Simplified multitime maximum principle

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Abstract. Many science and engineering problems can be formulated as optimization problems that are governed by m-flow type $PDEs$ (multitime evolution systems) and by cost functionals expressed as multiple integrals or curvilinear integrals. Our paper discuss the m-flow type PDE-constrained optimization problems, focussing on a simplified multitime maximum principle. This extends the simplified single-time maximum principle of Pontryaguin in the $ODEs$ case (curves) to include the case of $PDEs$ (submanifolds).

In Section 1 the idea of multitime is motivated. In Section 2 a multitime maximum principle, for the case of multiple integral functionals, is stated and proved. A version of multitime maximum principle, for the case of curvilinear integral functionals, is formulated in Section 3. Though a multiple integral functional is mathematically equivalent to a curvilinear integral functional (Section 4), their meaning is totally different in real life problems. A multitime maximum principle approach of variational calculus is presented in Section 5.

1 Multitime concept

The adjective multitime was introduced in Physics by Dirac (1932), and later was used in Mathematics by [4], [6], [8], [12]-[16], [19], [20], [22], etc. To underline the sense of this adjective, we collect the following remarks.

1) A space coordinate is merely an index numbering freedom degrees, and the time coordinate is usual the physical time in which the systems evolves. Such classical theory is satisfactory unless we turn our attention to relativistic problems (chiral fields, sine-Gordon PDE, etc). Moreover, in some physical problems we use a two-time $t = (t^1, t^2)$, where t^1 means the intrinsic time and t^2 is the observer time. Also there are a lot of problems where is no reason to prefer one coordinate to another. In this sense, we refer to *multitime geometric evolutions* and *multitime optimal control*

problems, where *multitime* means a vector parameter of evolution. Here we can include the description of torsion of prismatic bars, the maximization of the area surface for given width and diameter etc.

2) Multitime wave functions were first considered by Dirac in 1932 via m-time evolution equations $i\hbar\dfrac{\partial\psi}{\partial t^\alpha} = H_\alpha\psi$. The Dirac PDE system is consistent (completely integrable) if and only if $[H_\alpha, H_\beta] = 0$ for $\alpha \neq \beta$. The consistency condition is easy to achieve for non-interacting particles and tricky in the presence of interaction. But, until now, nobody attempted to write down consistent multitime equations for many interacting particles, although this would seem an immediate and highly relevant problem if one desires a manifestly covariant formulation of relativistic quantum mechanics.

3) The oscillators are very important in engineering and communications. For example, voltage-controlled oscillators, phase-locked loops, lasers, etc., abound in wireless and optical systems. A new approach for analyzing frequency and amplitude modulation in oscillators was realized recently using a novel concept, *warped time*, within a multitime PDE framework. To explain this idea from our point of view, we start with a single-time wave front $y(t) = \sin\left(\frac{2\pi}{T_1}t\right)\sin\left(\frac{2\pi}{T_2}t\right)$, $T_1 = 0.02\,s$; $T_2 = 1\,s$, where the two tones are at frequencies $f_1 = \dfrac{1}{T_1} = 50\,Hz$ and $f_2 = \dfrac{1}{T_2} = 1\,Hz$. Here there are 50 times faster varying sinusoids of period T_1 modulated by a slowly-varying sinusoid of period T_2. Then we build a "*two-variable representation*" of $y(t)$, obtained by the rules: for the "fast-varying" parts of $y(t)$, the time t is replaced by a new variable t^1; for the "slowly-varying" parts, by t^2. It appears a new periodic function of two variables, $\hat{y}(t^1, t^2) = \sin\left(\frac{2\pi}{T_1}t^1\right)\sin\left(\frac{2\pi}{T_2}t^2\right)$, motivated by the wide separated time scales. Inspection of the two-time (two-variable) wavefront $\hat{y}(t^1, t^2)$ directly provides information about the slow and fast variations of $y(t)$ more naturally and conveniently than $y(t)$ itself.

4) The known evolution laws in physical theories are *single-time evolution laws* (*ODEs*) or *multitime evolution laws* (PDEs). To change a single-time evolution into a *multitime evolution* it is enough to change the *ODEs* into *PDEs* accepting that the time t is a C^∞ function of certain parameters, let say $t = t(s^1, ..., s^m)$.

The PDE constraints often present significant challenges for optimization principles [2]-[6], [8], [9]. The multivariable maximum principle was studied in the presence of PDE constraints, starting with the papers [12]-[22]. This approach extends the single-time Pontryaguin's model [1], [7], [10], [11].

In this paper, we are looking for a multitime maximum principle, i.e., for necessary conditions of optimality. Our formulation and a proof mimic those that were applied to single-time maximum principle. A simplified version of this problem, obtained after years of debates in my research group and in conferences, is presented in this paper.

2   m-Flow type constrained optimization problem with multiple integral functional

Let us analyze a multitime optimal control problem based on a multiple integral cost functional and m-flow type PDE constraints:

$$\max_{u(\cdot),x_{t_0}} I(u(\cdot)) = \int_{\Omega_{0,t_0}} X(t,x(t),u(t))dt \qquad (1)$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X_\alpha^i(t,x(t),u(t)), i = 1,...,n; \alpha = 1,...,m, \qquad (2)$$

$$u(t) \in \mathcal{U}, \ t \in \Omega_{0,t_0}; \ x(0) = x_0, \ x(t_0) = x_{t_0}. \qquad (3)$$

