Gasdynamic regularity: some classifying geometrical remarks

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Dedicated to the 70-th anniversary of Professor Constantin Udriste

Abstract. Two distinct genuinely nonlinear contexts [isentropic; strictly anisentropic of a particular type] are considered for a hyperbolic quasilinear system of a gasdynamic type. To each of these two contexts a pair of two classes of solutions ["wave" solutions; "wave-wave regular interaction" solutions] is associated. A parallel is then considered between the isentropic pair of classes and the strictly anisentropic pair of classes, making evidence of some consonances, and, concurrently, of some nontrivial significant contrasts.

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1 Introduction

Some significant descriptions of gasdynamic evolutions are essentially based upon two types of genuinely nonlinear ingredients: "wave" solutions, and, respectively, "wave-wave interaction" solutions. The two mentioned types of ingredients are considered in the literature in two theoretical versions: a qualitative version, and, respectively, an analytical version.

The present paper deals with some aspects of the analytical version; see [2] to refer this version to the qualitative version. The qualitative version (used, for example, in the study of the 2D Riemann problem; see [14]) structures a "portrait" of a 2D wave-wave interaction of a special type [irregular; orthogonal] to which four wave solutions contribute − each of them in the form of an 1D simple waves solution.

The analytical version is constructive. Two distinct contexts [isentropic; strictly anisentropic of a particular type] are considered for a hyperbolic system of a gasdynamic type. The present paper takes into account two *genuinely nonlinear* constructions: an "algebraic" one [of a Burnat type; centered on a duality connection between the hodograph character and the physical character] and, respectively, a "differential" one [of a Martin type; centered on a Monge−Ampère type representation]. To *each* of the two mentioned contexts a pair of two classes of solutions [wave solutions; wave-wave interaction solutions] is associated − via a corresponding significant and specific

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intermediate construction. In the analytical construction the wave-wave interaction solutions are associated with a *regular* character.

The two mentioned constructions show some distinct, complementary, valences. • The "algebraic" approach appears to be essential for some isentropic multidimensional extensions with a classifying potential. • The "differential" approach appears, in its turn, to be essential for some strictly anisentropic descriptions. • The two mentioned constructions are associated with some distinct dimensional characterizations: the "algebraic" approach allows multidimensional objects, while the "differential" approach is restricted to two independent variables.

• We consider, to begin with, the "differential" approach via a comparison between two significant versions of it − a [hyperbolic] unsteady one-dimensional version, and an [elliptic-hyperbolic] steady two-dimensional one [which appears again to show a hyperbolic character in presence of a supersonic description]. • Finally, the "differential" approach is parallelled to the "algebraic" approach − making evidence of some consonances, and, concurrently, of some nontrivial significant contrasts. Some remarks concerning the ***fragility*** of the regular passage from isentropic to anisentropic are included.

2 "Algebraic" approach of a Burnat type. Genuine nonlinearity restrictions

2.1 Introduction

For the multidimensional first order hyperbolic system of a gasdynamic type

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u)\frac{\partial u_j}{\partial x_k} = 0, \quad 1 \le i \le n \tag{2.1}$$

the "algebraic" approach (Burnat [1]) starts with identifying *dual* pairs of directions $\vec{\beta}, \vec{\kappa}$ [we write $\vec{\kappa} \rightleftharpoons \vec{\beta}$] connecting [via their duality relation] the hodograph [= in the hodograph space H of the entities u] and physical [= in the physical space E of the independent variables] characteristic details. The duality relation at $u^* \in H$ has the form:

$$\sum_{j=1}^{n}\sum_{k=0}^{m} a_{ijk}(u^*)\beta_k\kappa_j = 0, \quad 1 \le i \le n. \tag{2.2}$$

Here $\vec{\beta}$ is an *exceptional* direction [= *normal characteristic* direction (orthogonal in the physical space E to a characteristic character)]. A direction $\vec{\kappa}$ dual to an exceptional direction $\vec{\beta}$ is said to be a *hodograph characteristic* direction. The reality of exceptional / hodograph characteristic directions implied in (2.2) is concurrent with the hyperbolicity of (2.1).

Example 2.1. For the *one-dimensional* strictly hyperbolic version of system (2.1) a *finite* number n of dual pairs $\vec{\kappa}_i \rightleftharpoons \vec{\beta}_i$ consisting in $\vec{\kappa}_i = \vec{R}_i$ and $\vec{\beta}_i = \Theta_i(u)[-\lambda_i(u), 1]$, where $\vec{R}_i$ is a right eigenvector of the $n \times n$ matrix a and λ_i is an eigenvalue of a, are available ($i = 1, ..., n$). Each dual pair associates in this case, at each $u^* \in \mathcal{R}$ [for a suitable region $\mathcal{R} \subset H$], to a vector $\vec{\kappa}$ a *single* dual vector $\vec{\beta}$.

Example 2.2 (Peradzyński [12]). For the *two-dimensional* isentropic version of (2.1) an *infinite* number of dual pairs are available at each $u^*\in H$. Each dual pair associates, at the mentioned u^*, to a vector $\vec{\kappa}$ a *single* dual vector $\vec{\beta}$.

Example 2.3 (Peradzyński [13]). For the isentropic description corresponding to the *three-dimensional* version of (2.1) an *infinite* number of dual pairs are available at each $u^*\in H$. Each dual pair associates, at the mentioned u^*, to a vector $\vec{\kappa}$ a *finite* [constant, $\neq 1$] number of k independent exceptional dual vectors $\vec{\beta}_j$, $1\leq j\leq k$; and therefore has the structure $\vec{\kappa} \rightleftharpoons (\vec{\beta}_1,\ldots,\vec{\beta}_k)$.

