1$, $||u_2|| = 1$. Using the equalities (4), we obtain $u_1^i p_i^1 = 2\mu$, $u_1^i p_i^2 = 0$, $u_2^i p_i^1 = 0$, $u_2^i p_i^2 = 2\mu$, which confirm that u_1 is orthogonal to p^2 and u_2 is orthogonal to p^1.

The relations $p_i^1 = \lambda^1(-2x^i) + 2\mu u_1^i$, $i = 1, 2$, $p_3^1 = 0$ produce

$$p_1^1 = -2\lambda^1 r\cos\theta^1 - 2\mu\sin\theta^1,\ p_2^1 = -2\lambda^1 r\sin\theta^1 + 2\mu\cos\theta^1,\ p_3^1 = 0.$$

Similarly, the relations $p_i^2 = \lambda^2(-2x^i) + 2\mu u_2^i$, $i = 1, 2$, $p_3^2 = 2\mu$ give

$$p_1^2 = -2\lambda^2 r\cos\theta^1,\ p_2^2 = -2\lambda^2 r\sin\theta^1,\ p_3^2 = 2\mu.$$

Using the partial derivative operators

$$\frac{\partial}{\partial \theta^1} = r\frac{\partial}{\partial t^1},\ \frac{\partial}{\partial \theta^2} = \frac{\partial}{\partial t^2},$$

the adjoint equations (ADJ') become

$$\frac{\partial p_1^1}{\partial \theta^1} + r\frac{\partial p_1^2}{\partial \theta^2} = \lambda^\gamma(-2ru_\gamma^1),\ \frac{\partial p_2^1}{\partial \theta^1} + r\frac{\partial p_2^2}{\partial \theta^2} = \lambda^\gamma(-2ru_\gamma^2),\ \frac{\partial p_3^1}{\partial \theta^1} + r\frac{\partial p_3^2}{\partial \theta^2} = 0.$$

Replacing p^1, p^2, it follows the PDE system

$$(r\frac{\partial\lambda^1}{\partial\theta^1}+r^2\frac{\partial\lambda^2}{\partial\theta^2}+\mu)\cos\theta^1+\frac{\partial\mu}{\partial\theta^1}\sin\theta^1=0$$

$$(r\frac{\partial\lambda^1}{\partial\theta^1}+r^2\frac{\partial\lambda^2}{\partial\theta^2}+\mu)\sin\theta^1+\frac{\partial\mu}{\partial\theta^1}\cos\theta^1=0$$

and hence the parameters λ^γ and μ are determined as solution of PDE system in the next theorem

Theorem 3.1. *We consider the problem of smallest area surface, evolving with unit areal speed, touching a cylinder. Then the evolution on the cylinder is characterized by the parameters λ^γ and μ related by the PDEs*

$$r\frac{\partial\lambda^1}{\partial\theta^1}+r^2\frac{\partial\lambda^2}{\partial\theta^2}+\mu=0,\ \frac{\partial\mu}{\partial\theta^1}=0.$$

The particular solution

$$\lambda^1=-\frac{\mu\theta^1+k^1}{r},\ \lambda^2=k^2=\text{constant},\ \mu=\text{constant}$$

gives

$$p_1^1(\theta)=2(\mu\theta^1+k^1)\cos\theta^1-2\mu\sin\theta^1$$
$$p_2^1(\theta)=2(\mu\theta^1+k^1)\sin\theta^1+2\mu\cos\theta^1,\ p_3^1=0;$$
$$p_1^2(\theta)=-2k^2r\cos\theta^1,\ p_2^2(\theta)=-2k^2r\sin\theta^1,\ p_3^2(\theta)=2\mu,$$

for $\theta=(\theta^1,\theta^2)$.

The evolution (PDE) on the interval $\omega_0\le\theta^1\le\omega_1$ shows that the surface of evolution is a cylindric quadrilateral fixed by the initial point $x^i(\omega_0,\theta^2)=x_0^i$ generator and with the terminal point $x^i(\omega_1,\theta^2)=x_1^i$ generator.

