Geometrization of the scleronomic Riemannian mechanical systems

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Abstract. The paper is devoted to the geometrical theory, on the phase space, of the classical concept of scleronomic Riemannian mechanical systems in the general case when the external forces depend on the material points and their velocities. We discuss the canonical semispray, the nonlinear connection, the metrical connection, the electromagnetic field and the almost Hermitian model of the mentioned mechanical system. Based on the methods of Lagrange geometry we prepare here the framework for the investigation of the geometrical theory of Riemannian mechanical systems whose external forces depend on the accelerations of order $k \geq 1$.

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1 Introduction

The geometrization of the scleronomic Riemannian mechanical systems with external forces depending on the velocity of the material point has been developed by Joseph Klein. His paper [9], published in 1962 in Annales de l'Institut Fourier, still remains the essential reference in this subject. Later on, the problem was treated by other well known mathematicians, see [5, 10, 11, 12]. A modern study of these mechanical systems, using the methods of Lagrange theory, can be found in the papers of R.Miron [13], I.Bucataru and R.Miron [3] and in the recent book "Finsler-Lagrange geometry. Applications to dynamical systems" by I.Bucataru and R.Miron, appeared in the Publishing House of Romanian Academy [2].

The geometrical theory of these systems is built on the base manifold TM of the tangent bundle of the configurations space M. Hereafter we call TM the *phase space*.

A scleronomic Riemannian mechanical system (SRMS on short) is a triple $\Sigma_{\mathcal{R}} = (M, T, F_e)$, where M is a real n-dimensional differentiable manifold, called the configuration space, n is called the freedom degree of the system, $T = \frac{1}{2}g_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt}$ is

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the kinetic energy of a given Riemannian space $\mathcal{R}^n = (M, g)$ and $F_e(x, \frac{dx}{dt})$ are the external forces *a-priori* given. Obviously, the nonconservative Riemannian mechanical systems are particular cases of SRMS.

In this case, the Lagrange equations are expressed by

$$\text{(I)} \qquad \frac{d}{dt}\Big(\frac{\partial L}{\partial y^i}\Big) - \frac{\partial L}{\partial x^i} = F_i(x, y), \quad y^i = \frac{dx^i}{dt}.$$

Fundamental results in the geometric theory of SRMS on the phase space TM has been obtained by J. Klein ([9]), E. Cartan, A. Lichnerowicz, I. Bucataru, R. Miron ([2]) and others.

An important geometrical object is the canonical semi-spray S determined by $\Sigma_{\mathcal{R}}$ only and whose integral curves are the solution curves of the Lagrange equations (I). Therefore, the geometrical theory of the SRMS $\Sigma_{\mathcal{R}}$ is the differential geometry of the pair (TM, S). It follows that the nonlinear connection N determined by S is the canonical nonlinear connection of $\Sigma_{\mathcal{R}}$.

Let us recall here one important property that characterizes the canonical semi-spray S:

The vector field S defined on the phase space TM is the only one satisfying the equation

$$i_S\omega = -dT + \sigma,$$

where $\omega = 2g_{ij}\delta y^j \wedge dx^i$ is the almost symplectic structure of $\Sigma_{\mathcal{R}}$ and $\sigma = F_i(x, y)dx^i$ is the 1-form on TM of the external forces F_e.

However, some other important problems have not been studied yet, for example the study of the vector field S as dynamical system on TM, the evolution nonlinear connection N determined by S, the structure equations of the N-metrical connection, the electromagnetic field of $\Sigma_{\mathcal{R}}$ (in the case when $\mathcal{R}^n = (M, g_{ij})$ is a pseudo-Riemannian space), the gravitational field $g_{ij}(x)$ studied by means of the N-metrical connection of $\Sigma_{\mathcal{R}}$, the almost Hermitian model of $\Sigma_{\mathcal{R}}$ on TM. Only a part of these topics will be studied in the present paper.

The methods used in the study of geometrical theory of SRMS are borrowed from the geometry of Lagrange spaces of order $k \geq 1$. We will consider in the following only the case $k = 1$. The case $k > 1$ will be studied in future by considering the following problem.

Study the geometry of the scleronomic Riemannian mechanical systems $\Sigma_{\mathcal{R}} = (M, T, F_e)$ whose external forces F_e depend on the material points x and on their accelerations of order 1,2,...k, namely, $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, ..., $\frac{d^kx}{dt^k}$.

The difficulty here consists in finding the Lagrange equations of order k for $\Sigma_{\mathcal{R}}$. The solution of the problem is based on the prolongation of order $k \geq 1$ of the Riemannian space $\mathcal{R}^n = (M, g)$, ([15]).

This category of SRMS will be studied in a forthcoming paper.

2 Preliminaries. Semisprays on TM

Let M be an n-dimensional, real differentiable manifold, named the *configuration space*, n being the number of degrees of freedom. A point of M will be denoted by x and its local coordinates by (x^i), $(i = 1, 2, \dots, n)$. The velocity of x with respect to the time t will be denoted by $\dot{x} = \frac{dx}{dt}$ and its coordinates by $\dot{x}^i = \frac{dx^i}{dt}$. These notations are usual in Lagrange geometry, ([16]).

Let us consider the tangent bundle (TM, π, M), whose total space TM is a $2n$-dimensional differentiable manifold called the *phase space* (we point out that in classical mechanics the cotangent space T^*M is considered as the phase space). We prefer this naming for TM because it is more intuitive.

In the following we denote by $(x^i, \frac{dx^i}{dt}) = (x^i, y^i)$ the coordinates of a point $u = (x, \frac{dx}{dt}) = (x, y) \in TM$. A change of local coordinates $(x^i, y^i) \longrightarrow (\widetilde{x}^i, \widetilde{y}^i)$ is given by

$$\widetilde{x}^i = \widetilde{x}^i(x^1, .., x^n),\ det\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) \neq 0,\ \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^j. \tag{2.1}$$

The Einstein summation convention is used for all $i, j, k, ... = 1, 2, .., n$.

The natural basis $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)$ of the tangent vector space T_uTM transforms with respect to (2.1) as follows:

$$\frac{\partial}{\partial x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial}{\partial \widetilde{x}^j} + \frac{\partial \widetilde{y}^j}{\partial x^i}\frac{\partial}{\partial \widetilde{y}^j}, \qquad \frac{\partial}{\partial y^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial}{\partial \widetilde{y}^j}. \tag{2.2}$$

From (2.2) we can see that the vector fields $\left(\frac{\partial}{\partial y^i}\right)$, $(i = 1, .., n)$ generate an integrable distribution V of local dimension n on TM, called the vertical distribution.

Also, from (2.1) and (2.2) we obtain that

$$\mathbb{C} = y^i \frac{\partial}{\partial y^i}.$$

is a vertical vector field on TM, called the *Liouville vector field*, with the property $\mathbb{C} \neq 0$ on $\widetilde{TM} = TM \setminus \{0\}$.

There exists a $\mathcal{F}(TM)$-linear application $J : \chi(TM) \longrightarrow \chi(TM)$ defined by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \qquad J\left(\frac{\partial}{\partial y^i}\right) = 0.$$

One can see that J is globally defined on TM and it has the following properties $KerJ = V$, $ImJ = V$ and $J^2 = 0$. The application J is called the *tangent structure* on TM.