Definition 2.4 (Burnat [1]). A curve $\mathcal{C}\subset H$ is said to be *characteristic* if it is tangent at each point of it to a characteristic direction $\vec{\kappa}$. A hypersurface $\mathcal{S}\subset H$ is said to be *characteristic* if it possesses at least a characteristic system of coordinates.

2.2 Genuine nonlinearity. Nondegeneracy. Simple waves solutions

Remark 2.5. As it is well-known (Lax [8]), in case of an one-dimensional strictly hyperbolic version of (2.1) a hodograph characteristic curve $\mathcal{C}\subset\mathcal{R}\subset H$, of index i, is said to be *genuinely nonlinear (gnl)* if the dual constructive pair $\vec{\kappa}_i \rightleftharpoons \vec{\beta}_i$ is restricted [the restriction is on the *pair* !] by $\vec{\kappa}_i(u)\diamond\vec{\beta}_i(u) \equiv \vec{R}_i(u)\cdot\mathrm{grad}_u\lambda_i(u)\neq 0$ in $\mathcal{R}$; see Example 2.1. This condition transcribes the requirement $\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\alpha}\neq 0$ along *each* hodograph characteristic curve $\mathcal{C}$.

Definition 2.6. We naturally extend the *gnl* character of a hodograph characteristic curve $\mathcal{C}$ to the cases corresponding to Examples 2.2 and 2.3, by requiring along $\mathcal{C}$: $\left|\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}\alpha}\right|\neq 0$ and, respectively, $\sum\limits_{\mu=1}^{k}\left|\frac{\mathrm{d}\vec{\beta}_\mu}{\mathrm{d}\alpha}\right|\neq 0$.

Definition 2.7*a*. A solution of (2.1) whose hodograph is laid along a *gnl* characteristic *curve* is said to be a *simple waves solution* (here below also called *wave*). The *gnl* character implies a *nondegeneracy* [resulting from a "funning out"] of such a solution.

Here are three types of simple waves solutions, presented in an implicit form − respectively associated, in presence of a *gnl* character, to the Examples 2.1−2.3 above [$\alpha(x,t)$ results from the implicit function theorem; solution $U\circ\alpha$ is structured by (2.2)]

$$\left.\begin{array}{c} u=U[\alpha(x,t)];\ \alpha=\theta(\xi),\ \ \xi=x-\zeta_i(\alpha)t,\\ u=U[\alpha(x,t)];\ \alpha=\theta(\xi),\ \xi=\sum_{\nu=0}^{m}\beta_\nu[U(\alpha)]x_\nu=\sum_{\nu=0}^{m}\beta_\nu\,\{U[\theta(\xi)]\}x_\nu,\\ u=U[\alpha(x,t)];\ \alpha=\theta(\xi_1,\ldots,\xi_k),\ \xi_j=\sum_{\nu=0}^{m}\beta_{j\nu}[U(\alpha)]x_\nu;\ 1\leq j\leq k \end{array}\right\}\ \begin{array}{c}\dfrac{\mathrm{d}U}{\mathrm{d}\alpha}=\vec{\kappa}\\ \text{along } \mathcal{C}.\end{array}$$

These representations indicate that a simple waves solution is constant in E over some straightlines / planes [cf. $\xi=\mathrm{constant}$; for 1D or 2D] or include some planar substructures [corresponding to ξ_j; for 3D].

2.3 Genuine nonlinearity: a constructive extension. Nondegeneracy. Riemann−Burnat invariants. A subclass of wave-wave regular interaction solutions

Remark 2.8. Let $R_1, \ldots, R_p$ be *gnl* characteristic coordinates on a given p-dimensional characteristic region $\mathcal{R}$ of a hodograph hypersurface $\mathcal{S}$ with the normal $\vec{n}$. Solutions of the ***intermediate*** system

$$\frac{\partial u_l}{\partial x_s}=\sum_{k=1}^{p}\eta_k\kappa_{kl}(u)\beta_{ks}(u),\ u\in\mathcal{R};\ 1\leq l\leq n,\ 0\leq s\leq m;\ \ \vec{\kappa}_k\perp\vec{n},\ \ 1\leq k\leq p \tag{2.3}$$

appear to concurrently satisfy the system (2.1) [we carry (2.3) into (2.1) and take into account (2.2)]. This indicates a key importance of the "algebraic" concept of dual pair.

Definition 2.7*b*. A solution of (2.1) whose hodograph is laid on a characteristic hypersurface is said to correspond to a *wave-wave regular interaction* if its hodograph possesses a *gnl* system of coordinates *and* there exists a set of ***Riemann−Burnat invariants*** $R(x)$, structuring the dependence on x of the solution u by a *regular* interaction representation

$$u_l=u_l[R_1(x_0,...,x_m),...,R_p(x_0,...,x_m)],\ 1\leq l\leq n. \tag{2.4}$$

Remark 2.9. We consider next a ***subclass*** of the wave-wave solutions of (2.1). This subclass results whether (2.1) is replaced by (2.3) in Definition 2.7*b* [because, the solutions of (2.3) concurrently satisfy (2.1); cf. Remark 2.8]. To construct this subclass we have to put together (2.3) and (2.4). We compute from (2.4)