Ingredients: $t = (t^\alpha) = (t^1,...,t^m) \in R_+^m$ is the *multitime* (multi-parameter of evolution); $dt = dt^1 \cdots dt^m$ is the *volume element* in R_+^m; Ω_{0,t_0} is the parallelepiped fixed by the diagonal opposite points $0 = (0,...,0)$ and $t_0 = (t_0^1,...,t_0^m)$ which is equivalent to the *closed interval* $0 \le t \le t_0$ via the *product order* on R_+^m; $x : \Omega_{0,t_0} \to R^n$, $x(t) = (x^i(t))$ is a C^2 *state vector*; $u : \Omega_{0,t_0} \to U \subset R^k$, $u(t) = (u^a(t))$, $a = 1,...,k$ is a C^1 *control vector*; the *running cost* $X(t,x(t),u(t))$ is a C^1 *nonautonomous Lagrangian*; $X_\alpha(t,x(t),u(t)) = (X_\alpha^i(t,x(t),u(t)))$ are C^1 vector fields satisfying the complete integrability conditions (*m*-flow type problem), i.e., $D_\beta X_\alpha = D_\alpha X_\beta$ (D_α is the total derivative operator) or

$$\left(\frac{\partial X_\alpha}{\partial u^a}\delta_\beta^\gamma - \frac{\partial X_\beta}{\partial u^a}\delta_\alpha^\gamma\right)\frac{\partial u^a}{\partial t^\gamma} = [X_\alpha, X_\beta] + \frac{\partial X_\beta}{\partial t^\alpha} - \frac{\partial X_\alpha}{\partial t^\beta},$$

where $[X_\alpha, X_\beta]$ means the *bracket* of vector fields. This hypothesis selects the set of all admissible controls (satisfying the complete integrability conditions)

$$\mathcal{U} = \left\{ u : R_+^m \to U \,\middle|\, D_\beta X_\alpha = D_\alpha X_\beta \right\}$$

and the admissible states.

We introduce a *costate variable* or *Lagrange multiplier matrix* $p = (p_i^\alpha)$ and a new Lagrangian

$$L(t,x(t),u(t),p(t)) = X(t,x(t),u(t)) + p_i^\alpha(t)[X_\alpha^i(t,x(t),u(t)) - \frac{\partial x^i}{\partial t^\alpha}(t)].$$

The PDE-constrained optimization problem (1)-(3) is changed into another optimization problem

$$\max_{u(\cdot),x_{t_0}} \int_{\Omega_{0,t_0}} L(t,x(t),u(t),p(t))dt$$

subject to

$$u(t) \in \mathcal{U}, \ p(t) \in \mathcal{P}, \ t \in \Omega_{0,t_0}, \ x(0) = x_0, x(t_0) = x_{t_0},$$

where the set $\mathcal{P}$ will be defined later. The *control Hamiltonian*

$$H(t,x(t),u(t),p(t)) = X(t,x(t),u(t)) + p_i^\alpha(t)X_\alpha^i(t,x(t),u(t)),$$

i.e.,

$$H = L + p_i^\alpha \frac{\partial x^i}{\partial t^\alpha} \text{ (modified Legendrian duality)},$$

allows to rewrite this new problem as

$$\max_{u(\cdot), x_{t_0}} \int_{\Omega_{0,t_0}} [H(t, x(t), u(t), p(t)) - p_i^\alpha(t) \frac{\partial x^i}{\partial t^\alpha}(t)] dt$$

subject to

$$u(t) \in \mathcal{U}, \ p(t) \in \mathcal{P}, \ t \in \Omega_{0,t_0}, \ x(0) = x_0, x(t_0) = x_{t_0}.$$

Suppose that there exists a continuous control $\hat{u}(t)$ defined over the parallelepiped Ω_{0,t_0} with $\hat{u}(t) \in \text{Int}\,\mathcal{U}$ which is an optimum point in the previous problem. Now consider a variation $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t)$, where h is an arbitrary continuous vector function. Since $\hat{u}(t) \in \text{Int}\,\mathcal{U}$ and a continuous function over a compact set Ω_{0,t_0} is bounded, there exists $\epsilon_h > 0$ such that $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t) \in \text{Int}\,\mathcal{U}, \ \forall |\epsilon| < \epsilon_h$. This ϵ is used in our variational arguments.

Define $x(t, \epsilon)$ as the m-sheet of the state variable corresponding to the control variable $u(t, \epsilon)$, i.e.,

$$\frac{\partial x^i}{\partial t^\alpha}(t, \epsilon) = X_\alpha^i(t, x(t, \epsilon), u(t, \epsilon)), \forall t \in \Omega_{0,t_0}$$

and $x(0, \epsilon) = x_0$. For $|\epsilon| < \epsilon_h$, we define the function

$$I(\epsilon) = \int_{\Omega_{0,t_0}} X(t, x(t, \epsilon), u(t, \epsilon)) dt.$$

Since the function $u(t, \epsilon)$ is admissible, it follows that the function $x(t, \epsilon)$ is admissible. On the other hand, the control $\hat{u}(t)$ must be optimal. Therefore $I(\epsilon) \leq I(0), \ \forall |\epsilon| < \epsilon_h$.

For any continuous vector function $p = (p_i^\alpha) : \Omega_{0,t_0} \to R^{nm}$, we have

$$\int_{\Omega_{0,t_0}} p_i^\alpha(t) [X_\alpha^i(t, x(t, \epsilon), u(t, \epsilon)) - \frac{\partial x^i}{\partial t^\alpha}(t, \epsilon)] dt = 0.$$

Necessarily, we must use the Lagrange function which includes the variations

$$L(t, x(t, \epsilon), u(t, \epsilon), p(t)) = X(t, x(t, \epsilon), u(t, \epsilon))$$

$$+ p_i^\alpha(t)[X_\alpha^i(t, x(t, \epsilon), u(t, \epsilon)) - \frac{\partial x^i}{\partial t^\alpha}(t, \epsilon)]$$

and the associated function

$$I(\epsilon) = \int_{\Omega_{0,t_0}} L(t, x(t, \epsilon), u(t, \epsilon), p(t)) dt.$$

Suppose that the costate variable p is of class C^1. Also we introduce the control Hamiltonian

$$H(t, x(t, \epsilon), u(t, \epsilon), p(t)) = X(t, x(t, \epsilon), u(t, \epsilon)) + p_i^\alpha(t) X_\alpha^i(t, x(t, \epsilon), u(t, \epsilon))$$

corresponding to the variation. Then we rewrite

$$I(\epsilon) = \int_{\Omega_{0,t_0}} [H(t, x(t, \epsilon), u(t, \epsilon), p(t)) - p_i^\alpha(t) \frac{\partial x^i}{\partial t^\alpha}(t, \epsilon)] dt.$$

