Equivalence of multitime optimal control problems

Constantin Udrişte

Abstract. Many science and engineering problems can be formulated as optimization problems that are governed by m-flow type $PDEs$ (multitime evolution systems) and by cost functionals expressed as curvilinear integrals or multiple integrals. Though these functionals are mathematically equivalent on m-intervals, their meaning is totally different in real life problems. Our paper discusses the m-flow type PDE-constrained optimization problems of Mayer, Lagrange and Bolza, focussing on their equivalence. Section 1 formulates the Mayer problem with a terminal cost functional. In Section 2, the idea of equivalence is motivated for the Mayer, Lagrange and Bolza problems, based on curvilinear integral cost, using the curvilinear primitive. In Section 3, similar results are proved for the Mayer, Lagrange and Bolza problems, based on multiple integral cost, using both the curvilinear primitive and the hyperbolic primitive. Section 4 shows that curvilinear integral functionals and multiple integral functionals are equivalent on m-intervals.

M.S.C. 2000: 93C20, 93C35, 49K20, 49J20.
Key words: PDE-constrained optimal control; multitime Mayer-Lagrange-Bolza problems; multiple or curvilinear integral functional; geometric evolution.

1 Multitime optimal control problem of Mayer

We introduce the states

$$x = (x^i) \in \mathbb{R}^n,\ i = 1, ..., n,$$

the controls

$$u = (u^a) \in \mathbb{R}^q,\ a = 1, ..., q,$$

the hyperparallelipiped $\Omega_{0t_0} \subset \mathbb{R}^m_+$ fixed by the diagonal opposite points $0, t_0 \in \mathbb{R}^m_+$, the evolution parameter (multitime)

$$s = (s^\alpha) \in \Omega_{0t_0},\ \alpha = 1, ..., m$$

and a controlled multitime completely integrable evolution (m-flow)

$$\frac{\partial x^i}{\partial s^\alpha}(s) = X^i_\alpha(s, x(s), u(s)),\ x(0) = x_0, \tag{1.1}$$

Balkan Journal of Geometry and Its Applications, Vol.15, No.1, 2010, pp. 155-162.

where $X_\alpha(s, x(s), u(s)) = (X^i_\alpha(s, x(s), u(s)))$ are C^1 vector fields satisfying the complete integrability conditions (m-flow type problem), i.e., $D_\beta X_\alpha = D_\alpha X_\beta$ (D_α is the total derivative operator) or

$$\left(\frac{\partial X_\alpha}{\partial u^a}\delta^\gamma_\beta - \frac{\partial X_\beta}{\partial u^a}\delta^\gamma_\alpha\right)\frac{\partial u^a}{\partial s^\gamma} = [X_\alpha, X_\beta] + \frac{\partial X_\beta}{\partial s^\alpha} - \frac{\partial X_\alpha}{\partial s^\beta},$$

where $[X_\alpha, X_\beta]$ means the *bracket* of vector fields. The hypothesis on the vector fields X_α selects the set of all admissible controls (satisfying the complete integrability conditions)

$$\mathcal{U} = \left\{ u : \mathbb{R}^m_+ \to U \,\middle|\, D_\beta X_\alpha = D_\alpha X_\beta \right\}$$

and the admissible states.

The *multitime problem of Mayer* is to determine a control function $u(\cdot)$, in an appropriate set of functions, to maximize the terminal cost functional

$$P(u(\cdot)) = g(t_0, x(t_0)), \tag{1.2}$$

where $g : \Omega_{0t_0} \times \mathbb{R}^n \to \mathbb{R}$ is a smooth function. Mayer problems arise when there is a particular emphasis on the final multitime t_0 and/or final state $x(t_0)$, with m-flow constraints.

The multidimensional evolution of m-flow type is characteristic for differential geometry optimal problems, but also for engineering or economic optimal problems.