$$\frac{\partial u_l}{\partial x_s}=\sum_{k=1}^{p}\frac{\partial u_l}{\partial R_k}\cdot\frac{\partial R_k}{\partial x_s}=\sum_{k=1}^{p}\kappa_{kl}(u)\frac{\partial R_k}{\partial x_s},\ \ 1\leq l\leq n;\ 0\leq s\leq m \tag{2.5}$$

and compare (2.5) with (2.3), taking into account the independence of the characteristic directions $\vec{\kappa}_k$. It results that for a wave-wave regular interaction solution in the mentioned subclass, $R_i(x)$ in (2.4) must fulfil a *reasonable* (overdetermined and Pfaff) system

$$\frac{\partial R_k}{\partial x_s}=\eta_k\beta_{ks}[u(R)],\ \ 1\leq k\leq p,\ 0\leq s\leq m. \tag{2.6}$$

Sufficient restrictions for solving (2.6) are proposed in [6], [7], [12], [13].

A wave-wave regular interaction reflects the *nondegenerate* nature of the *gnl* hodographs of the interacting simple waves solutions. The "algebraic" characterization of a wave-wave regular interaction will be regarded to correspond to a case of ["algebraic"] nondegeneracy. The *gnl* character of the contributing simple waves solutions results in an ad hoc *gnl* character of the wave-wave regular interaction solution constructed. Importance of a ***criterion of selection***, in favor of the genuine nonlinearity, when a ***hybrid*** nature [concurrently implying a linear degeneracy] could be present, is discussed and exemplified in [2].

3 "Differential" approach of a Martin type

A gasdynamic Riemann−Lax invariance analysis associated to the systems (3.1) and (3.3) indicates that an "algebraic" construction of simple waves solutions or wave-wave regular interaction solutions appears to be essentially *isentropic* ([2]: 4.6, Example 5.1).

• To a ***particular*** strictly anisentropic context [characterized in sections 3.3, 3.4 to be "pseudo isentropic" and *gnl*] we associate in §3 a "differential" Martin type approach ([10], [11]; see sections 3.1, 3.2) centered on a Monge−Ampère type of representation of solution. This is done in two significant versions [unsteady one-dimensional, supersonic steady two-dimensional].

• In an isentropic context the Martin type approach *persists* [due to its "pseudo isentropic" character] and appears to coincide with the Burnat type approach. It still *essentially replaces* the Burnat type approach in a strictly anisentropic context. The nature of this replacement significantly depends on the possibility of Martin's approach to characterize, in a strictly anisentropic context, some "differential" *analogues* of the "algebraic" simple waves solutions or wave-wave regular interaction solutions. Such a Martin characterization appears to be active only in presence of a *geometrical linearization* (described in §4; in the sense of [11]). In presence of such a linearization a parallel between Burnat's approach and Martin's approach is included in §4.

3.1 An unsteady one-dimensional version. First details

For the unsteady one-dimensional conservative anisentropic form of (2.1)

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v_x)}{\partial x}=0,\ \ \frac{\partial(\rho v_x)}{\partial t}+\frac{\partial}{\partial x}\left(\rho v_x^2+p\right)=0,\ \ \frac{\partial(\rho S)}{\partial t}+\frac{\partial(\rho v_x S)}{\partial x}=0,\ \ S=S(p,\rho) \quad (3.1)$$

(in usual notations: ρ, v_x, p, S are respectively the mass density, fluid velocity, pressure and entropy density) a Martin type approach uses, to begin with, the first two equations $(3.1)_{1,2}$ to introduce (Martin [10]) the functions $\psi, \tilde{\xi}$ and ξ cf.

$$\mathrm{d}x=\frac{1}{\rho}\mathrm{d}\psi+v_x\mathrm{d}t,\quad \mathrm{d}\tilde{\xi}=v_x\mathrm{d}\psi-p\,\mathrm{d}t;\quad \xi=\tilde{\xi}+pt,\quad \mathrm{d}\xi=v_x\mathrm{d}\psi+t\mathrm{d}p. \quad (3.2)$$

3.2 A steady two-dimensional version. First details

For the steady two-dimensional conservative anisentropic form of (2.1)

$$\begin{gathered}\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}=0,\ \ \frac{\partial}{\partial x}\left(\rho v_x^2+p\right)+\frac{\partial}{\partial y}\left(\rho v_x v_y\right)=0,\\ \frac{\partial}{\partial x}\left(\rho v_x v_y\right)+\frac{\partial}{\partial y}\left(\rho v_y^2+p\right)=0,\ \ \frac{\partial(\rho v_x S)}{\partial x}+\frac{\partial(\rho v_y S)}{\partial y}=0;\ \ S=S(p,\rho)\end{gathered} \quad (3.3)$$

a Martin type approach uses, to begin with, the first three equations $(3.3)_{1,2,3}$ to introduce (Martin [10]) the functions $\tilde{\xi}, \tilde{\eta}$ cf.

$$\mathrm{d}\tilde{\xi}=-(\rho v_x v_y)\mathrm{d}x+(\rho v_x^2+p)\mathrm{d}y,\quad \mathrm{d}\tilde{\eta}=-(\rho v_x^2+p)\mathrm{d}x+(\rho v_x v_y)\mathrm{d}y,$$

the functions ξ, η

$$\xi=\tilde{\xi}-py,\ \ \eta=\tilde{\eta}+px,$$

and the stream function ψ, to get

$$\mathrm{d}\psi = -(\rho v_y)\mathrm{d}x + (\rho v_x)\mathrm{d}y,\quad \mathrm{d}\xi = v_x\mathrm{d}\psi - y\mathrm{d}p,\quad \mathrm{d}\eta = v_y\mathrm{d}\psi + x\mathrm{d}p. \tag{3.4}$$