3.2 Approaching and leaving the cylinder

Now we must put together the previous results. So suppose $x(t)\in N=R^3\setminus C$ for $t\in\Omega_{0s_0}\cup(\Omega_{0\tau}\setminus\Omega_{0s_1})$ and $x(t)\in\partial C$ for $t\in\Omega_{0s_1}\setminus\Omega_{0s_0}$. For $t\in\Omega_{0s_0}\cup(\Omega_{0\tau}\setminus\Omega_{0s_1})$, the 2-sheet of evolution $x(\cdot)$ consists in two pieces. Particularly, it can be a union of two planar sheets. Suppose the first planar sheet touchs the cylinder at the point $x(s_0)$. In this case, we can take

$$p_1^1=-\cos\phi_0,\ p_2^1=\sin\phi_0,\ p_3^1=0,$$

for the tangency angle ϕ_0 from initial point x_0, and

$$p_1^2=0,\ p_2^2=0,\ p_3^2=1.$$

By the jump conditions, the vectors $p^1(\cdot)$, $p^2(\cdot)$ are continuous when the evolution 2-sheet $x(\cdot)$ hits the boundary ∂C at the two-time s_0. In other words, we must have the identities

$$2k^1\cos\theta_0^1-\sin\theta_0^1+\theta_0^1\cos\theta_0^1=-\cos\phi_0$$
$$2k^1\sin\theta_0^1+\cos\theta_0^1+\theta_0^1\sin\theta_0^1=\sin\phi_0$$
$$2k^2r\cos\theta_0^1=0,\ 2k^2r\sin\theta_0^1=0,\ 2\mu=1,$$

i.e., $2k^1 = -\theta_0^1,\ \theta_0^1 + \phi_0 = \dfrac{\pi}{2},\ k^2 = 0,\ \mu = \dfrac{1}{2}$. The last two equalities show that the optimal (particularly, planar) 2-sheet is tangent to the cylinder along the generator $x^1 = x^1(s_0),\ x^2 = x^2(s_0), x^3 \in R$.

Let us analyse what happen with the evolution 2-sheet as it leaves the boundary ∂C at the point $x(s_1)$. We then have

$$p_1^1(\theta_1^{1-}, \theta^2) = -\theta_0^1 \cos\theta_1^1 - \sin\theta_1^1 + \theta_1^1 \cos\theta_1^1,$$
$$p_2^1(\theta_1^{1-}, \theta^2) = -\theta_0^1 \sin\theta_1^1 - \cos\theta_1^1 + \theta_1^1 \sin\theta_1^1,$$
$$p_3^1(\theta_1^{1-}, \theta^2) = 0;\ \ p_1^2(\theta_1^{1-}, \theta^2) = 0,\ p_2^2(\theta_1^{1-}, \theta^2) = 0,\ p_3^2(\theta_1^{1-}, \theta^2) = 1.$$

The formulas for k^1, λ^1, λ^2 imply $\lambda^1(\theta_1^1, \theta^2) = \dfrac{\theta_0^1 - \theta_1^1}{2r}$, $\lambda^2(\theta_1^1, \theta^2) = k^2$. The jump theory gives

$$p^\alpha(\theta_1^{1+}, \theta^2) = p^\alpha(\theta_1^{1-}, \theta^2) - \lambda^\alpha(\theta_1^1, \theta^2)\nabla f(x(\theta_1^1, \theta^2))$$

for $f(x) = r^2 - (x^1)^2 - (x^2)^2$. Then

$$\lambda^1(\theta_1^1, \theta^2)\nabla f(x(\theta_1^1, \theta^2)) = (\theta_1^1 - \theta_0^1)\begin{pmatrix} \cos\theta_1^1 \\ \sin\theta_1^1 \\ 0 \end{pmatrix}.$$

In this way, $p_1^1(\theta_1^{1+}, \theta^2) = -\sin\theta_1^1$, $p_2^1(\theta_1^{1+}, \theta^2) = \cos\theta_1^1$, and so the planar 2-sheet of evolution is tangent to the boundary ∂C along the generator by the point $x(s_1)$. If we apply the usual two-time maximum principle after $x(\cdot)$ leaves the cylinder C, we find

$$p_1^1 = \text{constant} = -\cos\phi_1,\ p_2^1 = \text{constant} = -\sin\phi_1;\ p_1^2 = 0,\ p_2^2 = 0.$$

Therefore $-\cos\phi_1 = -\sin\theta_1^1,\ -\sin\phi_1 = -\cos\theta_1^1$ and so $\phi_1 + \theta_1^1 = \pi,\ k^2 = 0$.

Open Problem. What happen when the surface in the exterior of the cylinder is a non-planar minimal sheet?