To evaluate the multiple integral

$$\int_{\Omega_{0,t_0}} p_i^\alpha(t) \frac{\partial x^i}{\partial t^\alpha}(t,\epsilon) dt,$$

we integrate by parts, via the divergence formula

$$\frac{\partial}{\partial t^\alpha}(p_i^\alpha x^i) = \frac{\partial p_i^\alpha}{\partial t^\alpha} x^i + p_i^\alpha \frac{\partial x^i}{\partial t^\alpha},$$

obtaining

$$\int_{\Omega_{0,t_0}} p_i^\alpha(t) \frac{\partial x^i}{\partial t^\alpha}(t,\epsilon) dt = \int_{\Omega_{0,t_0}} \frac{\partial}{\partial t^\alpha}(p_i^\alpha(t) x^i(t,\epsilon)) dt - \int_{\Omega_{0,t_0}} \frac{\partial p_i^\alpha}{\partial t^\alpha}(t) x^i(t,\epsilon) dt.$$

Now we apply the divergence integral formula

$$\int_{\Omega_{0,t_0}} \frac{\partial}{\partial t^\alpha}(p_i^\alpha(t) x^i(t,\epsilon)) dt = \int_{\partial\Omega_{0,t_0}} \delta_{\alpha\beta} p_i^\alpha(t) x^i(t,\epsilon) n^\beta(t) d\sigma,$$

where $(n^\beta(t))$ is the unit normal vector to the boundary $\partial\Omega_{0,t_0}$. Substituting, we find

$$I(\epsilon) = \int_{\Omega_{0,t_0}} [H(t, x(t,\epsilon), u(t,\epsilon), p(t)) + \frac{\partial p_j^\alpha}{\partial t^\alpha}(t) x^j(t,\epsilon)] dt$$

$$- \int_{\partial\Omega_{0,t_0}} \delta_{\alpha\beta} p_i^\alpha(t) x^i(t,\epsilon) n^\beta(t) d\sigma.$$

Differentiating with respect to ϵ, it follows

$$I'(\epsilon) = \int_{\Omega_{0,t_0}} [H_{x^j}(t, x(t,\epsilon), u(t,\epsilon), p(t)) + \frac{\partial p_j^\alpha}{\partial t^\alpha}(t)] x_\epsilon^j(t,\epsilon) dt$$

$$+ \int_{\Omega_{0,t_0}} H_{u^a}(t, x(t,\epsilon), u(t,\epsilon), p(t)) h^a(t) dt - \int_{\partial\Omega_{0,t_0}} \delta_{\alpha\beta} p_i^\alpha(t) x_\epsilon^i(t,\epsilon) n^\beta(t) d\sigma.$$

Evaluating at $\epsilon = 0$, we find

$$I'(0) = \int_{\Omega_{0,t_0}} [H_{x^j}(t, x(t), \hat{u}(t), p(t)) + \frac{\partial p_j^\alpha}{\partial t^\alpha}(t)] x_\epsilon^j(t,0) dt$$

$$+ \int_{\Omega_{0,t_0}} H_{u^a}(t, x(t), \hat{u}(t), p(t)) h^a(t) dt - \int_{\partial\Omega_{0,t_0}} \delta_{\alpha\beta} p_i^\alpha(t) x_\epsilon^i(t,0) n^\beta(t) d\sigma.$$

where $x(t)$ is the m-sheet of the state variable corresponding to the optimal control $\hat{u}(t)$.

We need $I'(0) = 0$ for all $h(t) = (h^a(t))$. On the other hand, the functions $x_\epsilon^i(t,0)$ are the components of the solution of the Cauchy problem

$$\nabla_t x_\epsilon^i(t,0) = X_x(t, x(t,0), u(t)) \cdot x_\epsilon(t,0) + X_u(t, x(t,0), u(t)) \cdot h(t),$$

$$t \in \Omega_{0,t_0}, \ x_\epsilon(0,0) = 0$$

and hence they depend on $h(t)$. To overpass this difficulty, we define $\mathcal{P}$ as the set of solutions of the boundary value problem

$$\frac{\partial p_j^\alpha}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^j}(t, x(t), \hat{u}(t), p(t)), \ \forall t \in \Omega_{0,t_0}, \tag{4}$$

$$\delta_{\alpha\beta} p_j^\alpha(t) n^\beta(t)|_{\partial\Omega_{0,t_0}} = 0, \ \text{(orthogonality or tangency)}.$$

Therefore

$$H_{u^a}(t, x(t), \hat{u}(t), p(t)) = 0, \ \forall t \in \Omega_{0,t_0}. \tag{5}$$

Moreover

$$\frac{\partial x^j}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p_j^\alpha}(t, x(t), \hat{u}(t), p(t)), \ \forall t \in \Omega_{0,t_0}, \ x(0) = x_0. \tag{6}$$

Remarks. (i) The algebraic system (5) describes the critical points of the Hamiltonian with respect to the control variable. (ii) The *PDEs* (4) and (6) and the condition (5) are Euler-Lagrange *PDEs* associated to the new Lagrangian.

Summarizing, we obtain a *multitime maximum principle* similar to the *single-time Pontryaguin maximum principle*.