3.3 The unsteady one-dimensional version: anisentropic details

Remark 3.1*a* [concerning an unsteady one-dimensional solution with shock]. A continuous [smooth] anisentropic [strictly adiabatic] flow results behind a shock discontinuity of non-constant continuous [smooth] velocity which penetrates into a region of uniform flow. For such a flow, entropy $S(p,\rho)$ in $(3.1)_3$ is a function of ψ alone, $F(\psi)$, determined by the shock conditions. Prescription of F as a function of ψ provides an algebraic relation between p, ρ, ψ throughout the anisentropic flow region. • Such a ***particular*** anisentropic flow shows a "pseudo isentropic" and *gnl* character − thus allowing a Monge−Ampère type description (3.6)−(3.8) for the system (3.1) [see Remarks 3.2 and 3.3*a,b,c* here below].

We follow Martin [10] to seek for solutions of (3.1) which fulfil the (natural; see [5]) requirement

$$\frac{\partial p}{\partial t}\frac{\partial \psi}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \psi}{\partial t} \neq 0, \tag*{$(3.5)_{x,t}$}$$

use (3.5) to select [Martin] p and ψ as new independent variables in place of x and t, and compute from (3.2)

$$\frac{\partial x}{\partial \psi} = v_x\frac{\partial t}{\partial \psi} + \frac{1}{\rho},\ \frac{\partial x}{\partial p} = v_x\frac{\partial t}{\partial p},\ v_x = \frac{\partial \xi}{\partial \psi},\ t = \frac{\partial \xi}{\partial p}. \tag{3.6}$$

On eliminating x from $(3.6)_{1,2}$ and taking $(3.6)_{3,4}$ into account it results that ξ must fulfil the hyperbolic Monge−Ampère equation

$$\frac{\partial^2\xi}{\partial p^2}\frac{\partial^2\xi}{\partial \psi^2} - \left(\frac{\partial^2\xi}{\partial p\partial\psi}\right)^2 = -\zeta^2(p,\psi) \equiv \frac{\partial}{\partial p}\left(\frac{1}{\rho}\right) \equiv -\frac{1}{\rho^2 c^2} \tag{3.7}$$

where $\rho = \rho(p,\psi)$ and $c(p,\psi) = \sqrt{\left(\frac{\partial\rho}{\partial p}\right)_S^{-1}}$ is an ad hoc sound speed. Finally, we compute from (3.6)

$$x = \int \left(\frac{\partial \xi}{\partial \psi}\frac{\partial^2\xi}{\partial p\partial\psi} + \frac{1}{\rho}\right)\mathrm{d}\psi + \left(\frac{\partial \xi}{\partial \psi}\frac{\partial^2\xi}{\partial p^2}\right)\mathrm{d}p. \tag{3.8}$$

Remark 3.2. For *any* smooth solution $\xi(p,\psi)$ of (3.7) we get from (3.6), (3.8)

$$v_x = v_x(p,\psi),\ x = x(p,\psi),\ t = t(p,\psi). \tag{3.9}$$

On reversing [cf. (3.5)] $(3.9)_{2,3}$ into $p = p(x,t)$, $\psi = \psi(x,t)$ and carrying this into $(3.9)_1$ we get a form $p(x,t)$, $v_x(x,t)$, $\psi(x,t)$ of the corresponding anisentropic solution of (3.1).

Remark 3.3 ([3], [4]). (*a*) The hyperbolicity of (3.1) corresponds to the hyperbolicity of (3.7). (*b*) On prescribing F we will not find the streamlines $\mathcal{C}_0$ among the physical characteristic fields of (3.1) [a "pseudo isentropic" aspect]. (*c*) The two families of characteristics $\overline{\mathcal{C}}_\mp$ of (3.7) in the plane p,ψ appear to correspond to the two families of sound characteristics $\mathcal{C}_\pm$ in the plane x,t [a "pseudo isentropic" aspect].

3.4 The steady two-dimensional version: anisentropic details

Remark 3.1b [concerning a steady two-dimensional solution with shock]. A continuous [smooth] anisentropic [strictly adiabatic] ***rotational*** flow results behind a curved shock discontinuity from a region of uniform flow ahead. For such a flow, entropy $S(p,\rho)$ in $(3.3)_4$ and the Bernoulli type function $e+\frac{p}{\rho}+\frac{1}{2}V^2$ [e is the density of the internal energy, $V^2=v_x^2+v_y^2$; according to Crocco's form of (3.3)] are functions of ψ alone, $F(\psi)$, respectively $H(\psi)$, determined by the shock conditions. Prescription of F and H as functions of ψ provides two algebraic relations among p,ρ,V^2,ψ throughout the anisentropic flow region. • Such a ***particular*** anisentropic flow shows a "pseudo isentropic" and *gnl* character − thus allowing a Monge−Ampère type description (3.10), $(3.14)_{\xi,\eta}$ for the system (3.3) [see Remarks 3.4*a,b,c* here below].