4 Touching, approaching and leaving a sphere

4.1 Touching a sphere

Let us take as obstacle the sphere $B:\ f(x) = r^2 - \delta_{ij}x^i x^j \geq 0,\ x = (x^1, x^2, x^3),\ i, j = 1, 2, 3$. Suppose $x(t) \in \partial B$ for $t \in \Omega_{0s_1} \setminus \Omega_{0s_0}$. In this case we use the modified version of two-time maximum principle.

We introduce the set $N = R^3 \setminus B:\ f(x) = r^2 - \delta_{ij}x^i x^j \leq 0$ and we build the functions $c_\alpha(x, u) = \dfrac{\partial f}{\partial x^i}(x) X_\alpha^i(x, u),\ \alpha = 1, 2$, i.e., $c_\alpha(x, u) = -2\delta_{ij}x^i u_\alpha^j$. Let us use the two-time maximum principle using the constraints

$$c_1(x, u) = -2\delta_{ij}x^i u_1^j = 0,\ c_2(x, u) = -2\delta_{ij}x^i u_2^j = 0$$

$$q(u) = ||u_1||^2||u_2||^2 - < u_1, u_2 >^2 - 1 = 0.$$

Then the condition
$$\frac{\partial p_i^\alpha}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i} + \lambda^\gamma(t)\frac{\partial c_\gamma}{\partial x^i}$$
is reduced to

$$(ADJ'') \qquad \frac{\partial p_i^\alpha}{\partial t^\alpha}(t) = \lambda^\gamma(t)(-2u_\gamma^i).$$

The condition of critical point for the control u becomes $\dfrac{\partial H}{\partial u} = \lambda^\gamma \dfrac{\partial c_\gamma}{\partial u}$, i.e.,
$$\frac{\partial H}{\partial u_1^i} = \lambda^1 \frac{\partial c_1}{\partial u_1^i},\ \frac{\partial H}{\partial u_2^i} = \lambda^2 \frac{\partial c_2}{\partial u_2^i},$$
or

$$(5) \qquad \begin{aligned} p_i^1 &= \lambda^1(-2x^i) + 2\mu(||u_2||^2 u_1^i - < u_1, u_2 > u_2^i) \\ p_i^2 &= \lambda^2(-2x^i) + 2\mu(||u_1||^2 u_2^i - < u_1, u_2 > u_1^i). \end{aligned}$$

We recall that $x(t) \in \partial B$ means $\delta_{ij}x^i x^j = r^2$. Consequently,
$$x^i p_i^1 = \lambda^1(-2r^2),\ x^i p_i^2 = \lambda^2(-2r^2).$$

To develop further our ideas, we accept that the sphere B is represented by the parametrization
$$x^1 = r\cos\theta^1\cos\theta^2,\ x^2 = r\sin\theta^1\cos\theta^2,\ x^3 = r\sin\theta^2.$$

We use the partial velocities (orthogonal vectors)
$$\frac{\partial x^1}{\partial\theta^1} = -r\sin\theta^1\cos\theta^2,\ \frac{\partial x^2}{\partial\theta^1} = r\cos\theta^1\cos\theta^2,\ \frac{\partial x^3}{\partial\theta^1} = 0$$
$$\frac{\partial x^1}{\partial\theta^2} = -r\cos\theta^1\sin\theta^2,\ \frac{\partial x^2}{\partial\theta^2} = -r\sin\theta^1\sin\theta^2,\ \frac{\partial x^3}{\partial\theta^2} = r\cos\theta^2.$$
Then the area formula
$$\int\!\!\int_{\sigma\subset\partial B} d\sigma = r^2\int_0^{\theta^1}\int_0^{\theta^2}\cos\theta^2 d\theta^1 d\theta^2 = r^2\theta^1\sin\theta^2 = t^1t^2$$
suggests to take $t^1 = r\theta^1,\ t^2 = r\sin\theta^2$. On the other hand, the evolution $PDEs$ are transformed in
$$\frac{\partial x^i}{\partial\theta^1} = \frac{\partial x^i}{\partial t^1}\frac{\partial t^1}{\partial\theta^1} + \frac{\partial x^i}{\partial t^2}\frac{\partial t^2}{\partial\theta^1} = r\frac{\partial x^i}{\partial t^1} = ru_1^i$$
$$\frac{\partial x^i}{\partial\theta^2} = \frac{\partial x^i}{\partial t^1}\frac{\partial t^1}{\partial\theta^2} + \frac{\partial x^i}{\partial t^2}\frac{\partial t^2}{\partial\theta^2} = r\cos\theta^2\frac{\partial x^i}{\partial t^2} = r\cos\theta^2 u_2^i,$$
evidentiating the controls
$$u_1 :\ u_1^1 = -\sin\theta^1\cos\theta^2,\ u_1^2 = \cos\theta^1\cos\theta^2,\ u_1^3 = 0$$
$$u_2 :\ u_2^1 = -\cos\theta^1\ \tan\theta^2,\ u_2^2 = -\sin\theta^1\tan\theta^2,\ u_2^3 = 1,$$