Theorem 1. (Simplified multitime maximum principle; necessary conditions) *Suppose that the problem of maximizing the functional (1) subject to the PDE constraints (2) and to the conditions (3), with X, X_α^i of class C^1, has an interior solution $\hat{u}(t) \in \mathcal{U}$ which determines the m-sheet of state variable $x(t)$. Then there exists a C^1 costate $p(t) = (p_i^\alpha(t))$ defined over Ω_{0,t_0} such that the relations (4), (5), (6) hold.*

Theorem 2. (Sufficient conditions) *Consider the problem of maximizing the functional (1) subject to the PDE constraints (2) and to the conditions (3), with X, X_α^i of class C^1. Suppose that an interior solution $\hat{u}(t) \in \mathcal{U}$ and the corresponding m-sheet of state variable $x(t)$ satisfy the relations (4), (5), (6). If for the resulting costate variable $p(t) = (p_i^\alpha(t))$ the control Hamiltonian $H(t, x, u, p)$ is jointly concave in (x, u) for all $t \in \Omega_{0,t_0}$, then $\hat{u}(t)$ and the corresponding $x(t)$ achieve the unique global maximum of (1).*

Proof. Let us have in mind that we must maximize the functional (1) subject to the evolution system (2) and the conditions (3). We fix a pair $(\hat{x}, \hat{u})$, where $\hat{u}$ is a candidate optimal m-sheet of the controls and $\hat{x}$ is a candidate optimal m-sheet of the states. Calling $\hat{I}$ the values of the functional for $(\hat{x}, \hat{u})$, let us prove that

$$\hat{I} - I = \int_{\Omega_{0,t_0}} (\hat{X} - X) dt \geq 0,$$

where the strict inequality holds under strict concavity. Denoting $\hat{H} = H(\hat{x}, \hat{p}, \hat{u})$ and $H = H(x, \hat{p}, u)$, we find

$$\hat{I} - I = \int_{\Omega_{0,t_0}} \left((\hat{H} - \hat{p}_i^\alpha \frac{\partial \hat{x}^i}{\partial t^\alpha}) - (H - \hat{p}_i^\alpha \frac{\partial x^i}{\partial t^\alpha}) \right) dt.$$

Integrating by parts, we obtain

$$\hat{I} - I = \int_{\Omega_{0,t_0}} \left((\hat{H} + \hat{x}^i \frac{\partial \hat{p}_i^\alpha}{\partial t^\alpha}) - (H + x^i \frac{\partial \hat{p}_i^\alpha}{\partial t^\alpha}) \right) dt$$

$$+ \int_{\partial \Omega_{0,t_0}} (\delta_{\alpha\beta} \hat{p}_i^\alpha(t) x^i(t) n^\beta(t) - \delta_{\alpha\beta} \hat{p}_i^\alpha(t) \hat{x}^i(t) n^\beta(t)) d\sigma.$$

Taking into account that any admissible m-sheet has the same initial and terminal conditions as the optimal m-sheet, we derive

$$\hat{I} - I = \int_{\Omega_{0,t_0}} \left((\hat{H} - H) + \frac{\partial \hat{p}_i^\alpha}{\partial t^\alpha} (\hat{x}^i - x^i) \right) dt.$$

The definition of concavity and the maximum principle imply

$$\int_{\Omega_{0,t_0}} \left((\hat{H} - H) + \frac{\partial \hat{p}_i^\alpha}{\partial t^\alpha} (\hat{x}^i - x^i) \right) dt$$

$$\geq \int_{\Omega_{0,t_0}} \left((\hat{x}^i - x^i)(\frac{\partial \hat{H}}{\partial x^i} + \frac{\partial \hat{p}_i^\alpha}{\partial t^\alpha}) + (\hat{u}^a - u^a)\frac{\partial \hat{H}}{\partial u^a} \right) dt = 0.$$

This last equality is true since all "^" functions satisfy the multitime maximum principle. In this way, $\hat{I} - I \geq 0$.

Theorem 3. (Sufficient conditions) *Consider the problem of maximizing the functional (1) subject to the PDE constraints (2) and to the conditions (3), with X, X_α^i of class C^1. Suppose that an interior solution $\hat{u}(t) \in \mathcal{U}$ and the corresponding m-sheet of state variable $x(t)$ satisfy the relations (4), (5), (6). Giving the resulting costate variable $p(t) = (p_i^\alpha(t))$, we define $M(t,x,p) = H(t,x,\hat{u}(t),p)$. If $M(t,x,p)$ is concave in x for all $t \in \Omega_{0,t_0}$, then $\hat{u}(t)$ and the corresponding $x(t)$ achieve the unique global maximum of (1).*

Remark. The Theorems 2 and 3 can be extended immediately to *incave functionals*.

Examples. 1) We consider the problem

$$\max_{u(\cdot),x_1} I(u(\cdot)) = - \int_{\Omega_{0,1}} (x(t) + u_1(t)^2 + u_2(t)^2) dt^1 dt^2$$

subject to

$$\frac{\partial x}{\partial t^\alpha}(t) = u_\alpha(t), \ \alpha = 1,2, \ x(0,0) = 0, x(1,1) = x_1 = \text{free}.$$

This problem means to find an optimal control $u = (u_1, u_2)$ to bring the (PDE) dynamical system from the origin $x(0,0) = 0$ at two-time $t^1 = 0, t^2 = 0$ to a terminal point $x(1,1) = x_1$, which is unspecified, at two-time $t^1 = 1, t^2 = 1$, such as to maximize the objective functional. Also the complete integrability condition imposes $\frac{\partial u_1}{\partial t^2} = \frac{\partial u_2}{\partial t^1}$. The control Hamiltonian is

$$H(x(t), u(t), p(t)) = -(x(t) + u_1(t)^2 + u_2(t)^2) + p^1(t)u_1(t) + p^2(t)u_2(t).$$

Since
$$\frac{\partial H}{\partial u_\alpha} = -2u_\alpha + p^\alpha, \ \frac{\partial^2 H}{\partial u_\alpha^2} = -2 < 0, \ \frac{\partial^2 H}{\partial u_\alpha \partial u_\beta} = 0,$$

the critical point $p^\alpha = 2u_\alpha$ is a maximum point. Then the PDE $\dfrac{\partial p^\alpha}{\partial t^\alpha} = -\dfrac{\partial H}{\partial x}$ reduces to $\dfrac{\partial p^1}{\partial t^1} + \dfrac{\partial p^2}{\partial t^2} = 1$. Also, since the point $x(1,1) = x_1$ is unspecified, the transversality conditions imply $p^1(t)n^1(t) + p^2(t)n^2(t)|_{\partial\Omega_{0,1}} = 0$.