We follow Martin [10] to parallel section 3.3 and seek for solutions of (3.3) which fulfil the [natural] requirement $(3.5)_{x,y}$, use (3.10) to select [Martin] p and ψ as new independent variables in place of x and y, and compute from $(3.4)_{2,3}$

$$x=\frac{\partial\eta}{\partial p},\ \ y=-\frac{\partial\xi}{\partial p},\ \ v_x=\frac{\partial\xi}{\partial\psi},\ \ v_y=\frac{\partial\eta}{\partial\psi},\tag{3.10}$$

and from $(3.4)_1$

$$v_x\frac{\partial y}{\partial\psi}-v_y\frac{\partial x}{\partial\psi}=\frac{1}{\rho},\quad v_x\frac{\partial y}{\partial p}-v_y\frac{\partial x}{\partial p}=0.\tag{3.11}$$

We then transcribe (3.11) via (3.10) cf.

$$\frac{\partial\xi}{\partial\psi}\frac{\partial^2\xi}{\partial p\partial\psi}+\frac{\partial\eta}{\partial\psi}\frac{\partial^2\eta}{\partial p\partial\psi}+\frac{1}{\rho(p,\psi)}=0,\qquad \frac{\partial\xi}{\partial\psi}\frac{\partial^2\xi}{\partial p^2}+\frac{\partial\eta}{\partial\psi}\frac{\partial^2\eta}{\partial p^2}=0\ .\tag{3.12}$$

Finally we integrate $(3.12)_1$ with respect to p and obtain

$$\left(\frac{\partial\xi}{\partial\psi}\right)^2+\left(\frac{\partial\eta}{\partial\psi}\right)^2=\mathcal{F}(p,\psi),\ \ \frac{\partial\xi}{\partial\psi}\frac{\partial^2\xi}{\partial p^2}+\frac{\partial\eta}{\partial\psi}\frac{\partial^2\eta}{\partial p^2}=0;\ \ \mathcal{F}(p,\psi)=2\mathcal{G}(\psi)-2\int^p\frac{\mathrm{d}p}{\rho}\tag{3.13}$$

where $\mathcal{F}$ is determined by the shock conditions. We solve simultaneously for $\frac{\partial\eta}{\partial\psi}$ and $\frac{\partial^2\eta}{\partial p^2}$ in (3.13) and carry the result into $\frac{\partial^2}{\partial p^2}\left(\frac{\partial\eta}{\partial\psi}\right)=\frac{\partial}{\partial\psi}\left(\frac{\partial^2\eta}{\partial p^2}\right)$ in order to eliminate η in favor of ξ. We are led to a Monge−Ampère type equation for ξ:

$$4\mathcal{F}\left[\left(\frac{\partial^2\xi}{\partial p\partial\psi}\right)^2-\frac{\partial^2\xi}{\partial p^2}\frac{\partial^2\xi}{\partial\psi^2}\right]-4\left(\frac{\partial\xi}{\partial\psi}\frac{\partial\mathcal{F}}{\partial p}\right)\frac{\partial^2\xi}{\partial p\partial\psi}+2\left(\frac{\partial\xi}{\partial\psi}\frac{\partial\mathcal{F}}{\partial\psi}\right)\frac{\partial^2\xi}{\partial p^2}+\left\{\left(\frac{\partial\mathcal{F}}{\partial p}\right)^2-2\left[\mathcal{F}-\left(\frac{\partial\xi}{\partial\psi}\right)^2\right]\frac{\partial^2\mathcal{F}}{\partial p^2}\right\}=0\tag{3.14_ξ}$$

where $\rho=\rho(p,\psi)$ and $c(p,\psi)=\sqrt{\left(\frac{\partial\rho}{\partial p}\right)_S^{-1}}$ is an ad hoc sound speed.

As the system (3.12) is symmetric in ξ and η it results that η must fulfil the same Monge−Ampère type equation (3.14). A given solution ξ of $(3.14)_\xi$ is paired by a computed [cf. (3.13)] solution η of $(3.14)_\eta$.

The characteristic directions for (3.14) in the plane p,ψ are given ([5]) by

$$\left(\frac{\mathrm{d}p}{\mathrm{d}\psi}\right)_\pm=\frac{2\mathcal{F}\frac{\partial^2\xi}{\partial p\partial\psi}-\frac{\partial\mathcal{F}}{\partial p}\frac{\partial\xi}{\partial\psi}\pm\sqrt{\Delta}}{-2\mathcal{F}\frac{\partial^2\xi}{\partial p^2}}\ ,\quad \Delta=\frac{4}{\rho^2c^2}v_y^2\left(V^2-c^2\right)\ ,\quad V^2=v_x^2+v_y^2\ .\tag{3.15}$$

We have

$$\left.\begin{array}{l} -V^2\mathrm{d}y + \dfrac{1}{\rho c}\left[cv_x \pm v_y\sqrt{V^2-c^2}\right]\mathrm{d}\psi = 0 \\ v_y\left[v_y\mathrm{d}v_x - v_x\mathrm{d}v_y \mp \dfrac{1}{\rho c}\sqrt{V^2-c^2}\mathrm{d}p\right] = 0 \end{array}\right\} \text{ along the characteristics } \overline{\mathcal{C}}_\pm \text{ of (3.14).} \tag{3.16}$$

Remark 3.4*a*. In contrast with the unsteady one-dimensional case, the system (3.3) and the Monge−Ampère equation (3.14) show an ***elliptic-hyperbolic*** character generally ([4]). Still, cf. (3.16), they show both a *hyperbolic* character for a *supersonic* flow. This aspect pairs the one-dimensional Remark 3.3*a*.