A semispray on the manifold TM is a vector field $S \in \chi(TM)$ with the property

$$J(S) = \mathbb{C}.$$

Locally, a semispray S can be expressed as follows:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}. \tag{2.3}$$

The functions $G^i(x,y)$ give the *coefficients of the semispray S*. The system of functions $\{G^i(x,y)\}$, $(i = 1,..,n)$, determine a geometric object field on TM whose transformation rule, with respect to (2.1), is:

$$2\widetilde{G^i} = 2\frac{\partial \widetilde{x}^i}{\partial x^j}G^j - \frac{\partial \widetilde{y}^i}{\partial x^j}y^j. \tag{2.4}$$

Conversely, a geometric object field G^i having the transformation rule (2.4) determines, by (2.3), a semispray S. **Example.** For a Riemannian space $\mathcal{R}^n = (M, g_{ij}(x))$, the system of functions $2\check{G}^i(x,y) = \gamma^i{}_{jk}(x)y^jy^k$ gives us the coefficients of a semispray, $\gamma^i{}_{jk}(x)$ being the Christoffel symbols of g_{ij}. A semispray S is a spray if and only if its coefficients G^i are 2-homogeneous functions with respect to y^i (see [17, p.8]). Hence this semispray reduces to a spray.

We remark that the difference between two semisprays S and S' on TM is a vertical vector field.

It follows that the vector field S' on TM, defined by $S' = S + F_e$, is a semispray, where S is a semispray and F_e is a vertical vector field.

Therefore, by fixing a semispray S and choosing in a suitable way a vertical vector field F_e, we obtain any semispray S' on TM.

3 Nonlinear connections on TM

As we have seen in the previous section, there exists a vertical distribution V on TM of local dimension n, which is integrable. It is naturally to question the existence of distributions N on TM, complementary to the vertical distribution V. The answer to this question is affirmative, these distributions exist.

A regular distribution N on TM, complementary to the vertical distribution V, is called a *nonlinear connection.* Using it, the tangent space T_uTM splits in a direct sum of the vertical subspace V_u and an horizontal (complementarity) subspace N_u:

$$T_uTM = N_u \oplus V_u, \tag{3.1}$$

for any $u \in TM$.

A detailed theory of nonlinear connections can be found in the monograph [17].

The distribution N of a nonlinear connection is called the *horizontal distribution.*

It can be easily seen that if M is paracompact then there exist nonlinear connections on TM.

In the local coordinates (x^i, y^i), the nonlinear connection N can be characterized by the local basis:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i(x,y)\frac{\partial}{\partial y^j}. \tag{3.2}$$

The system of functions $N^j{}_i(x,y)$ is called the *system of coefficients of the nonlinear connection N* ([2, 16, 17]). With respect to a change of coordinates (2.1) the coefficients $N^j{}_i(x,y)$ transforms by the rule:

$$\frac{\partial \widetilde{x}^j}{\partial x^k}N^k{}_i = \widetilde{N}^j{}_k\frac{\partial \widetilde{x}^k}{\partial x^i} + \frac{\partial \widetilde{y}^j}{\partial x^i}. \tag{3.3}$$

Conversely, if a system of functions $N_i^j(x,y)$ on TM is given, having the rule of transformation (3.3), then it determines a nonlinear connection N on TM.

Remark that N is a *linear connection* if the functions $N^j{}_i(x,y)$ are linear with respect to the variables y^i, namely $N_j^i(x,y) = \Gamma^i_{jk}(x)y^k$.

The following formulas hold good:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^k{}_{ij}\frac{\partial}{\partial y^k}, \quad \text{where } R^k{}_{ij} = \frac{\delta N^k{}_i}{\delta x^j} - \frac{\delta N^k{}_j}{\delta x^i}. \tag{3.4}$$

We point out that $R^k{}_{ij}$ is a d-tensor field (d- means "distinguished", [16, 17]) and it is called *the curvature tensor* of the nonlinear connection N.

From here it immediately follows that N is an integrable distribution if and only if $R^k{}_{ij} = 0$.

The d-tensor field defined by

$$t^k_{ij} = \frac{\partial N^k{}_i}{\partial y^j} - \frac{\partial N^k{}_j}{\partial y^i}$$

is called *the (weak) torsion tensor* of the nonlinear connection N.

We emphasize that a nonlinear connection N determines some important geometric structures on TM as: the *almost complex structure* $\mathbb{F}$, the *almost product structure* $\mathbb{P}$ and the *adjoint structure* $\Theta = \mathbb{F} + J$ (see [2, 16]). Namely, we have

$$J \circ \mathbb{P} = J, \qquad \mathbb{P} \circ J = -J, \qquad \mathbb{F}^2 = -Id, \qquad \Theta^2 = 0,$$

$$\mathbb{F} \circ J + J \circ \mathbb{F} = \Theta \circ J + J \circ \Theta = h + v = Id,$$

where h and v are the projections determined by N and V, respectively.

Recall that the system of vectors $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ is an adapted basis to the direct decomposition (3.1). Its dual basis is $(dx^i, \delta y^i)$, where

$$\delta y^i = dy^i + N^i{}_j(x,y)dx^j. \tag{3.5}$$

Using the formulas (3.2), (3.5), it follows that the previous structures can be expressed in local coordinates by the tensors:

$$J = \frac{\partial}{\partial y^i} \otimes dx^i, \quad \Theta = \frac{\delta}{\delta x^i} \otimes \delta y^i,$$

$$\mathbb{F} = \frac{\delta}{\delta x^i} \otimes \delta y^i - \frac{\partial}{\partial y^i} \otimes dx^i, \quad \mathbb{P} = \frac{\delta}{\delta x^i} \otimes dx^i - \frac{\partial}{\partial y^i} \otimes \delta y^i.$$

Any semispray S, having the coefficients G^i, determines a nonlinear connection N with the coefficients:

$$N^i{}_j(x,y) = \frac{\partial G^i}{\partial y^j}(x,y).$$

If $N^i{}_j(x,y)$ are the coefficients of a nonlinear connection N, then $G^i(x,y) = \frac{1}{2}N^i{}_j(x,y)y^j$ are the coefficients of a semispray given by

$$S = y^i\frac{\partial}{\partial x^i} - N^j{}_i y^i \frac{\partial}{\partial y^j} = y^i\frac{\delta}{\delta x^i},$$

called the *associated semispray* of the nonlinear connection N.

4 The dynamical covariant derivative

The dynamical covariant derivative with respect to the nonlinear connection N is introduced as derivative operator in the d-tensor fields algebra([2]).

Let us consider a nonlinear connection N having the coefficients $N^i{}_j$ and the associated semispray $S = y^i \dfrac{\delta}{\delta x^i}$.

We introduce here *the dynamical covariant derivative* corresponding to the pair (S, N) as the operator $\nabla : \chi^v(TM) \longrightarrow \chi^v(TM)$ defined by:

$$\nabla\left(X^i \frac{\partial}{\partial y^i}\right) = \left(SX^i + X^j N^i{}_j\right) \frac{\partial}{\partial y^i}.$$

Consequently, for any $f \in \mathcal{F}(TM)$, $\forall X \in \chi^v(TM)$, we have $\nabla f = Sf$, $\nabla(X+Y) = \nabla X + \nabla Y$, $\nabla(fX) = \nabla f X + f \nabla X$.