We continue by solving the boundary value problem

$$\frac{\partial p^1}{\partial t^1} + \frac{\partial p^2}{\partial t^2} = 1, \ \frac{\partial p^1}{\partial t^2} = \frac{\partial p^2}{\partial t^1}, \ p^1(t)n^1(t) + p^2(t)n^2(t)|_{\partial\Omega_{0,1}} = 0.$$

Consequently the components of the optimal control $u(t) = (u_1(t), u_2(t))$ are harmonic functions satisfying the boundary conditions $u_1(0, t^2) = u_1(1, t^2) = 0$, $u_2(t^1, 0) = u_2(t^1, 1) = 0$. Also the dynamical system $dx = u_1(t)dt^1 + u_2(t)dt^2$ gives $x(t) - x(0) = \displaystyle\int_{\Gamma_{0,t}} u_1(s)ds^1 + u_2(s)ds^2$.

2) We consider the problem

$$\max_{u(\cdot), x_1} I(u(\cdot)) = -\frac{1}{2}x(1,1)^2 - \frac{1}{2}\int_{\Omega_{0,1}} (u_1(t)^2 + u_2(t)^2)dt^1 dt^2$$

subject to

$$\frac{\partial x}{\partial t^\alpha}(t) = -u_\alpha(t), \ \alpha = 1, 2, \ x(0,0) = 1.$$

This problem means to find an optimal control $u = (u_1, u_2)$ to bring the (PDE) dynamical system from the point $x(0,0) = 1$ at two-time $t^1 = 0, t^2 = 0$ to a terminal point $x(1,1) = x_1$, at two-time $t^1 = 1, t^2 = 1$, such as to maximize the objective functional. Also the complete integrability condition imposes $\dfrac{\partial u_1}{\partial t^2} = \dfrac{\partial u_2}{\partial t^1}$. The control Hamiltonian is

$$H(x(t), u(t), p(t)) = -\frac{1}{2}(u_1(t)^2 + u_2(t)^2) - p^\alpha(t)u_\alpha(t).$$

Since
$$\frac{\partial H}{\partial u_\alpha} = -u_\alpha - p^\alpha, \ \frac{\partial^2 H}{\partial u_\alpha^2} = -1 < 0, \ \frac{\partial^2 H}{\partial u_\alpha \partial u_\beta} = 0,$$

the critical point $p^\alpha = -u_\alpha$ is a maximum point. Then the PDE $\dfrac{\partial p^\alpha}{\partial t^\alpha} = -\dfrac{\partial H}{\partial x} = 0$ reduces to $\dfrac{\partial p^1}{\partial t^1} + \dfrac{\partial p^2}{\partial t^2} = 0$. The transversality condition implies

$$p^1(t)n^1(t) + p^2(t)n^2(t)|_{\partial\Omega_{0,1}} = 0.$$

We continue by solving the Dirichlet problem

$$\frac{\partial p^1}{\partial t^1} + \frac{\partial p^2}{\partial t^2} = 0, \ \frac{\partial p^1}{\partial t^2} = \frac{\partial p^2}{\partial t^1}, \ p^1(t)n^1(t) + p^2(t)n^2(t)|_{\partial\Omega_{0,1}} = 0.$$

Consequently the components of the optimal control $u(t) = (u_1(t), u_2(t))$ are harmonic functions satisfying suitable boundary conditions. Also the dynamical system $dx = -u_1(t)dt^1 - u_2(t)dt^2$ gives the corresponding evolution

$$x(t) - x(0) = -\int_{\Gamma_{0,t}} u_1(s)ds^1 + u_2(s)ds^2.$$

3) **Transport** *PDEs* **for Air Traffic Flow** (see also [4]). The flow of aircraft can be analyzed and controlled using an Eulerian viewpoint of the airspace. In our formulation we use two parameters of evolution: s=position, t=time. The variable of the state is the density of aircraft $\rho(s,t)$, which represents the number of aircraft per unit length of jetway. The control variable is the speed $v(s,t)$ which the air traffic controller can prescribe to the aircraft located at position s and time t.

Given a speed field $v(s,t)$, the density of aircraft $\rho(s,t)$ satisfies the continuity *PDE* $\dfrac{\partial \rho}{\partial t}(s,t) + \dfrac{\partial(\rho v)}{\partial s}(s,t) = 0$. We would like to determine the speed field which maximizes the number of aircraft landing at the destination airport under the constraint that the density does not exceed the safety density ρ_{max}. Mathematically,

$$\max_{v(\cdot)} I(v(\cdot)) = \int_{\Omega_{0,A}} \rho(s,t)v(s,t)dsdt$$

subject to

$$\frac{\partial \rho}{\partial t}(s,t) + \frac{\partial(\rho v)}{\partial s}(s,t) = 0, \ \rho \le \rho_{max}, \ v_{min} \le v \le v_{max},$$

where $A = (L,T)$, L = value of s for final destination, T = final time, $I(v(\cdot))$= total number of aircraft landing, v_{min}, v_{max} = bounds of authorized aircraft speed. The new Lagrangian

$$L = \rho(s,t)v(s,t) + p(s,t)\left(\frac{\partial \rho}{\partial t}(s,t) + \frac{\partial(\rho v)}{\partial s}(s,t)\right) + \mu^1(s,t)(\rho_{max} - \rho(s,t))$$

$$+\mu^2(s,t)(v(s,t) - v_{min}) + \mu^3(s,t)(v_{max} - v(s,t))$$

produces the Hamiltonian

$$H = \left(-\rho(s,t) - \mu^2(s,t) + \mu^3(s,t)\right)v(s,t) - \mu^1(s,t)(\rho_{max} - \rho(s,t))$$

$$+\mu^2(s,t)v_{min} - \mu^3(s,t)v_{max},$$

of degree one with respect to the control v. The sign of the switching function $\sigma = -\rho(s,t) - \mu^2(s,t) + \mu^3(s,t)$ decides an optimal control. The adjoint *PDEs* (obtained from $I'(0) = 0$) are

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial s} = v - \mu^1, \ \rho\frac{\partial p}{\partial s} = \rho + \mu^2 - \mu^3.$$

Open problem: Study optimal control problems subject to

$$PDE: \ X_\beta^\alpha(x(t), u(t))\frac{\partial f^\beta}{\partial t^\alpha}(x(t), u(t)) = 0.$$

Bibliographical note. For strong contributions to optimal control problems, which have influenced our point of view, see [1]-[4], [6]-[11].