with $< u_1, u_2 >= 0$, $||u_1|| = \cos\theta^2$, $||u_2|| = \dfrac{1}{\cos\theta^2}$. Using the equalities (5), we obtain $u_1^i p_i^1 = 2\mu$, $u_1^i p_i^2 = 0$, $u_2^i p_i^1 = 0$, $u_2^i p_i^2 = 2\mu$, which confirm that u_1 is orthogonal to p^2 and u_2 is orthogonal to p^1.

We have

$$p_i^1 = \lambda^1(-2x^i) + 2\mu\,||u_2||^2 u_1^i,\; p_i^2 = \lambda^2(-2x^i) + 2\mu\,||u_1||^2 u_2^i,\; i = 1, 2, 3,$$

or explicitely

$$p_1^1 = \lambda^1(-2r\,\cos\theta^1\cos\theta^2) - 2\mu\,\frac{\sin\theta^1}{\cos\theta^2}$$
$$p_2^1 = \lambda^1(-2r\,\sin\theta^1\cos\theta^2) + 2\mu\,\frac{\cos\theta^1}{\cos\theta^2}$$
$$p_3^1 = \lambda^1(-2r\,\sin\theta^2)$$
$$p_1^2 = \lambda^2(-2r\,\cos\theta^1\,\cos\theta^2) - 2\mu\,\cos\theta^1\,\sin\theta^2\,\cos\theta^2$$
$$p_2^2 = \lambda^2(-2r\,\sin\theta^1\,\cos\theta^2) - 2\mu\,\sin\theta^1\,\sin\theta^2\,\cos\theta^2$$
$$p_3^2 = \lambda^2(-2r\,\sin\theta^2) + 2\mu\,(\cos\theta^2)^2.$$

Using the partial derivative operators

$$\frac{\partial}{\partial\theta^1} = r\frac{\partial}{\partial t^1},\; \frac{\partial}{\partial\theta^2} = r\cos\theta^2\frac{\partial}{\partial t^2},$$

the adjoint equations (ADJ'') become

$$\frac{\partial p_i^1}{\partial\theta^1}\cos\theta^2 + \frac{\partial p_i^2}{\partial\theta^2} = \lambda^\gamma(-2ru_\gamma^i\cos\theta^2).$$

Replacing p^1, p^2, it follows the PDE system

$$\frac{\partial\lambda^1}{\partial\theta^1}(-2r\,\cos\theta^1\,(\cos\theta^2)^2) - 2\frac{\partial\mu}{\partial\theta^1}\,\sin\theta^1$$
$$+\frac{\partial\lambda^2}{\partial\theta^2}(-2r\,\cos\theta^1\,\cos\theta^2) - 2\frac{\partial\mu}{\partial\theta^2}\,\cos\theta^1\,\sin\theta^2\,\cos\theta^2 - 4\mu\,\cos\theta^1\,(\cos\theta^2)^2 = 0,$$
$$\frac{\partial\lambda^1}{\partial\theta^1}(-2r\,\sin\theta^1\,(\cos\theta^2)^2) + 2\frac{\partial\mu}{\partial\theta^1}\,\cos\theta^1$$
$$+\frac{\partial\lambda^2}{\partial\theta^2}(-2r\,\sin\theta^1\,\cos\theta^2) - 2\frac{\partial\mu}{\partial\theta^2}\,\sin\theta^1\,\sin\theta^2\,\cos\theta^2 - 4\mu\,\sin\theta^1\,(\cos\theta^2)^2 = 0,$$
$$\frac{\partial\lambda^1}{\partial\theta^1}(-2r\,\sin\theta^2\,\cos\theta^2) + \frac{\partial\lambda^2}{\partial\theta^2}(-2r\,\sin\theta^2\,) + 2\frac{\partial\mu}{\partial\theta^2}\,(\cos\theta^2)^2 - 4\mu\,\cos\theta^2\,\sin\theta^2 = 0,$$