From here we deduce that

$$\nabla\left(\frac{\partial}{\partial y^i}\right) = N^j{}_i \frac{\partial}{\partial y^j}.$$

As usual ([2]) we can extend the operator ∇ to the algebra of d-tensor fields. Therefore, for a d-vector field X^i we have

$$\nabla X^i = X^i{}_{|} = SX^i + X^j N^i{}_j,$$

and for a d-one form ω_i

$$\nabla \omega_i = \omega_{i|} = S\omega_i - \omega_j N^j_i.$$

Analogously, for a covariant d-tensor field, we have, for instance

$$g_{ij|} = \nabla g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = Sg_{ij} - g_{sj} N^s{}_i - g_{is} N^s{}_j.$$

If g_{ij} is a metric tensor on TM, then N is called **metrical** with respect to g_{ij} if $g_{ij|} = 0$. Equivalently, we have $Sg_{ij} = g_{sj} N^s{}_i + g_{is} N^s{}_j$.

Let $c : t \in I \subset \mathbb{R} \longrightarrow c(t) = \left(x^i(t)\right) \in M$ be a curve on M and $\widetilde{c} = \left(x^i(t), \dfrac{dx^i}{dt}(t)\right)$ its extension to TM.

The curve c is an autoparallel curve of the nonlinear connection N if the curve $\widetilde{c}$ is an horizontal curve on TM. In other words, c is an autoparallel curve if and only if

$$\frac{d^2 x^i}{dt^2} + N^i{}_j(x, \frac{dx}{dt}) \frac{dx^j}{dt} = 0.$$

If N is the nonlinear connection $N^i{}_j(x, y) = \gamma^i{}_{jk}(x) y^k$, then for $\forall X^i(x)$ we have

$$\nabla X^i = y^k X^i{}_{|k} = y^k \left(\frac{\partial X^i}{\partial x^k} + X^m \gamma^i{}_{mk}\right),$$

where $X^i{}_{|k}$ is the ∇ operator of h-covariant derivative of $X^i(x, y)$ with respect to the N-linear connection $(\gamma^i{}_{jk}(x), 0)$, [16]. The theory of the N-linear connections and of the corresponding covariant derivatives is well known ([16], [17]).

5 Scleronomic Riemannian mechanical systems

The definition of mechanical systems which external forces depend on material point and its velocity, given by J. Klein ([9]) requires the study of the geometry of the phase space. The geometrization of these systems can not be done using Riemannian techniques only ([13], [18], [19], [20]) as in the classical case, when the external forces does not depend on velocities.

In order to overpass this difficulty, we shall apply the methods of Lagrange geometry ([2], [13], [16], [17]) considering the external forces to be a vertical vector field on the phase space. A good example are the so-called Liouville-Riemannian mechanical systems, where the external forces are given in the form $F_e = a(x,y)\,\mathbb{C}$, $a \in \mathcal{F}(TM)$, $a \neq 0$.

Generally, the main idea is to determine a canonical semispray on the phase space, which depends on the considered mechanical system only and which integral curves are the evolution curves of the mechanical system. Thus, one can regard the geometry of the canonical semispray as the geometry of the considered mechanical system.

Following J. Klein ([9]) we can give

Definition 5.1. A scleronomic Riemannian mechanical system (a SRSM on short) is a triple $\Sigma_{\mathcal{R}} = (M, T, F_e)$, where

- M is an n-dimensional differentiable manifold, called the configuration space;
- $T = \dfrac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j$ is the kinetic energy;
- $g_{ij}(x)$ is the metric tensor of a given Riemannian (or pseudo-Riemannian space) $\mathcal{R}^n = (M, g_{ij}(x))$;
- $F_e(x,y) = F^i(x,y)\dfrac{\partial}{\partial y^i}$ is a vertical vector field given on the phase space TM. F_e called the external forces field.

The covariant components of the external forces $F_e(x,y)$ are given by

$$F_i(x,y) = g_{ij}(x)F^j(x,y).$$

Examples.

1. The SRMS $\Sigma_{\mathcal{R}}$, where $F_e(x,y) = a(x,y)\,\mathbb{C}$, $a \neq 0$, $a \in \mathcal{F}(TM)$. Thus $F^i(x,y) = a(x,y)y^i$. This $\Sigma_{\mathcal{R}}$ can be called the Liouville SRMS, [3].
2. The SRMS $\Sigma_{\mathcal{R}}$, where $F_e(x,y) = F^i(x)\frac{\partial}{\partial y^i}$, and $F_i(x) = grad_i f(x)$, called conservative systems.
3. The SRMS $\Sigma_{\mathcal{R}}$, where $F_e(x,y) = F^i(x)\frac{\partial}{\partial y^i}$, but $F_i(x) \neq grad_i f(x)$, called non-conservative systems.

Remarks.

1. A conservative system $\Sigma_{\mathcal{R}}$ is called *a Lagrangian system* by J. Klein ([9]).
2. One should pay attention to not make confusion of this kind of mechanical systems with the "Lagrangian mechanical systems" $\Sigma_L = (M, L(x,y), F_e(x,y))$ introduced by R. Miron ([15]), where $L : TM \to \mathbb{R}$ is a regular Lagrangian.

Starting from Definition 5.2, in a very similar manner as in the geometrical theory of mechanical systems, one introduces

Postulate. The evolution equations of a SRSM $\Sigma_{\mathcal{R}}$ are the *Lagrange equations*:

$$\frac{d}{dt}\frac{\partial T}{\partial y^i} - \frac{\partial T}{\partial x^i} = F_i(x,y), \qquad y^i = \frac{dx^i}{dt}. \tag{5.1}$$

This postulate will be geometrically justified by the existence of a semi-spray S on TM whose integral curves are given by the equations (5.1). Therefore, the integral curves of Lagrange equations will be called the *evolution curves* of the SRSM $\Sigma_{\mathcal{R}}$.

Remarks.

In classical Analytical Mechanics, the coordinates (x^i) of a material point $x \in M$ are denoted by (q^i), and the velocities $y^i = \dfrac{dx^i}{dt}$ by $\dot{q}^i = \dfrac{dq^i}{dt}$. However, we prefer to use the notations (x^i) and (y^i) which are often used in the geometry of the tangent manifold TM ([2], [16], [17], [20], [21]).

The external forces $F_e(x,y)$ five rise to the one-form

$$\sigma = F_i(x,y)dx^i. \tag{5.2}$$

Since F_e is a (vertical vector field it follows that σ is semibasic one form. Conversely, if σ from (5.2) is semibasic one form, then $F_e = F^i(x,y)\frac{\partial}{\partial y^i}$, with $F^i = g^{ij}F_j$, is a vertical vector field on the manifold TM. J Klein ([9]) introduced the the external forces by means of a one-form σ, while R. Miron in [15] defined F_e as a vertical vector field on TM.

The SRMS $\Sigma_{\mathcal{R}}$ is a regular mechanical system because the Hessian matrix with elements $\dfrac{\partial^2 T}{\partial y^i \partial y^j} = g_{ij}(x)$ is nonsingular.