Remark 3.4*b*. On prescribing F and H we will not find the streamlines among the physical characteristic fields of (3.3) ([4]). This "pseudo isentropic" aspect pairs the one-dimensional Remark 3.3*b*.

Remark 3.4*c*. The Mach lines $\mathcal{C}_\pm$ of (3.3) in the physical plane and the characteristics $\overline{\mathcal{C}}_\pm$ [(3.15)] of the Monge−Ampère equation (3.14) are in correspondence ([4]). In fact, we get from $(3.16)_1$ and $(3.4)_1$

$$-V^2\mathrm{d}y+\frac{1}{\rho c}\left[cv_x\pm v_y\sqrt{V^2-c^2}\right](v_x\mathrm{d}y+v_y\mathrm{d}x) = 0 \text{ along the characteristics } \overline{\mathcal{C}}_\pm \text{ of (3.14)}$$

which results in

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{cv_x \pm v_y\sqrt{V^2-c^2}}{cv_y \mp v_x\sqrt{V^2-c^2}} = \frac{v_xv_y \pm c\sqrt{V^2-c^2}}{v_x^2 - c^2} = \lambda_\pm \text{ along the characteristics } \overline{\mathcal{C}}_\pm \text{ of (3.14)}$$

where λ_+ and λ_- are the Mach eigenvalues of the system (3.3). This "pseudo isentropic" aspect pairs the one-dimensional Remark 3.3*c*.

4 Martin linearization. A classifying parallel between Burnat's and Martin's approaches

4.1 Unsteady one-dimensional version. Pseudo simple waves solution. Pseudo wave-wave interaction solution. Riemann−Martin invariants

In case of anisentropic systems (3.1) and (3.3) some "differential" *analogues* of the "algebraic" simple waves solutions or wave-wave regular interaction solutions could be constructed, in presence of a "pseudo isentropic" and *gnl* character, by a *geometrical linearization* approach associated to a Martin type construction ([11]).

Such a linearization − we call it a ***Martin linearization*** − becomes active whether we can find for the Monge−Ampère type equation (3.7) / (3.14) associated to (3.1) / (3.3) a pair of intermediate integrals $\mathcal{F}_\pm\left(p, \psi, \xi, \frac{\partial\xi}{\partial p}, \frac{\partial\xi}{\partial\psi}\right)$, *linear* in ξ. The presence of such a pair appears to be a constructive ***intermediate*** element associated to a Martin type approach; we notice that, similarly, the Burnat construction was based upon an intermediate element [(2.3)]. • There are ***few*** cases of Martin linearization available in the literature (see for example Martin [11] or Ludford [9]).

• In presence of such a pair we have $\mathcal{F}_{\pm}=\text{constant}=R_{\pm}$ along a characteristic $\overline{\mathcal{C}}_{\pm}$ and we must distinguish between the circumstances (a) when $R_{\pm}$ depend on the characteristic $\overline{\mathcal{C}}_{\pm}$, and (b) when R_{+} or R_{-} are overall constants.

• In the case (a) we may use $R_{\pm}$ as new independent variables. It can be shown in this case (Martin [11]) that the entities p^{-1}, v_x, ψ^{-1}, t fulfil various Euler−Poisson−Darboux *linear* equations

$$\frac{\partial^2 w}{\partial R_+ \partial R_-} - \frac{\nu}{R_+ - R_-}\left(\frac{\partial w}{\partial R_+} - \frac{\partial w}{\partial R_-}\right) = 0, \quad \text{constant } \nu$$

to which well-known representations of solutions are associated; we present these representations by

$$p=p(R_+,R_-),\ \psi=\psi(R_+,R_-),\ v_x=v_x(R_+,R_-);\quad t=t(R_+,R_-),\ x=x(R_+,R_-) \quad (4.1)$$

where $x(R_+,R_-)$ results by quadratures [see (3.8)]. Reversing $(4.1)_{4,5}$ into $R_{\pm} = R_{\pm}(x,t)$ will induce a form of solution $(4.1)_{1,2,3}$, parallel to (2.4) [as $R_{\pm}$ have a characteristic nature]. We call $R_{\pm}(x,t)$ ***Riemann−Martin invariants***.

• In the case (b) we notice that a solution $\xi(p,\psi)$ of the *linear* equation $\mathcal{F}_+ \equiv R_+$ or $\mathcal{F}_- \equiv R_-$ will automatically fulfil (3.7). We have to follow, in this case, Remark 3.2 to describe a solution of (3.1); we call such solution a *pseudo simple waves solution*. See [10] for some numerical remarks.