3 m-Flow type constrained optimization problem with curvilinear integral cost functional

The cost functionals of mechanical work type are very important for applications (see our papers [12]-[22]). This is the reason to analyze a multitime optimal control problem based on a path independent curvilinear integral as cost functional and on PDE constraints of m-flow type :

$$\max_{u(\cdot), x_{t_0}} J(u(\cdot)) = \int_{\Gamma_{0,t_0}} X^0_\alpha(t, x(t), u(t)) dt^\alpha \tag{7}$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t), u(t)), i = 1, ..., n; \alpha = 1, ..., m, \tag{8}$$

$$u(t) \in \mathcal{U}, \ t \in \Omega_{0,t_0}; x(0) = x_0, x(t_0) = x_{t_0}. \tag{9}$$

Ingredients: $t = (t^\alpha) \in R^m_+$ is the *multitime* (multi-parameter of evolution); Γ_{0,t_0} is an arbitrary C^1 curve joining the diagonal opposite points $0 = (0, ..., 0)$ and $t_0 = (t^1_0, ..., t^m_0)$ in Ω_{0,t_0}; $x : \Omega_{0,t_0} \to R^n$, $x(t) = (x^i(t))$ is a C^2 *state vector*; $u : \Omega_{0,t_0} \to U \subset R^k$, $u(t) = (u^a(t))$, $a = 1, ..., k$ is a C^1 *control vector*; the *running cost* $X^0_\alpha(t, x(t), u(t)) dt^\alpha$ is a nonautonomous closed (completely integrable) *Lagrangian 1-form* , i.e., it satisfies $D_\beta X^0_\alpha = D_\alpha X^0_\beta$ (D_α is the total derivative operator) or

$$\left(\frac{\partial X^0_\alpha}{\partial u^a} \delta^\gamma_\beta - \frac{\partial X^0_\beta}{\partial u^a} \delta^\gamma_\alpha \right) \frac{\partial u^a}{\partial t^\gamma} = X^i_\alpha \frac{\partial X^0_\beta}{\partial x^i} - X^i_\beta \frac{\partial X^0_\alpha}{\partial x^i} + \frac{\partial X^0_\beta}{\partial t^\alpha} - \frac{\partial X^0_\alpha}{\partial t^\beta};$$

the vector fields $X_\alpha(t, x(t), u(t)) = (X^i_\alpha(t, x(t), u(t)))$ are of class C^1 and satisfy the complete integrability conditions (m-flow type problem) , i.e., $D_\beta X_\alpha = D_\alpha X_\beta$ or

$$\left(\frac{\partial X_\alpha}{\partial u^a} \delta^\gamma_\beta - \frac{\partial X_\beta}{\partial u^a} \delta^\gamma_\alpha \right) \frac{\partial u^a}{\partial t^\gamma} = [X_\alpha, X_\beta] + \frac{\partial X_\beta}{\partial t^\alpha} - \frac{\partial X_\alpha}{\partial t^\beta},$$

where $[X_\alpha, X_\beta]$ means the *bracket* of vector fields. This hypothesis selects the set of all admissible controls (satisfying the complete integrability conditions)

$$\mathcal{U} = \left\{ u : R^m_+ \to U \,\middle|\, D_\beta X^0_\alpha = D_\alpha X^0_\beta, \ D_\beta X_\alpha = D_\alpha X_\beta \right\}$$

and the set of admissible states.

We introduce a *costate variable* or *Lagrange multiplier function* $p = (p_i)$ such that the new Lagrange 1-form

$$L_\alpha(t, x(t), u(t), p(t)) = X^0_\alpha(t, x(t), u(t)) + p_i(t)[X^i_\alpha(t, x(t), u(t)) - \frac{\partial x^i}{\partial t^\alpha}(t)]$$

be closed. The PDE constrained optimization problem (7)-(9) is replaced by another optimization problem

$$\max_{u(\cdot),x_{t_0}} \int_{\Gamma_{0,t_0}} L_\alpha(t,x(t),u(t),p(t))dt^\alpha$$

subject to

$$u(t) \in \mathcal{U}, \ p(t) \in \mathcal{P}, \ t \in \Omega_{0,t_0}, \ x(0)=x_0, x(t_0)=x_{t_0},$$

where the set $\mathcal{P}$ will be defined later. If we use the *control Hamiltonian 1-form*

$$H_\alpha(t,x(t),u(t),p(t)) = X_\alpha^0(t,x(t),u(t)) + p_i(t)X_\alpha^i(t,x(t),u(t)),$$

$$H_\alpha = L_\alpha + p_i \frac{\partial x^i}{\partial t^\alpha} \text{ (nonstandard duality)},$$

we can rewrite

$$\max_{u(\cdot),x_{t_0}} \int_{\Gamma_{0,t_0}} [H_\alpha(t,x(t),u(t),p(t)) - p_i(t)\frac{\partial x^i}{\partial t^\alpha}(t)]dt^\alpha$$

subject to

$$u(t) \in \mathcal{U}, \ p(t) \in \mathcal{P}, \ t \in \Omega_{0,t_0}, \ x(0)=x_0, x(t_0)=x_{t_0}.$$

Suppose that there exists a continuous control $\hat{u}(t)$ defined over Ω_{0,t_0} with $\hat{u}(t) \in \text{Int}\,\mathcal{U}$ which is optimum in the previous problem. Now consider a variation $u(t,\epsilon) = \hat{u}(t)+\epsilon h(t)$, where h is an arbitrary continuous vector function. Since $\hat{u}(t) \in \text{Int}\,\mathcal{U}$ and a continuous function over a compact set Ω_{0,t_0} is bounded, there exists $\epsilon_h > 0$ such that $u(t,\epsilon) = \hat{u}(t) + \epsilon h(t) \in \text{Int}\,\mathcal{U}, \ \forall |\epsilon| < \epsilon_h$. This ϵ is used in the next variational arguments.