We have the following important result.

Proposition 5.1. *The system of evolution equations* (5.1) *are equivalent to the following second order differential equations:*

$$\frac{d^2x^i}{dt^2} + \gamma^i{}_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt} = F^i(x, \frac{dx}{dt}), \tag{5.3}$$

where $\gamma^k_{ij}(x)$ are the Christoffel symbols of the metric tensor $g_{ij}(x)$.

In general, for a SRMS $\Sigma_{\mathcal{R}}$, the system of differential equations (5.3) is not autoadjoint. Consequently, it can not be written as the Euler-Lagrange equations for a certain Lagrangian.

In the case of conservative SRMS, with $F_i(x) = -\frac{\partial U(x)}{\partial x^i}$, here $U(x)$ a potential function, the equations (5.1) can be written as Euler-Lagrange equations for the Lagrangian $T+U$. They have $T+U = \textit{constant}$ as a prime integral.

This is the reason that the nonconservative SRMS $\Sigma_{\mathcal{R}}$, with F_e depending on $y^i = \dfrac{dx^i}{dt}$ cannot be studied by the methods of classical mechanics. A good geometrical theory of the SRMS $\Sigma_{\mathcal{R}}$ should be based on the geometry of the phase space TM.

From (5.3) we can see that in the canonical parametrization $t = s$ (s being the arc length in the Riemannian space $\mathcal{R}^n$), we obtain the following result:

Proposition 5.2. *If the external forces are identically zero, then the evolution curves of the system $\Sigma_{\mathcal{R}}$ are the geodesics of the Riemannian space $\mathcal{R}^n$.*

In the following we will study how the evolution equations change when the space $\mathcal{R}^n = (M, g)$ is replaced by another Riemannian space $\bar{\mathcal{R}}^n = (M, \bar{g})$ such that:

1° $\mathcal{R}^n$ and $\bar{\mathcal{R}}^n$ have the same parallelism of directions;
2° $\mathcal{R}^n$ and $\bar{\mathcal{R}}^n$have same geodesics;
3° $\bar{\mathcal{R}}^n$ is conformal to $\mathcal{R}^n$.

In each of these cases, the Levi-Civita connections of these two Riemmanian spaces are transformed by the rule:

1° $\quad \bar{\gamma}^i{}_{jk}(x) = \gamma^i{}_{jk}(x) + \delta^i_j \alpha_k(x);$
2° $\quad \bar{\gamma}^i{}_{jk}(x) = \gamma^i{}_{jk}(x) + \delta^i_j \alpha_k(x) + \delta^i_k \alpha_j(x);$
3° $\quad \bar{\gamma}^i{}_{jk}(x) = \gamma^i{}_{jk}(x) + \delta^i_j \alpha_k(x) + \delta^i_k \alpha_j(x) - g_{jk}(x)\alpha^i(x),$

where $\alpha_k(x)$ is an arbitrary covector field on M, and $\alpha^i(x) = g^{ij}(x)\alpha_j(x)$.

It follows that the evolution equations (5.3) change to the evolution equations of the system $\Sigma_{\bar{\mathcal{R}}}$ as follows:

1° In the first case we obtain

$$\frac{d^2x^i}{dt^2} + \gamma^i{}_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt} = -\alpha\frac{dx^i}{dt} + F^i(x, \frac{dx}{dt}); \qquad \alpha = \alpha_k(x)\frac{dx^k}{dt}$$

Therefore, even though $\Sigma_{\mathcal{R}}$ is a conservative system, the mechanical system $\Sigma_{\bar{\mathcal{R}}}$ is nonconservative system having the external forces

$$\bar{F}_e = \left(-\alpha(x,y)y^i + F^i(x,y)\right)\frac{\partial}{\partial y^i}, \quad \alpha = \alpha_k(x)\frac{dx^k}{dt}.$$

2° In the second case we have

$$\frac{d^2x^i}{dt^2} + \gamma^i{}_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt} = -2\alpha\frac{dx^i}{dt} + F^i(x, \frac{dx}{dt}); \qquad \alpha = \alpha_k(x)\frac{dx^k}{dt}$$

and

$$\bar{F}_e = \left(-2\alpha(x,y)y^i + F^i(x,y)\right)\frac{\partial}{\partial y^i}, \quad \alpha = \alpha_k(x)\frac{dx^k}{dt}.$$

3° In the third case $\bar{F}_e$ is

$$\bar{F}_e = \left\{2(-\alpha y^i + T\alpha^i) + F^i\right\}\frac{\partial}{\partial y^i}, \quad \alpha = \alpha_k(x)\frac{dx^k}{dt}, \ \alpha^i(x) = g^{ij}(x)\alpha_j(x).$$

The previous properties lead to examples with very interesting properties.

6 Examples of scleronomic mechanical systems

Recall that in the case of classical *conservative mechanical systems* we have $F_e =$ *grad* $\mathcal{U}$, where $\mathcal{U}(x)$ is a potential function. Therefore, the Lagrange equations are given by

$$\frac{d}{dt}\frac{\partial}{\partial y^i}(T + \mathcal{U}) - \frac{\partial}{\partial x^i}(T + \mathcal{U}) = 0.$$

We obtain from here a prime integral $T+\mathcal{U}=h(constant)$ which give us *the energy conservation law.*

In the nonconservative case we have numerous examples suggested by 1°, 2°, 3° from the previous section, where we take $F^i(x,y)=0$.

Other examples of SRMS can be obtained as follows

1. Take $F_e=-\beta(x,y)y^i\dfrac{\partial}{\partial y^i}$, where $\beta=\beta_i(x)y^i$ is determined by the electromagnetic potentials $\beta_i(x)$, $(i=1,..,n)$.

2. Take $F_e=2(T-\beta)y^i\dfrac{\partial}{\partial y^i}$, where $\beta=\beta_i(x)y^i$ and T is the kinetic energy.

3. In the three-body problem, M. Barbosu and B. Elmabsout [4] applied the conformal transformation, [22], $d\bar{s}^2=(T+\mathcal{U})ds^2$ to the classic Lagrange equations and had obtained a nonconservative mechanical system with external force field $F_e=2(T+U)y^i\dfrac{\partial}{\partial y^i}$.

4. The external forces $F_e=F^i(x)\dfrac{\partial}{\partial y^i}$ lead to classical nonconservative Riemannian mechanical systems. For instance, for $F_i=-grad_i\,\mathcal{U}+R_i(x)$ where $R_i(x)$ are the resistance forces, and the configuration space M is $\mathbb{R}^3$.

5. If $M=\mathbb{R}^3$, $T=\dfrac{1}{2}m\delta_{ij}y^iy^j$ and $F_e=F^i(x)\dfrac{\partial}{\partial y^i}$, then the evolution equations are $m\dfrac{d^2x^i}{dt^2}=F^i(x)$, which is *the Newton's law.*

6. The harmonic oscillator.

   $M=\mathbb{R}^n,\ g_{ij}=\delta_{ij},\ F_i=-\omega_i^2x^i$ (the summation convention is not applied) and ω_i are positive numbers, $(i=1,..,n)$. The functions $h_i=(x^i)^2+\omega_i^2x^i$, and $H=\sum_{i=1}^n h_i$ are prime integrals.