equivalent to

$$\frac{\partial\lambda^1}{\partial\theta^1}(-2r\cos\theta^2) + \frac{\partial\lambda^2}{\partial\theta^2}(-2r) - 2\frac{\partial\mu}{\partial\theta^2}\sin\theta^2 - 4\mu\cos\theta^2 = 0,\; \frac{\partial\mu}{\partial\theta^1} = 0,$$

$$\frac{\partial\lambda^1}{\partial\theta^1}(-2r\sin\theta^2\;\cos\theta^2) + \frac{\partial\lambda^2}{\partial\theta^2}(-2r\,\sin\theta^2\,) - 2\frac{\partial\mu}{\partial\theta^2}\,\cos^2\theta^2 + 4\mu\,\cos\theta^2\,\sin\theta^2 = 0$$

or to the system in the next

Theorem 4.1. *We consider the problem of smallest area surface, evolving with unit areal speed, touching a sphere. Then the evolution on the sphere is characterized by the parameters λ^γ and μ related by the PDEs*

$$\frac{\partial \lambda^1}{\partial \theta^1}(-r\cos\theta^2) + \frac{\partial \lambda^2}{\partial \theta^2}(-r) - 2\mu\cos\theta^2 = 0,\ \frac{\partial \mu}{\partial \theta^1} = 0,\ \frac{\partial \mu}{\partial \theta^2} = 0.$$

It follows $\mu =$ constant. The particular solution

$$\lambda^1 = -\frac{2\mu\theta^1 + k^1}{r},\ \lambda^2 = -\frac{k^2}{2r}$$

produces

$$p_1^1(\theta) = (2\mu\theta^1 + k^1)(2\,\cos\theta^1\cos\theta^2) - 2\mu\,\frac{\sin\theta^1}{\cos\theta^2}$$

$$p_2^1(\theta) = (2\mu\theta^1 + k^1)(2\,\sin\theta^1\cos\theta^2) + 2\mu\,\frac{\cos\theta^1}{\cos\theta^2}$$

$$p_3^1(\theta) = (2\mu\theta^1 + k^1)(2\,\sin\theta^2);$$

$$p_1^2(\theta) = k^2(\cos\theta^1\,\cos\theta^2) - 2\mu\,\cos\theta^1\,\sin\theta^2\,\cos\theta^2$$

$$p_2^2(\theta) = k^2(\sin\theta^1\,\cos\theta^2) - 2\mu\,\sin\theta^1\,\sin\theta^2\,\cos\theta^2$$

$$p_3^2(\theta) = k^2(\sin\theta^2) + 2\mu\,(\cos\theta^2)^2,\ \ \theta = (\theta^1, \theta^2).$$

4.2 Approaching and leaving the sphere

Now we must put together the previous results. So suppose $x(t) \in N = R^3 \setminus B$ for $t \in \Omega_{0s_0} \cup (\Omega_{0\tau} \setminus \Omega_{0s_1})$ and $x(t) \in \partial C$ for $t \in \Omega_{0s_1} \setminus \Omega_{0s_0}$.

For $t \in \Omega_{0s_0} \cup (\Omega_{0\tau} \setminus \Omega_{0s_1})$, the 2-sheet of evolution $x(\cdot)$ consists in two pieces. Particularly, it can be a union of two planar sheets. Suppose the first planar sheet touchs the sphere at the point $x(s_0)$. In this case, we can take

$$p_1^1 = -\cos\phi_0,\ p_2^1 = \sin\phi_0,\ p_3^1 = 0,$$

for the tangency angle ϕ_0 from the initial point x_0, and

$$p_1^2 = 0,\ p_2^2 = 0,\ p_3^2 = 1.$$

By the jump conditions, the vectors $p^1(\cdot)$, $p^2(\cdot)$ are continuous when the evolution 2-sheet $x(\cdot)$ hits the boundary ∂B at the two-time s_0. In other words, we must have the identities

$$(2\mu\theta_0^1 + k^1)(2\,\cos\theta_0^1\cos\theta_0^2) - 2\mu\,\frac{\sin\theta_0^1}{\cos\theta_0^2} = -\cos\phi_0$$