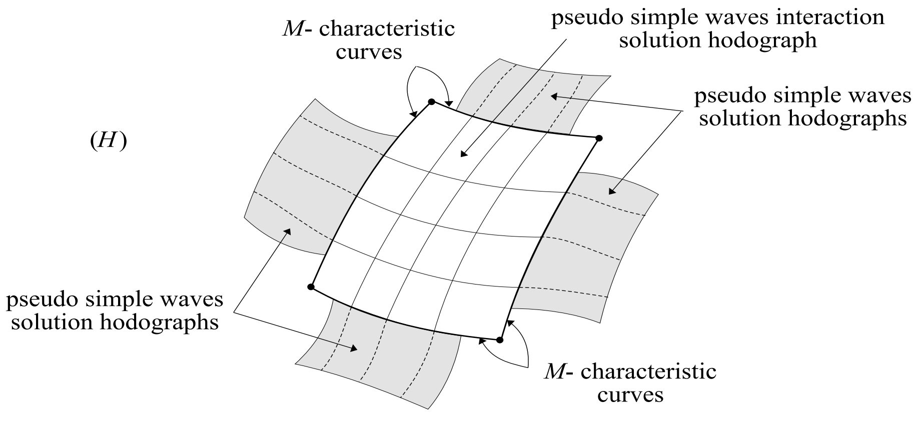

FIGURE 1

• Solution (4.1) might be regarded as *pseudo nondegenerate* [a formal regular interaction of *pseudo* simple waves solutions]. The image of a characteristic $\mathcal{C} \subset E$ on the hodograph of a solution of (3.1) will be said to be a *M*- characteristic. The hodograph of a formal regular interaction of pseudo simple waves solutions will be then made by glueing, along suitable *M*- characteristics, a hodograph (4.1) with some suitable hodographs of pseudo simple waves solutions; see Figure 1.

• The anisentropic solutions of (3.1) which do not belong to a linearization case will not show a regularity structure (4.1).

4.2 The steady two-dimensional version

For the Monge−Ampère type equation (3.15) a study of these authors on the Martin linearization [possibility, details] is in progress: identifying complementary restrictions.

4.3 Pseudo simple waves solution: an one-dimensional example

A case of Martin linearization is associated to $\zeta = \frac{\psi^{\nu-1}}{p^{\nu+1}}$ $\left(\nu = -\frac{\gamma-1}{2\gamma}, \text{integral}\,\nu,\ \nu \neq 0,1\right)$ in (3.7). To this ζ two intermediate integrals of (3.7), $\mathcal{F}_\pm \equiv p\frac{\partial\xi}{\partial p} + \psi\frac{\partial\xi}{\partial\psi} - \xi \pm \frac{1}{\nu}\left(\frac{\psi}{p}\right)^\nu$, correspond. We satisfy $\mathcal{F}_+ \equiv R_+ = 0$ by $\xi = \frac{1}{\nu}\left(\frac{\psi}{p}\right)^\nu$, and calculate from (3.6), (3.8) (see [4])

$$x = -\frac{\nu+1}{2\nu+1}\frac{\psi^{2\nu-1}}{p^{2\nu+1}}, \quad t = -\frac{\psi^\nu}{p^{\nu+1}}, \quad v_x = \frac{\psi^{\nu-1}}{p^\nu}, \quad \rho = (2\nu+1)\frac{p^{2\nu+1}}{\psi^{2\nu-2}},$$

which leads to

$$p = -\left(\frac{\nu+1}{2\nu+1}\right)^\nu \frac{t^{2\nu-1}}{(-x)^\nu}, \quad v_x = \frac{2\nu+1}{\nu+1}\frac{x}{t}, \quad \psi = -\left(\frac{\nu+1}{2\nu+1}\right)^{\nu+1}\frac{t^{2\nu+1}}{(-x)^{\nu+1}}. \tag{4.2}$$

This is a [local] pseudo simple waves solution of (3.1) corresponding to a certain region $\mathcal{D} \subset E$ (for example, a region of $t > 0$, $x < 0$). For this solution the assumption (3.5) holds.

Now, we proceed with the details. We compute

$$c = \frac{1}{\zeta\rho} = \frac{1}{2\nu+1}\frac{\psi^{\nu-1}}{p^\nu} = \frac{1}{2\nu+1}v_x \tag{4.3}$$

and notice that the *explicit* equations of the [physical] field lines $\mathcal{C}_-, \mathcal{C}_+, \mathcal{C}_0$ [of these, only $\mathcal{C}_\pm$ have a characteristic character (see Remark 3.3)] through a point $(x^*, t^*) \in \mathcal{D}$ result, cf. (4.2), (4.3), by respectively integrating the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x(x,t) - \theta_\alpha c(x,t) = k_\alpha\frac{x}{t}, \quad \alpha = -,0,+, \quad (\theta_-,\theta_0,\theta_+) = (-1,0,1) \ \text{ along}\, \mathcal{C}_- \tag{4.4}$$

where for $1 < \gamma < \frac{5}{3}$ we have $k_- = \dfrac{2\nu}{\nu+1} = -2\dfrac{\gamma-1}{\gamma+1}$, $k_0 = \dfrac{2\nu+1}{\nu+1} = \dfrac{2}{\gamma+1}$, $k_+ = 2$. We get from (4.4)

$$|x| = K_\alpha|t|^{k_\alpha}; \quad K_\alpha = \log\frac{|x^*|}{|t^*|^{k_\alpha}}, \quad \alpha = -,0,+, \ \text{ along } \mathcal{C}_\alpha \ni (x^*, t^*). \tag{4.5}$$

Remark 4.1. We notice that a pseudo simple waves solution has a two-dimensional hodograph [see (3.5)] and for it none of the characteristic fields $\mathcal{C}_\pm$ in the physical plane x, t is made of straightlines generally [see (4.5)]. This is ***in contrast*** with some "algebraic" aspects [see Definition 2.7a and the final lines of section 2.2].