7. Suggested by the example 6, we consider a system $\Sigma_{\mathcal{R}}$ with $F_e=-\omega(x)\,\mathbb{C}$, where $\omega(x)$ is a positive function and $\mathbb{C}$ is the Liouville vector field.

   The evolution equations, in the case $M=\mathbb{R}^n$, are given by $\dfrac{d^2x^i}{dt^2}+\omega(x)\dfrac{dx^i}{dt}=0$. Putting $y^i=\dfrac{dx^i}{dt}$, we can write $\dfrac{dy^i}{dt}+\omega(x)y^i=0, \qquad (i=1,..,n)$. So, we obtain $y^i=C^ie^{-\int\omega(x(t))dt}$ and therefore $x^i=C_0^i+C^i\displaystyle\int e^{-\int\omega(x(t))dt}dt$.

8. We can consider the systems $\Sigma_{\mathcal{R}}$ having $F_e=a^i_{jk}(x)y^jy^k\dfrac{\partial}{\partial y^i}$, where $a^i_{jk}(x)$ is a symmetric tensor field on M. The external force field F_e has homogeneous components of degree 2 with respect to y^i.

9. Relativistic nonconservative mechanical systems can be obtained for a Minkowski metric in the space-time $\mathbb{R}^4$.

10. A particular case of example 1 above is the case when the external force field coefficients $F^i(x,y)$ are linear in y^i, i.e. $F_e = F^i(x,y)\frac{\partial}{\partial y^i} = Y^i_k(x)y^k\frac{\partial}{\partial y^i}$, where $Y : TM \to TM$ is a fiber diffeomorphism called *Lorentz force*, namely for any $x \in M$, we have $Y_x : T_xM \to T_xM$, $\quad Y_x(\frac{\partial}{\partial x^i}) = Y^j_i(x)\frac{\partial}{\partial x^j}$.

    Let us remark that in this case, formally, we can write the Lagrange equations of this SRMS in the form

$$\nabla_{\dot\gamma}\dot\gamma = Y(\dot\gamma),$$

    where ∇ is the Levi-Civita connection of the Riemannian space (M,g) and $\dot\gamma$ is the tangent vector along the evolution curves $\gamma : [a,b] \to M$.

    This type of SRMS is important because of the global behavior of its evolution curves.

    Let us denote by S the evolutionary semispray, i.e. S is a vector field on TM which is tangent to the canonical lift $\hat\gamma = (\gamma, \dot\gamma)$ of the evolution curves, [7], (see the following section for a detailed discussion on the evolution semispray).

    We will denote by T^c the energy levels of the Riemannian metric g, i.e.

$$T^c = \{(x,y) \in TM : T(x,y) = \frac{c^2}{2}\},$$

    where T is the kinetic energy of g, and c is a positive constant. One can easily see that T^c is the hypersurface in TM of constant Riemannian length vectors, namely for any $X = (x,y) \in T^c$, we must have $|X|_g = c$, where $|X|_g$ is the Riemannian length of the vector field X on M.

    If we restrict ourselves for a moment to the two dimensional case, then it is known that for sufficiently small values of c the restriction of the flow of the semispray S to T^c contains no less than two closed curves when M is the 2-dimensional sphere, and at least three otherwise ([8]). These curves projected to the base manifold M will give closed evolution curves for the given Riemannian mechanical system.

    A detailed study of Riemannian non-conservative mechanical systems will be included in a forthcoming paper.

7 The evolution semispray of the mechanical system $\Sigma_{\mathcal{R}}$

Let us assume that F_e is global defined on M, and consider the mechanical system $\Sigma_{\mathcal{R}} = (M,T,F_e)$. We have

Theorem 7.1. *[9] The following properties hold good:*

1° *The quantity S defined by*

$$S = y^i\frac{\partial}{\partial x^i} - (2\overset{\circ}{G^i} - F^i)\frac{\partial}{\partial y^i}, \tag{7.1}$$

where $2\overset{\circ}{G^i} = \gamma^i{}_{jk}y^jy^k$ is a vector field on the phase space TM.

2° *S is a semispray, which depends on $\Sigma_{\mathcal{R}}$ only.*

3° *The integral curves of the semispray S are the evolution curves of the system $\Sigma_{\mathcal{R}}$.*

Proof. 1° Writing S in the form

$$S = \overset{\circ}{S} + F_e,$$

where $\overset{\circ}{S}$ is the canonical semispray with the coefficients $\overset{\circ}{G^i}$, we can see immediately that S is a vector field on TM.

2° Since $\overset{\circ}{S}$ is a semispray and F_e a vertical vector field, it follows S is a semispray. From (7.1) we can see that S depends on Σ, only.

3° The integral curves of S are given by

$$\frac{dx^i}{dt} = y^i; \qquad \frac{dy^i}{dt} + 2\overset{\circ}{G^i}(x,y) = F^i(x,y).$$

Replacing y^i in the second equation we obtain (5.1) □

S will be called *the evolution* or *canonical semispray* of the nonconservative Riemannian mechanical system $\Sigma_{\mathcal{R}}$. In the terminology of J. Klein, S is the dynamical system of $\Sigma_{\mathcal{R}}$.

Based on S we can develop the geometry of the mechanical system $\Sigma_{\mathcal{R}}$ on TM.

Let us remark that S can also be written as follows:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

with the coefficients

$$2G^i = 2\overset{\circ}{G^i} - F^i.$$

We point out that S is homogeneous of degree 2 if and only if $F^i(x,y)$ is 2-homogeneous with respect to y^i. This property is not satisfied in the case $\dfrac{\partial F^i}{\partial y^j} \equiv 0$, and it is satisfied for examples 1 and 8 from section 6. We have:

Theorem 7.2. *The variation of the kinetic energy T of a mechanical system $\Sigma_{\mathcal{R}}$, along the evolution curves* (5.1)*, is given by:* $\dfrac{dT}{dt} = F_i \dfrac{dx^i}{dt}$.

Proof. A straightforward computation gives

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial T}{\partial y^i}\frac{dy^i}{dt} = \left(\frac{d}{dt}\frac{\partial T}{\partial y^i} - F_i\right)\frac{dx^i}{dt} + \frac{\partial T}{\partial y^i}\frac{dy^i}{dt} = \\ &= \frac{d}{dt}\left(y^i\frac{\partial T}{\partial y^i}\right) - F_i\frac{dx^i}{dt} = 2\frac{dT}{dt} - F_i\frac{dx^i}{dt},\end{aligned}$$

and therefore the relation holds good. □

Corollary 7.3. $T = \text{constant}$ *along the evolution curves if and only if the Liouville vector $\mathbb{C}$ and the external force F_e are orthogonal vectors along the evolution curves of Σ.*

Corollary 7.4. *If $F_i = grad_i\, \mathcal{U}$ then $\Sigma_{\mathcal{R}}$ is conservative and $T + \mathcal{U} = h$ (constant) on the evolution curves of $\Sigma_{\mathcal{R}}$.*

If the external forces F_e are dissipative, i.e. $\langle \mathbb{C}, F_e \rangle \leq 0$, then from the previous theorem, it follows a result of Bucataru-Miron (see [2, 3]):