2 Multitime optimal control problems of Lagrange and Bolza based on the curvilinear integral action

2.1 Multitime optimal control problem of Lagrange with curvilinear integral action

In the *multitime problem of Lagrange with curvilinear integral action*, the cost functional takes the form

$$P(u(\cdot)) = \int_{\Gamma_{0t_0}} L_\alpha(s, x(s), u(s))ds^\alpha, \tag{2.1}$$

where the *running cost* $L_\alpha(s, x(s), u(s))ds^\alpha$ is a nonautonomous closed (completely integrable) *Lagrangian 1-form* , i.e., it satisfies $D_\beta L_\alpha = D_\alpha L_\beta$ (D_α is the total derivative operator) or

$$\left(\frac{\partial L_\alpha}{\partial u^a}\delta^\gamma_\beta - \frac{\partial L_\beta}{\partial u^a}\delta^\gamma_\alpha\right)\frac{\partial u^a}{\partial s^\gamma} = X^i_\alpha\frac{\partial L_\beta}{\partial x^i} - X^i_\beta\frac{\partial L_\alpha}{\partial x^i} + \frac{\partial L_\beta}{\partial s^\alpha} - \frac{\partial L_\alpha}{\partial s^\beta}$$

and Γ_{0t_0} is an arbitrary C^1 curve joining the diagonal opposite points $0 = (0, ..., 0)$ and $t_0 = (t^1_0, ..., t^m_0)$ in Ω_{0t_0}. A problem of Lagrange reflects the situation where the cost accumulates with multitime as a "mechanical work".

2.2 Multitime optimal control problem of Bolza with sum action

A *multitime problem of Bolza with sum action* is a combination of problems of Mayer and Lagrange as the cost takes the form

$$P(u(\cdot)) = \int_{\Gamma_{0t_0}} L_\alpha(s, x(s), u(s))ds^\alpha + g(t_0, x(t_0)), \tag{2.2}$$

with $g : \Omega_{0t_0} \times \mathbb{R}^n \to \mathbb{R}$ a smooth function and $L_\alpha(s, x(s), u(s))ds^\alpha$ a closed Lagrange 1-form. The Bolza problems arise when there is a cumulative cost which increases during the control action but special emphasis is placed on the situation at the final multitime t_0.

2.3 Equivalence of the previous problems via curvilinear primitive

Mayer, Lagrange and Bolza multitime problems are all equivalent in that each of them can be converted to any other one via the curvilinear primitive. First, the Lagrange and Mayer problems are special cases of Bolza problems. Second, a Bolza problem can be transformed into a Mayer problem by introducing an extra component y for the state vector, which satisfies the PDEs (curvilinear primitive)

$$\frac{\partial y}{\partial t^\alpha}(t) = L_\alpha(t, x(t), u(t)), \;\; y(0) = 0.$$

Using this extra variable, the cost takes the Mayer form

$$P(u(\cdot)) = g(t_0, x(t_0)) + y(t_0).$$

Third, a Mayer problem can be converted into a Lagrange problem by rewriting the cost via the curvilinear primitive

$$P(u(\cdot)) = g(t_0, x(t_0)) = g(0, x(0)) + \int_{\Gamma_{0t_0}} D_\alpha g(s, x(s))ds^\alpha$$

$$= g(0, x(0)) + \int_{\Gamma_{0t_0}} \left(\frac{\partial g}{\partial s^\alpha}(s, x(s)) + \frac{\partial g}{\partial x^i}(s, x(s))X^i_\alpha(s, x(s), u(s)) \right) ds^\alpha.$$

Since the point $x(0)$ is fixed, the problem is to maximize the curvilinear integral cost

$$\overline{P}(u(\cdot)) = \int_{\Gamma_{0t_0}} \overline{L}_\alpha(s, x(s), u(s))ds^\alpha,$$

where

$$\overline{L}_\alpha(s, x(s), u(s)) = \frac{\partial g}{\partial s^\alpha}(s, x(s)) + \frac{\partial g}{\partial x^i}(s, x(s))X^i_\alpha(s, x(s), u(s)),$$

which is a problem of Lagrange based on the curvilinear integral action.

3 Multitime optimal control problems Lagrange and Bolza based on the multiple integral action

3.1 Multitime optimal control problem of Lagrange with multiple integral action

Sometimes the running cost functional in a *multitime Lagrange problem* appears as a multiple integral

$$Q(u(\cdot)) = \int_{\Omega_{0t_0}} L(s, x(s), u(s))ds, \tag{3.1}$$

where $ds = ds^1 \cdots ds^m$ is the *volume element* in $\mathbb{R}^m_+$ and the Lagrangian $L : \Omega_{0,t_0} \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is a smooth function. A problem of Lagrange is adapted to a situation where the cost accumulates with multitime as a "volume".