4.4 Algebraic approach and differential approach: some unsteady one-dimensional contrasts

In each of the cases of Martin linearization a parallel is possible, independent of the already mentioned Riemann−Lax invariance analysis, between the "algebraic" approach and the "differential" approach. In [3] it is computed, at each point of the hodograph (4.1), the following relation between the Burnat hodograph characteristic directions $\vec{\kappa}$ and the Martin hodograph characteristic directions $\vec{\mu}$

$$\vec{\mu}_{\pm} = \left(\frac{\partial p}{\partial R_{\pm}}, \frac{\partial v_x}{\partial R_{\pm}}, \frac{\partial S}{\partial R_{\pm}} \right)^t = \eta_{\mp} \vec{\kappa}_{\mp} + \widetilde{\eta}_{\mp} \vec{\kappa}_0 \tag{4.6}$$

where

$$S(R_+,R_-) \equiv F[\psi(R_+,R_-)], \;\; \eta_{\mp} = \frac{1}{\Lambda_{\mp}} \frac{\partial v_x}{\partial R_{\pm}}, \;\; \widetilde{\eta}_{\mp} = \frac{\partial S}{\partial R_{\pm}} \; .$$

We notice from (4.6) that at the hodograph points of a solution of (3.1) the M-characteristic fields and, respectively, the Burnat characteristic fields appear to be *distinct* generally in the mentioned strictly anisentropic context and can be shown to be *coincident* in the isentropic context (cf. $\widetilde{\eta}_{\mp} \equiv 0$).

Remark 4.2. Representation (4.1) corresponds, for a strictly anisentropic description, to an example of hodograph surface of (3.1) which ***is not*** a Burnat characteristic surface [Definition 2.4]. Still, incidentally and essentially for the linearized approach, this representation appears to be associated with an example of hodograph surface of (3.1) for which a characteristic character persists in a Martin sense.

4.5 Final remarks

Finding a solution to the systems (2.1)/(3.1)/(3.3) or, alternatively, to the Monge−Ampère type equations (3.7)/(3.14) is a hard task generally. This suggests considering suitable classes of solutions to these systems or, alternatively, to the mentioned Monge−Ampère type equations.

In case of the system (2.1) a pair of such classes puts together the simple waves solutions and the wave-wave regular interaction solutions − associated to an "algebraic" construction. • In case of the equations (3.7)/(3.14) a pair of such classes could be constructed "differentially" by a *linearization* approach. The classes in this pair appear to be respectively connected with the pseudo simple waves solutions or the pseudo wave-wave regular interaction solutions. • A ***classifying parallel*** is constructed between the two pairs of classes. It is noticed that this parallel *concurrently classifies* two gasdynamic contexts: an isentropic one, and, respectively, an anisentropic one.

The regular passage [which uses the two mentioned pairs of classes] from an isentropic description to an anisentropic description appears to be ***fragile***. This aspect is suggested by section 4.1 [noticing "few cases of Martin linearization"], section 4.2 [reporting "complementary restrictions concerning the Martin linearization"], Remarks 3.1a and 3.1b [indicating a *particular* character of the anisentropic flow considered], sections 3.1−3.4 [considering dimensional restrictions (two independent variables) − in contrast with Burnat's availability].

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References

[1] M. Burnat, *The method of characteristics and Riemann's invariants for multidimensional hyperbolic systems"*, Sibirsk. Math. J. 11 (1970), 279–309.

[2] L.F. Dinu, *Multidimensional wave-wave regular interactions and genuine nonlinearity*, Preprint Series of Newton Institute for Math. Sci. No. 29, Cambridge UK, 2006.

[3] L.F. Dinu and M.I. Dinu, *Nondegeneracy, from the prospect of wave-wave regular interactions of a gasdynamic type*, Proceedings of the International Conference on Mathematical and Numerical Aspects of Waves, 2007, University of Reading UK, p.461.

[4] L.F. Dinu and M.I. Dinu, *Martin's "differential" approach*, Preprint Series of Newton Institute for Math. Sci. No. 21, Cambridge UK, 2009.

[5] L.F. Dinu, *Mathematical Concepts in Nonlinear Gas Dynamics*, CRC Press, forthcoming monograph.

[6] E.V. Ferapontov and K.R. Khusnutdinova, *On integrability of (2+1)-dimensional quasilinear systems]*, Commun. Math. Phys. 248 (2004), 187–206.

[7] E.V. Ferapontov and K.R. Khusnutdinova, *The Haahtjes tensor and double waves in multidimensional systems of hydrodynamic type: a necessary condition for integrability*, Proc. Roy. Soc. A 462 (2006), 1197-1219,

[8] P.D. Lax, *Hyperbolic systems of conservation laws (II)*, Comm. Pure and Appl. Math. 10 (1957), 537–566.

[9] G.S.S. Ludford, *Generalized Riemann invariants*, Pacific J. Math. 5 (1955), 441–450.

[10] M.H. Martin, *A new approach to problems in two dimensional flow*, Quarterly Appl. Math. 8 (1950), 137–150.

[11] M.H. Martin, *The Monge–Ampère partial differential equation* $rt - s^2 + \lambda^2 = 0$, Pacific J. Math. 3 (1953), 165–187.

[12] Z. Peradzyński, *Nonlinear plane k-waves and Riemann invariants*, Bull. Acad. Polon. Sci., Ser. Sci. Tech. 19 (1971), 625–631.

[13] Z. Peradzyński, *Riemann invariants for the nonplanar k-waves*, Bull. Acad. Polon. Sci., Ser. Sci. Tech. 19 (1971), 717–724.

[14] T. Zhang and Y.-X. Zheng, *Conjecture on structure of solutions of Riemann problem for two-dimensional gas dynamics*, SIAM J. Math. Anal. 21 (1990), 593–630.

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