Corollary 7.5. *The kinetic energy T decreases along the evolution curves if and only if the external forces F_e are dissipative.*

8 The nonlinear connection of $\Sigma_{\mathcal{R}}$

Let us consider the evolution semispray S of $\Sigma_{\mathcal{R}}$ given by

$$S = y^i \frac{\partial}{\partial x^i} - (2 \overset{\circ}{G}{}^i - F^i) \frac{\partial}{\partial y^i}$$

with the coefficients

$$2G^i = 2 \overset{\circ}{G}{}^i - F^i.$$

Consequently, the evolution nonlinear connection N of the mechanical system $\Sigma_{\mathcal{R}}$ has the coefficients:

$$N^i{}_j = \overset{\circ}{N}{}^i{}_j - \frac{1}{2} \frac{\partial F^i}{\partial y^j} = \gamma^i_{jk} y^k - \frac{1}{2} \frac{\partial F^i}{\partial y^j}. \tag{8.1}$$

If the external forces F_e does not depend by velocities $y^i = \dfrac{dx^i}{dt}$, then $N = \overset{\circ}{N}$.

Since the energy of S is T (the kinetic energy), the Theorem 7.2 holds good in this case. The variation of T is given in Theorem 7.2 and hence we obtain that T is conserved along the evolution curves of $\Sigma_{\mathcal{R}}$ if and only if the vector field F_e and the Liouville vector field $\mathbb{C}$ are orthogonal. Recall that F_e is called *dissipative* if $\langle F_e, \mathbb{C} \rangle \leq 0$, or, equivalently, $g_{ij}(x) y^i F^j(x) \leq 0$ ([2, p. 211]).

Let us consider the helicoidal vector field (see Bucataru-Miron, [2, 3])

$$P_{ij} = \frac{1}{2} \left(\frac{\partial F_i}{\partial y^j} - \frac{\partial F_j}{\partial y^i} \right) \tag{8.2}$$

and the symmetric part of tensor $\dfrac{\partial F_i}{\partial y^j}$:

$$Q_{ij} = \frac{1}{2} \left(\frac{\partial F_i}{\partial y^j} + \frac{\partial F_j}{\partial y^i} \right).$$

On TM, P gives rise to the 2-form:

$$P = P_{ij}\ dx^i \wedge dx^j$$

and Q is the symmetric vertical tensor:

$$Q = Q_{ij}\ dx^i \otimes dx^j.$$

Denoting by ∇ the dynamical derivative with respect to the pair (S, N), one proves the theorem of Bucataru-Miron [2]:

Theorem 8.1. *For a scleronomic Riemannian mechanical system $\Sigma_{\mathcal{R}} = (M, T, F_e)$ the evolution nonlinear connection is the unique nonlinear connection that satisfies the following conditions:* $\nabla g = -\dfrac{1}{2}\ Q$, $\omega_L(hX, hY) = \dfrac{1}{2}P(X,Y)$, $\forall X, Y \in \chi(TM)$, *where*

$$\omega_L = 2g_{ij}\ \overset{\circ}{\delta}\ y^j \wedge dx^i$$

is the symplectic structure determined by the metric tensor g_{ij} and the nonlinear connection $\overset{\circ}{N}$.

The adapted basis of the distributions N and V is given by $\left(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^j} = \frac{\overset{\circ}{\delta}}{\delta x^i} + \frac{1}{2}\frac{\partial F^j}{\partial y^i}\frac{\partial}{\partial y^j}$$

and its dual basis $(dx^i, \delta y^i)$ has the 1-forms δy^i expressed by

$$\delta y^i = dy^i + N^i{}_j dx^j = \overset{\circ}{\delta} y^i - \frac{1}{2}\frac{\partial F^i}{\partial y^j}dx^j.$$

It follows that the curvature tensor $\mathcal{R}^i{}_{jk}$ of N (from (3.4)) is

$$\mathcal{R}^k{}_{ij} = \frac{\delta N^k{}_i}{\delta x^j} - \frac{\delta N^k{}_j}{\delta x^i} = \left(\frac{\delta}{\delta x^j}\frac{\partial}{\partial y^i} - \frac{\delta}{\delta x^i}\frac{\partial}{\partial y^j}\right)(\overset{\circ}{G}{}^k - \frac{1}{2}F^k)$$

and the torsion tensor of N is:

$$t^k{}_{ij} = \frac{\partial N^k{}_i}{\partial y^j} - \frac{\partial N^k{}_j}{\partial y^i} = \left(\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i} - \frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}\right)(\overset{\circ}{G}{}^k - \frac{1}{2}F^k) = 0.$$

These formulas have the following consequences.

1. The evolution nonlinear connection N of $\Sigma_{\mathcal{R}}$ is integrable if and only if the curvature tensor $\mathcal{R}^i{}_{jk}$ vanishes.

2. The nonlinear connection is torsion free, i.e. $t^k_{ij} = 0$.

The autoparallel curves of the evolution nonlinear connection N are given by the system of differential equations

$$\frac{d^2x^i}{dt^2} + N^i_j\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt} = 0,$$

which is equivalent to

$$\frac{d^2x^i}{dt^2} + \overset{\circ}{N}{}^i{}_j\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt} = \frac{1}{2}\frac{\partial F^i}{\partial y^j}\frac{dx^j}{dt}.$$

Under the initial conditions $(x_0, (\frac{dx}{dt})_0)$, locally this uniquely determines the autoparallel curves of N. If F^i is 2-homogeneous with respect to y^i, then the previous system coincides with the Lagrange equations (5.3).

Therefore, we have:

Theorem 8.2. *If the external forces F_e are 2-homogeneous with respect velocities $y^i = \dfrac{dx^i}{dt}$, then the evolution curves of $\Sigma_{\mathcal{R}}$ coincide to the autoparallel curves of the evolution nonlinear connection N of $\Sigma_{\mathcal{R}}$.*

In order to proceed further, we need the exterior differential of 1-forms δy^i. One obtains (see [17, p. 29])

$$d(\delta y^i) = dN^i{}_j \wedge dx^j = \frac{1}{2} R^i{}_{kj} dx^j \wedge dx^k + B^i{}_{kj} \delta y^j \wedge dx^k, \tag{8.3}$$

where $B^i{}_{kj} = B^i{}_{jk} = \dfrac{\partial^2 G^i}{\partial y^k \partial y^j}$ are the coefficients of the Berwald connection determined by the nonlinear connection N.