$$(2\mu\theta_0^1 + k^1)(2\,\sin\theta_0^1\cos\theta_0^2) + 2\mu\,\frac{\cos\theta_0^1}{\cos\theta_0^2} = \sin\phi_0$$

$$(2\mu\theta_0^1 + k^1)(2\,\sin\theta_0^2) = 0;$$

$$k^2(\cos\theta_0^1\,\cos\theta_0^2) - 2\mu\,\cos\theta_0^1\,\sin\theta_0^2\,\cos\theta_0^2 = 0$$

$$k^2(\sin\theta_0^1\,\cos\theta_0^2) - 2\mu\,\sin\theta_0^1\,\sin\theta_0^2\,\cos\theta_0^2 = 0$$

$$k^2(\sin\theta_0^2) + 2\mu\,(\cos\theta_0^2)^2 = 1,$$

i.e., $k^1 = -\theta_0^1,\ \theta_0^2 = 0,\ \theta_0^1 + \phi_0 = \dfrac{\pi}{2},\ k^2 = 0,\ \mu = \dfrac{1}{2}$. The last two equalities show that the optimal (particularly, planar) 2-sheet is tangent to the sphere at the point $(x^1(s_0), x^2(s_0), x^3(s_0))$.

Let us analyse what happen with the evolution 2-sheet as it leaves the boundary ∂B at the point $x(s_1)$. We then have

$$p_1^1(\theta_1^{1-}, \theta_1^{2-}) = (\theta_1^1 - \theta_1^0)(2\cos\theta_1^1\cos\theta_1^2) - 2\mu\,\frac{\sin\theta_1^1}{\cos\theta_1^2}$$

$$p_2^1(\theta_1^{1-}, \theta_1^{2-}) = (\theta_1^1 - \theta_1^0)(2\sin\theta_1^1\cos\theta_1^2) + 2\mu\,\frac{\cos\theta_1^1}{\cos\theta_1^2}$$

$$p_3^1(\theta_1^{1-}, \theta_1^{2-}) = 0,\ \ p_1^2(\theta_1^{1-}, \theta_1^{2-}) = 0,\ \ p_2^2(\theta_1^{1-}, \theta_1^{2-}) = 0,\ \ p_3^2(\theta_1^{1-}, \theta_1^{2-}) = 1.$$

The formulas for k^1, λ^1, λ^2 imply $\lambda^1(\theta_1^1, \theta^2) = \dfrac{\theta_0^1 - \theta_1^1}{2r}$, $\lambda^2(\theta_1^1, \theta^2) = k^2$. The jump theory gives

$$p^\alpha(\theta_1^{1+}, \theta_1^{2+}) = p^\alpha(\theta_1^{1-}, \theta_1^{2-}) - \lambda^\alpha(\theta_1^1, \theta_1^2)\nabla f(x(\theta_1^1, \theta_1^2))$$

for $f(x) = r^2 - \delta_{ij}x^ix^j$. Then

$$\lambda^1(\theta_1^1, \theta_1^2)\nabla f(x(\theta_1^1, \theta_1^2)) = (\theta_1^1 - \theta_0^1)\begin{pmatrix} \cos\theta_1^1\cos\theta_1^2 \\ \sin\theta_1^1\cos\theta_1^2 \\ \sin\theta_1^2 \end{pmatrix}.$$

In this way, $p_1^1(\theta_1^{1+}, \theta_1^{2+}) = -\sin\theta_1^1$, $p_2^1(\theta_1^{1+}, \theta_1^{2+}) = \cos\theta_1^1$, and so the planar 2-sheet of evolution is tangent to the boundary ∂B at the point $x(s_1)$. If we apply the usual two-time maximum principle after $x(\cdot)$ leaves the sphere B, we find

$$p_1^1 = \text{constant} = -\cos\phi_1,\ \ p_2^1 = \text{constant} = -\sin\phi_1;\ \ p_1^2 = 0,\ \ p_2^2 = 0.$$

Therefore $-\cos\phi_1 = -\sin\theta_1^1,\ -\sin\phi_1 = -\cos\theta_1^1$ and so $\phi_1 + \theta_1^1 = \pi,\ k^2 = 0$.

Open Problem. What happens when the surface in the exterior of the sphere is a non-planar minimal sheet?

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