Let us consider an arbitrary vector function $h(t)$ and define $x(t,\epsilon)$ as the m-sheet of the state variable corresponding to the control variable $u(t,\epsilon)$, i.e.,

$$\frac{\partial x^i}{\partial t^\alpha}(t,\epsilon) = X_\alpha^i(t,x(t,\epsilon),u(t,\epsilon)), \forall t \in \Omega_{0,t_0}, \ x(0,\epsilon) = x_0.$$

For $|\epsilon| < \epsilon_h$, we define the function

$$J(\epsilon) = \int_{\Gamma_{0,t_0}} X_\alpha^0(t,x(t,\epsilon),u(t,\epsilon))dt^\alpha.$$

Since the control function $u(t,\epsilon)$ is admissible, it follows that the evolution function $x(t,\epsilon)$ is admissible. On the other hand, the control $\hat{u}(t)$ is supposed to be optimal. Therefore $J(\epsilon) \leq J(0), \ \forall |\epsilon| < \epsilon_h$.

For any continuous function $p = (p_i) : \Omega_{0,t_0} \to R^n$, we have

$$\int_{\Gamma_{0,t_0}} p_i(t)[X_\alpha^i(t,x(t,\epsilon),u(t,\epsilon)) - \frac{\partial x^i}{\partial t^\alpha}(t,\epsilon)]dt^\alpha = 0.$$

The variations determine the closed Lagrange 1-form

$$L_\alpha(t, x(t, \epsilon), u(t, \epsilon), p(t)) = X_\alpha^0(t, x(t, \epsilon), u(t, \epsilon)) + p_i(t)[X_\alpha^i(t, x(t, \epsilon), u(t, \epsilon)) - \frac{\partial x^i}{\partial t^\alpha}(t, \epsilon)]$$

and the function

$$J(\epsilon) = \int_{\Gamma_{0,t_0}} L_\alpha(t, x(t, \epsilon), u(t, \epsilon), p(t)) dt^\alpha.$$

Suppose that the costate p is of class C^1. Also we introduce the control Hamiltonian 1-form

$$H_\alpha(t, x(t, \epsilon), u(t, \epsilon), p(t)) = X_\alpha^0(t, x(t, \epsilon), u(t, \epsilon)) + p_i(t) X_\alpha^i(t, x(t, \epsilon), u(t, \epsilon)).$$

Then we rewrite

$$J(\epsilon) = \int_{\Gamma_{0,t_0}} [H_\alpha(t, x(t, \epsilon), u(t, \epsilon), p(t)) - p_i(t) \frac{\partial x^i}{\partial t^\alpha}(t, \epsilon)] dt^\alpha.$$

To evaluate the curvilinear integral

$$\int_{\Gamma_{0,t_0}} p_i(t) \frac{\partial x^i}{\partial t^\alpha}(t, \epsilon) dt^\alpha,$$

we integrate by parts, via

$$\frac{\partial}{\partial t^\alpha}(p_i x^i) = \frac{\partial p_i}{\partial t^\alpha} x^i + p_i \frac{\partial x^i}{\partial t^\alpha},$$

obtaining

$$\int_{\Gamma_{0,t_0}} p_i(t) \frac{\partial x^i}{\partial t^\alpha}(t, \epsilon) dt^\alpha = (p_i(t) x^i(t, \epsilon))|_0^{t_0} - \int_{\Gamma_{0,t_0}} \frac{\partial p_i}{\partial t^\alpha}(t) x^i(t, \epsilon) dt^\alpha.$$

Substituting, we get the function

$$J(\epsilon) = \int_{\Gamma_{0,t_0}} [H_\alpha(t, x(t, \epsilon), u(t, \epsilon), p(t)) + \frac{\partial p_j}{\partial t^\alpha}(t) x^j(t, \epsilon)] dt^\alpha - p_i(t_0) x^i(t_0, \epsilon) + p_i(0) x^i(0, \epsilon).$$

It follows

$$J'(\epsilon) = \int_{\Gamma_{0,t_0}} [H_{\alpha x^j}(t, x(t, \epsilon), u(t, \epsilon), p(t)) + \frac{\partial p_j}{\partial t^\alpha}(t)] x_\epsilon^j(t, \epsilon) dt^\alpha$$

$$+ \int_{\Gamma_{0,t_0}} H_{\alpha u^a}(t, x(t, \epsilon), u(t, \epsilon), p(t)) h^a(t) dt^\alpha - p_i(t_0) x_\epsilon^i(t_0, \epsilon) + p_i(0) x_\epsilon^i(0, \epsilon).$$

Evaluating at $\epsilon = 0$, we find

$$J'(0) = \int_{\Gamma_{0,t_0}} [H_{\alpha x^j}(t, x(t), \hat{u}(t), p(t)) + \frac{\partial p_j}{\partial t^\alpha}(t)] x_\epsilon^j(t, 0) dt^\alpha$$

$$+ \int_{\Gamma_{0,t_0}} H_{\alpha u^a}(t, x(t), \hat{u}(t), p(t)) h^a(t) dt^\alpha - p_i(t_0) x_\epsilon^i(t_0, 0),$$

where $x(t)$ is the m-sheet of the state variable corresponding to the optimal control $\hat{u}(t)$. We need $J'(0) = 0$ for all $h(t) = (h^a(t))$. This is possible if we define $\mathcal{P}$ as the set of solutions of the terminal value problem

$$\frac{\partial p_j}{\partial t^\alpha}(t) = -\frac{\partial H_\alpha}{\partial x^j}(t, x(t), \hat{u}(t), p(t)), \ \forall t \in \Omega_{0,t_0}; \ p_j(t_0) = 0. \tag{10}$$

Therefore

$$H_{\alpha u^a}(t, x(t), \hat{u}(t), p(t)) = 0, \ \forall t \in \Omega_{0,t_0}. \tag{11}$$

Moreover

$$\frac{\partial x^j}{\partial t^\alpha}(t) = \frac{\partial H_\alpha}{\partial p_j}(t, x(t), \hat{u}(t), p(t)), \ \forall t \in \Omega_{0,t_0}; \ x(0) = x_0. \tag{12}$$

Remarks. (i) The algebraic system (11) describes the common critical points of the functions H_α with respect to the control variable u. (ii) The *PDEs* (10) and (12) and the relation (11) are Euler-Lagrange *PDEs* associated to the new Lagrangian 1-form.