9 The canonical metrical connection $C\Gamma(N)$

The coefficients of the canonical metrical connection $C\Gamma(N) = \left(F^i{}_{jk}, C^i{}_{jk}\right)$ are given by the generalized Christoffel symbols ([13]):

$$\begin{cases} F^i{}_{jk} = \dfrac{1}{2} g^{is} \left(\dfrac{\delta g_{sk}}{\delta x^j} + \dfrac{\delta g_{js}}{\delta x^k} - \dfrac{\delta g_{jk}}{\delta x^s} \right), \\ C^i{}_{jk} = \dfrac{1}{2} g^{is} \left(\dfrac{\partial g_{sk}}{\partial y^j} + \dfrac{\partial g_{js}}{\partial y^k} - \dfrac{\partial g_{jk}}{\partial y^s} \right), \end{cases}$$

where $g_{ij}(x)$ is the metric tensor of $\Sigma_{\mathcal{R}}$.
On the other hand, we have $\dfrac{\delta g_{jk}}{\delta x^i} = \dfrac{\partial g_{jk}}{\partial x^i}$ and $\dfrac{\partial g_{jk}}{\partial y^i} = 0$, and therefore we obtain:

Theorem 9.1. *The canonical metrical connection $C\Gamma(N)$ of the mechanical system $\Sigma_{\mathcal{R}}$ has the coefficients*

$$F^i{}_{jk}(x, y) = \gamma^i{}_{jk}(x), \qquad C^i{}_{jk}(x, y) = 0.$$

Let $\omega^i{}_j$ be the connection forms of $C\Gamma(N)$:

$$\omega^i{}_j = F^i{}_{jk} dx^k + C^i{}_{jk} \delta y^k = \gamma^i{}_{jk}(x) dx^k.$$

Then, we have ([13]):

Theorem 9.2. *The structure equation of $C\Gamma(N)$ can be expressed by*

$$\begin{cases} d(dx^i) - dx^k \wedge \omega^i{}_k &= - \overset{1}{\Omega}{}^i, \\ d(\delta y^i) - \delta y^k \wedge \omega^i{}_k &= - \overset{2}{\Omega}{}^i, \\ d\omega^i{}_j \; - \; \omega^k{}_j \wedge \omega^i{}_k &= - \Omega^i_j, \end{cases} \tag{9.1}$$

where the 2*-forms of torsion* $\overset{1}{\Omega}{}^i$, $\overset{2}{\Omega}{}^i$ *are as follows*

$$\overset{1}{\Omega}{}^i = C^i{}_{jk} dx^j \wedge \delta y^k = 0,$$
$$\overset{2}{\Omega}{}^i = R^i{}_{jk} dx^j \wedge dx^k + P^i{}_{jk} dx^j \wedge \delta y^k.$$

Here $R^i{}_{jk}$ *is the curvature tensor of* N *and* $P^i{}_{jk} = \gamma^i{}_{jk} - \gamma^i{}_{kj} = 0$.

The curvature 2-form Ω^i_j is given by

$$\Omega^i_j = \frac{1}{2} R_j{}^i{}_{kh}\ dx^k \wedge dx^h + P_j{}^i{}_{kh}\ dx^k \wedge \delta y^h + \frac{1}{2} S_j{}^i{}_{kh}\ \delta y^k \wedge \delta y^h,$$

where

$$\begin{aligned} R_h{}^i{}_{jk} &= \frac{\delta F^i{}_{hj}}{\delta x^k} - \frac{\delta F^i{}_{hk}}{\delta x^j} + F^s{}_{hj} F^i{}_{sk} - F^s{}_{hk} F^i{}_{sj} + C^i{}_{hs} R^s{}_{jk} = \\ &= \frac{\partial \gamma^i{}_{hj}}{\partial x^k} - \frac{\partial \gamma^i{}_{hk}}{\partial x^j} + \gamma^s{}_{hj}\gamma^i{}_{sk} - \gamma^s{}_{hk}\gamma^i{}_{sj} = \mathbf{r}_h{}^i{}_{jk} \end{aligned}$$

is the Riemannian tensor of curvature of the Levi-Civita connection $\gamma^i_{jk}(x)$ and the curvature tensors $P_j{}^i{}_{kh}$, $S_j{}^i{}_{kh}$ vanish. Therefore, the tensors of torsion of $C\Gamma(N)$ are

$$R^i{}_{jk},\ T^i{}_{jk} = 0,\ S^i{}_{jk} = 0,\ P^i{}_{jk} = 0,\ C^i{}_{jk} = 0 \tag{9.2}$$

and the curvature tensors of $C\Gamma(N)$ are

$$R_j{}^i{}_{kh}(x,y) = \mathbf{r}_j{}^i{}_{kh}(x),\ P_j{}^i{}_{kh}(x,y) = 0,\ S_j{}^i{}_{kh}(x,y) = 0. \tag{9.3}$$

The Bianchi identities can be obtained directly from (9.1), taking into account the conditions (9.2) and (9.3).

The h- and v-covariant derivatives of d-tensor fields with respect to $C\Gamma(N) = (\gamma^i{}_{jk}, 0)$ are expressed, for instance, by

$$\nabla_k t_{ij} = \frac{\delta t_{ij}}{\delta x^k} - \gamma^s{}_{ik} t_{sj} - \gamma^s{}_{jk} t_{is}, \quad \dot{\nabla}_k t_{ij} = \frac{\partial t_{ij}}{\partial y^k} - C^s{}_{ik} t_{sj} - C^s{}_{jk} t_{is} = \frac{\partial t_{ij}}{\partial y^k}.$$

Therefore, $C\Gamma(N)$ being a metric connection with respect to $g_{ij}(x)$, we have

$$\nabla_k\ g_{ij} = \overset{\circ}{\nabla}_k g_{ij} = 0,$$

($\overset{\circ}{\nabla}$ is the covariant derivative with respect to Levi-Civita connection of g_{ij}) and $\dot{\nabla}_k\ g_{ij} = 0$.

The deflection tensors of $C\Gamma(N)$ are $D^i_j = \nabla_j y^i = \dfrac{\delta y^i}{\delta x^j} + y^s \gamma^i{}_{sj} = -N^i{}_j + y^s \gamma^i{}_{sj}$ and $d^i{}_j = \dot{\nabla}_j\ y^i = \delta^i{}_j$.

The evolution nonlinear connection of a scleronomic Riemannian mechanical system Σ given by (8.1) implies $D^i{}_j = -\overset{\circ}{N}{}^i_j + \frac{1}{2}\frac{\partial F^i}{\partial y^j} + y^s \gamma^i_{sj} = \frac{1}{2}\frac{\partial F^i}{\partial y^j}$, where we have used $\overset{\circ}{D}_j\ y^i = \overset{\circ}{D}{}^i_j = 0$. It follows:

Proposition 9.3. *For a scleronomic Riemannian mechanical system the deflection tensors* $D^i{}_j$ *and* $d^i{}_j$ *of the connection* $C\Gamma(N)$ *are expressed by* $D^i{}_j = \dfrac{1}{2}\dfrac{\partial F^i}{\partial y^j}$, $d^i{}_j = \delta^i_j$.

10 The electromagnetism in the theory of the scleronomic Riemannian mechanical systems $\Sigma_{\mathcal{R}}$

In a scleronomic Riemannian mechanical system $\Sigma_{\mathcal{R}} = \Big(M, T, F_e\Big)$ whose external forces F_e depend on the point x and on the velocity $y^i = \dfrac{dx^i}{dt}$, the electromagnetic phenomena appears because the deflection tensors $D^i{}_j$ and the deflection tensor d^i_j are nonvanishing. Hence the d-tensors $D_{ij} = g_{ih}D^h{}_j$, $d_{ij} = g_{ih}\delta^h_j = g_{ij}$ determine the h-electromagnetic tensor $\mathcal{F}_{ij}$ and v-electromagnetic tensor f_{ij} by the formulas [15]:

$$\mathcal{F}_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), \qquad f_{ij} = \frac{1}{2}(d_{ij} - d_{ji}).$$

From the formula of deflection tensors, we have

Proposition 10.1. *The h- and v-tensor fields $\mathcal{F}_{ij}$ and f_{ij} are given by*

$$\mathcal{F}_{ij} = \frac{1}{2}P_{ij}, \qquad f_{ij} = 0. \tag{10.1}$$

where P_{ij} is the helicoidal tensor (8.2) *of* $\Sigma_{\mathcal{R}}$.