3.2 Multitime optimal control problem of Bolza with sum action

A *multitime problem of Bolza with sum action* is a combination of problems of Mayer and Lagrange as the cost functional takes the form of a sum

$$Q(u(\cdot)) = \int_{\Omega_{0t_0}} L(s, x(s), u(s))ds + g(t_0, x(t_0)), \tag{3.2}$$

with $L(s, x(s), u(s))$ and $g : \Omega_{0t_0} \times \mathbb{R}^n \to \mathbb{R}$ smooth functions. Bolza problems arise when there is a cumulative cost which increases during the control action but special emphasis is placed on the situation at the final multitime t_0.

3.3 Equivalence of the previous problems via curvilinear primitive

Mayer, Lagrange and Bolza multitime problems are all equivalent in that each of them can be converted to any other one via the curvilinear primitive. First, the Lagrange and Mayer problems are special cases of Bolza problems. Second, a Bolza problem can be transformed into a Mayer problem by adding a new variable y for the state vector, which satisfies the PDEs (curvilinear primitive)

$$\frac{\partial y}{\partial t^\alpha}(t) = Y_\alpha(t), \;\; y(0) = 0, \, t \in \Omega_{0t_0},$$

where the functions Y_α are defined as follows: introduce Ω^β_{0t} as the face $\beta = 1, ..., m$ of the hyperparallelipiped $\Omega_{0t} \subset \Omega_{0t_0}$, use $ds_\beta = i_{\frac{\partial}{\partial s^\beta}} ds$ as the *interior product* or *contraction* of the volume form ds with $\frac{\partial}{\partial s^\beta}$ and define

$$Y_\beta(t) = \frac{1}{m} \int_{\Omega^\beta_{0t}} L(s, x(s), u(s))|_{s^\beta = t^\beta} \;\; ds_\beta,$$

where $x(\cdot) = (x^i(\cdot))$ solves the initial m-flow. Of course,

$$y(t_0) = y(0) + \int_{\Gamma_{0t_0}} Y_\alpha(t)dt^\alpha = y(0) + \int_{\Omega_{0t_0}} L(t, x(t), u(t))dt.$$

Consequently, using the extravariable y, the cost takes the Mayer form

$$Q(u(\cdot)) = y(t_0) + g(t_0, x(t_0)).$$

Third, a Mayer problem can be converted into a Lagrange problem, based on a curvilinear integral action, by rewriting the cost as

$$P(u(\cdot)) = g(t_0, x(t_0)) = g(0, x(0)) + \int_{\Gamma_{0t_0}} D_\alpha g(s, x(s))ds^\alpha$$

$$= g(0, x(0)) + \int_{\Gamma_{0t_0}} \left(\frac{\partial g}{\partial s^\alpha}(s, x(s)) + \frac{\partial g}{\partial x^i}(s, x(s))X^i_\alpha(s, x(s), u(s)) \right) ds^\alpha.$$

To pass from the curvilinear integral action to the multiple integral action, we use the C^{m-1} Lagrangian 1-form

$$L_\alpha = \frac{\partial g}{\partial s^\alpha}(s, x(s)) + \frac{\partial g}{\partial x^i}(s, x(s))X^i_\alpha(s, x(s), u(s))$$

and we define the Lagrangian

$$\overline{L}(s, x(s), u(s)) = \frac{\partial^{m-1} L_\alpha}{\partial s^1 ... \partial \hat{s}^\alpha ... \partial s^m},$$

where the symbol " ˆ " posed over ∂s^α designates that ∂s^α is omitted. Since the point $x(0)$ is fixed, the problem is to maximize the cost

$$\overline{Q}(u(\cdot)) = \int_{\Omega_{0t_0}} \overline{L}(s, x(s), u(s))ds,$$

which is a Lagrange problem based on a multiple integral action.