Indeed, we have

$$\mathcal{F}_{ij} = \frac{1}{2}(D_{ij} - D_{ji}) = \frac{1}{4}\left(\frac{\partial F^i}{\partial y^j} - \frac{\partial F^j}{\partial y^i}\right) = \frac{1}{2}P_{ij}.$$

If we denote $R_{ijk} := g_{ih}R^h{}_{jk}$, then we can prove:

Theorem 10.2. *The electromagnetic tensor $\mathcal{F}_{ij}$ of the mechanical system $\Sigma_{\mathcal{R}} = \Big(M, T, F_e\Big)$ satisfies the following generalized Maxwell equations:*

$$\nabla_k\mathcal{F}_{ji} + \nabla_i\mathcal{F}_{kj} + \nabla_j\mathcal{F}_{ik} = -(R_{kji} + R_{ikj} + R_{jik}), \tag{10.2}$$

$$\dot{\nabla}_k\mathcal{F}_{ji} = 0. \tag{10.3}$$

Proof. Applying the Ricci identities to the Liouville vector field, we obtain

$$\nabla_i D^k{}_j - \nabla_j D^k{}_i = y^h \mathbf{r}_h{}^k{}_{ij} - R^k_{ij},\ \nabla_i d^k{}_j - \dot{\nabla}_j D^k{}_i = 0$$

and this leads to

$$\nabla_i D_{kj} - \nabla_j D_{ki} = y^h \mathbf{r}_{hkij} - R_{kij}, \tag{10.4}$$

$$\dot{\nabla}_j D_{ki} = 0. \tag{10.5}$$

By taking cyclic permutations of the indices i,k,j and adding in (10.4), using the identity $\mathbf{r}_{hijk} + \mathbf{r}_{hjki} + \mathbf{r}_{hkij} = 0$ we deduce (10.2), and analogously (10.3). $\square$

From equations (10.1) and (10.5) we obtain as consequences:

Corollary 10.3. *The electromagnetic tensor* $\mathcal{F}_{ij}$ *of the mechanical system* $\Sigma_{\mathcal{R}} = (M, T, F_e)$ *does not depend on the velocities* $y^i = \dfrac{dx^i}{dt}$.

Indeed, by means of (10.3) we have $\dot{\nabla}_j \mathcal{F}_{ik} = \frac{\partial \mathcal{F}_{ik}}{\partial y^j} = 0$. In other words, the helicoidal tensor P_{ij} of $\Sigma_{\mathcal{R}}$ does not depend on the velocities $y^i = \dfrac{dx^i}{dt}$.
We end the present section with a remark. This theory has applications to the mechanical systems given by Example 1 in Section 6.

Remark. The theory of gravitation given by the gravitational potential $g_{ij}(x)$, $(i, j = 1, .., n)$ can be studied in the same manner as in the book [16].

11 The almost Hermitian model of the SRMS $\Sigma_{\mathcal{R}}$

Let us consider a SRMS $\Sigma_{\mathcal{R}} = (M, T(x, y), F_e(x, y))$ endowed with the evolution nonlinear connection with coefficients N^i_j from (8.1) and with the canonical N-metrical connection $C\Gamma(N) = (\gamma^i_{jk}(x), 0)$. Thus, on the phase space $\widetilde{TM} = TM \setminus \{0\}$ we can determine an almost Hermitian structure $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$ which depends on the SRMS only.

Let $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ be the adapted basis to the distributions N and V and its adapted cobasis $(dx^i, \delta y^i)$, where

$$\frac{\delta}{\delta x^i} = \frac{\overset{\circ}{\delta}}{\delta x^i} + \frac{1}{4}\frac{\partial F^s}{\partial y^i}\frac{\partial}{\partial y^s}, \quad \delta y^i = \overset{\circ}{\delta} y^i - \frac{1}{4}\frac{\partial F^i}{\partial y^s} dx^s.$$

The lift of the fundamental tensor $g_{ij}(x)$ of the Riemannian space $\mathcal{R}^n = (M, g_{ij}(x))$ is defined by $\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j$, and the almost complex structure $\mathbb{F}$, determined by the nonlinear connection N, is expressed by $\mathbb{F} = -\dfrac{\partial}{\partial y^i} \otimes dx^i + \dfrac{\delta}{\delta x^i} \otimes \delta y^i$.

Thus, the following theorems hold good.

Theorem 11.1. *We have:*

1. *The pair* $(\widetilde{TM}, \mathbb{G})$ *is a pseudo-Riemannian space.*
2. *The tensor* $\mathbb{G}$ *depends on* $\Sigma_{\mathcal{R}}$ *only.*
3. *The distributions* N *and* V *are orthogonal with respect to* $\mathbb{G}$.

Theorem 11.2.

1. *The pair* $(\widetilde{TM}, \mathbb{F})$ *is an almost complex space.*
2. *The almost complex structure* $\mathbb{F}$ *depends on* $\Sigma_{\mathcal{R}}$ *only.*
3. $\mathbb{F}$ *is integrable on the manifold* $\widetilde{TM}$ *if and only if the d-tensor field* $R^i_{jk}(x, y)$ *vanishes.*

Theorem 11.3. *We have*

1. *The triple* $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$ *is an almost Hermitian space.*
2. *The space* H^{2n} *depends on* $\Sigma_{\mathcal{R}}$ *only.*
3. *The almost symplectic structure of* H^{2n} *is* $\omega = g_{ij}\delta y^i \wedge dx^j$.

If the almost symplectic structure ω is a symplectic one (i.e. $d\omega = 0$), then the space H^{2n} is almost Kählerian. On the other hand, using the formula (8.3) one obtains

$$d\omega = \frac{1}{3!}(R_{ijk} + R_{jki} + R_{kij})dx^i \wedge dx^j \wedge dx^k + \frac{1}{2}(g_{is}B^s_{jk} - g_{js}B^s_{ik})\delta y^k \wedge dx^j \wedge dx^i.$$

Therefore, we deduce

Theorem 11.4. *The almost Hermitian space* H^{2n} *is almost Kählerian if and only if the following relations hold good* $R_{ijk} + R_{jki} + R_{kij} = 0, \quad g_{is}B^s_{jk} - g_{js}B^s_{ik} = 0.$

The space $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$ is called the *almost Hermitian model* of the SRMS $\Sigma_{\mathcal{R}}$.

One can use the almost Hermitian model H^{2n} to study the geometrical theory of the mechanical system $\Sigma_{\mathcal{R}}$. For instance the Einstein equations of the SRMS $\Sigma_{\mathcal{R}}$ are the Einstein equations of the pseudo-Riemannian space $(\widetilde{TM}, \mathbb{G})$.

Remark. The previous theory can be applied without difficulties to the examples 1-8 in Section 6.

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