3.4 Equivalence of the previous problems via hyperbolic primitive

Mayer, Lagrange and Bolza multitime problems are all equivalent in that each of them can be converted to any other one via the hyperbolic primitive. First, the Lagrange and Mayer problems are special cases of Bolza problems. Second, a Bolza problem can be transformed into a Mayer problem by adding a new variable y for the state vector, which satisfies the PDEs (hyperbolic primitive)

$$\frac{\partial^m y}{\partial t^1 ... \partial t^m}(t) = L(t, x(t), u(t)),\ y(0) = 0,\ t \in \Omega_{0,t_0},$$

where $x(\cdot) = (x^i(\cdot))$ solves the initial m-flow. Of course,

$$y(t_0) = BT + \int_{\Omega_{0t_0}} L(t, x(t, u(t))dt,$$

where BT means boundary terms. Consequently, using the extravariable y, the cost takes the Mayer form

$$Q(u(\cdot)) = y(t_0) + g(t_0, x(t_0)).$$

Third, a Mayer problem can be converted into a Lagrange problem by rewriting the cost as a hyperbolic primitive

$$P(u(\cdot)) = g(t_0, x(t_0)) = g(0, x(0)) + BT+$$
$$+\int_{\Omega_{0t_0}} D_{1\dots m} g(s, x(s))|_{\text{m-flow}} ds.$$

where BT means boundary terms. Since the point $x(0)$ is fixed, the problem is to maximize the cost

$$\overline{Q}(u(\cdot)) = \int_{\Omega_{0t_0}} \overline{L}(s, x(s), u(s)) ds,$$

where

$$\overline{L}(s, x(s), u(s)) = D_{1\dots m} g(s, x(s))|_{\text{m-flow}},$$

which is a Lagrange problem based on a multiple integral action.

4 Equivalence between multiple and curvilinear integral functionals

A multitime evolution system can be used as a constraint in a problem of extremizing a *multitime cost functional*. On the other hand, the multitime cost functionals can be introduced at least in two ways:

- either using a path independent curvilinear integral ("mechanical work"),

$$P(u(\cdot)) = \int_{\Gamma_{0t_0}} L_\beta(t, x(t), u(t)) dt^\beta + g(t_0, x(t_0)),$$

where Γ_{0t_0} is an arbitrary C^1 curve joining the points 0 and t_0, the *running cost* $\omega = L_\beta(x(t), u(t))dt^\beta$ is an autonomous closed (completely integrable) *Lagrangian 1-form*, and g is the *terminal cost*;

- or using a multiple integral ("volume"),

$$Q(u(\cdot)) = \int_{\Omega_{0t_0}} L(t, x(t), u(t)) dt + g(t_0, x(t_0)),$$

where the *running cost* $L(t, x(t), u(t))$ is an autonomous continuous *Lagrangian*, and $g(t_0, x(t_0))$ is the *terminal cost*.

Let us show that the functional P is equivalent to the functional Q. This means that in a multitime optimal control problem we can choose the appropriate functional based on geometrical-physical meaning or other criteria.

Theorem 7 [15]. *The multiple integral*

$$I(t_0) = \int_{\Omega_{0t_0}} L(t, x(t), u(t)) dt,$$

with L *as continuous function, is equivalent to the curvilinear integral*

$$J(t_0) = \int_{\Gamma_{0t_0}} L_\beta(t, x(t), u(t))dt^\beta,$$

where $\omega = L_\beta(x(t), u(t))dt^\beta$ *is a closed (completely integrable) Lagrangian* 1*-form and the functions* L_β *have total derivatives of the form*

$$D_\alpha,\ D_{\alpha\beta}\,(\alpha < \beta),\ ...,\ D_{1...\hat{\alpha}...m},$$

where the symbol " $\hat{}$ *" posed over* α *designates that* α *is omitted.*

5 Conclusion

It is well-known that the single-time optimal control problems of Mayer, Lagrange and Bolza are equivalent [1]-[3].

The results in the previous sections show that the multitime optimal control problems of Mayer, Lagrange and Bolza, formulated in the sense of the papers [4]-[18], are equivalent via the curvilinear primitive or via the hyperbolic primitive. Their treatment in mathematics, in the continuous context, has had a slow evolution, the obstruction being the complete integrability conditions. In fact, the multitime maximum principle was established only recently [5], [13], [16]-[18], requiring a geometrical language.

Acknowledgements: Partially supported by University Politehnica of Bucharest and by Academy of Romanian Scientists.

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Author's address:

Constantin Udrişte
University Politehnica of Bucharest, Faculty of Applied Sciences,
Department of Mathematics-Informatics I,
313 Splaiul Independentei, 060042 Bucharest, Romania.
E-mail: udriste@mathem.pub.ro, anet.udri@yahoo.com