Summarizing, we obtain a new variant of *multitime maximum principle*.

Theorem 4. (Simplified multitime maximum principle; necessary conditions) *Suppose that the problem of maximizing the functional (7) subject to the PDE constraints (8) and to the conditions (9), with X_α^0, X_α^i of class C^1, has an interior solution $\hat{u}(t) \in \mathcal{U}$ which determines the m-sheet of state variable $x(t)$. Then there exists a C^1 costate $p(t) = (p_i(t))$ defined over Ω_{0,t_0} such that the relations (10), (11), (12) hold.*

Theorem 5. (Sufficient conditions) *Consider the problem of maximizing the functional (7) subject to the PDE constraints (8) and to the conditions (9), with X_α^0, X_α^i of class C^1. Suppose that an interior solution $\hat{u}(t) \in \mathcal{U}$ and the corresponding m-sheet of state variable $x(t)$ satisfy the relations (10), (11), (12). If, for the resulting costate variable $p(t) = (p_i(t))$, the control Hamiltonian 1-form $H_\alpha(t, x, u, p)$ is jointly concave in (x, u) for all $t \in \Omega_{0,t_0}$, then $\hat{u}(t)$ and the corresponding $x(t)$ achieve the unique global maximum of (7).*

Theorem 6. (Sufficient conditions) *Consider the problem of maximizing the functional (7) subject to the PDE constraints (8) and to the conditions (9), with X_α^0, X_α^i of class C^1. Suppose that an interior solution $\hat{u}(t) \in \mathcal{U}$ and the corresponding m-sheet of state variable $x(t)$ satisfy the relations (10), (11), (12). Giving the resulting costate variable $p(t) = (p_i(t))$, we define the 1-form $M_\alpha(t, x, p) = H_\alpha(t, x, \hat{u}(t), p)$. If the 1-form $M_\alpha(t, x, p)$ is concave in x for all $t \in \Omega_{0,t_0}$, then $\hat{u}(t)$ and the corresponding $x(t)$ achieve the unique global maximum of (7).*

Remark. The Theorems 5 and 6 can be extended immediately to *incave functionals*.

Example. Let $t = (t^1, t^2) \in \Omega_{0,1}$, where $0 = (0, 0), 1 = (1, 1)$ are diagonal opposite points in $\Omega_{0,1}$. Denote by $\Gamma_{0,1}$ an arbitrary C^1 curve joining the points 0 and 1. We consider the problem

$$\max_{u(\cdot),x_1} J(u(\cdot)) = -\int_{\Gamma_{0,1}} (x(t) + u_\beta(t)^2)dt^\beta$$

subject to

$$\frac{\partial x}{\partial t^\alpha}(t) = u_\alpha(t), \ \alpha = 1,2, \ x(0,0) = 0, \ x(1,1) = x_1 = \text{free}.$$

This problem means to find an optimal control $u = (u_1, u_2)$ to bring the (PDE) dynamical system from the origin $x(0,0) = 0$ at two-time $t^1 = 0, t^2 = 0$ to a terminal point $x(1,1) = x_1$, which is unspecified, at two-time $t^1 = 1, t^2 = 1$, such as to maximize the objective functional. Also the complete integrability conditions impose

$$\frac{\partial x}{\partial t^1} + 2u_2\frac{\partial u_2}{\partial t^1} = \frac{\partial x}{\partial t^2} + 2u_1\frac{\partial u_1}{\partial t^2}, \ \frac{\partial u_1}{\partial t^2} = \frac{\partial u_2}{\partial t^1}.$$

The control Hamiltonian 1-form is

$$H_\beta(x(t), u(t), p(t)) = -(x(t) + u_\beta(t)^2) + p(t)u_\beta(t).$$

Since

$$\frac{\partial H_\beta}{\partial u_\beta} = -2u_\beta + p, \ \frac{\partial^2 H_\beta}{\partial u_\beta^2} = -2 < 0,$$

the critical point $u_1 = u_2 = \dfrac{p}{2}$ is a maximum point. The $PDEs$ $\dfrac{\partial p}{\partial t^\alpha} = -\dfrac{\partial H_\alpha}{\partial x}$ reduces to $\dfrac{\partial p}{\partial t^\alpha} = 1$. Also, since the point $x(1,1) = x_1$ is unspecified, the transversality condition implies $p(1) = 0$. It follows the costate $p(t) = t^1 + t^2 - 2$, the optimal control $\hat{u}_1(t) = \hat{u}_2(t) = \dfrac{1}{2}(t^1 + t^2 - 2)$ and the corresponding evolution $x(t) = \dfrac{(t^1)^2 + (t^2)^2}{4} + \dfrac{t^1t^2}{2} - (t^1 + t^2)$.

4 Equivalence between multiple and curvilinear integral functionals

A multitime evolution system can be used as a constraint in a problem of extremizing a *multitime cost functional*. On the other hand, the multitime cost functionals can be introduced at least in two ways:

- either using a path independent curvilinear integral,

$$P(u(\cdot)) = \int_{\Gamma_{0,t_0}} X_\beta^0(x(t), u(t))dt^\beta + g(x(t_0)),$$

where Γ_{0,t_0} is an arbitrary C^1 curve joining the points 0 and t_0, the *running cost* $\omega = X_\beta^0(x(t), u(t))dt^\beta$ is an autonomous closed (completely integrable) *Lagrangian 1-form*, and g is the *terminal cost*;

- or using a multiple integral,

$$Q(u(\cdot)) = \int_{\Omega_{0,t_0}} X(x(t), u(t))dt + g(x(t_0)),$$

where the *running cost* $X(x(t), u(t))$ is an autonomous continuous *Lagrangian*, and g is the *terminal cost*.

Let us show that the functional P is equivalent to the functional Q. This means that in a multitime optimal control problem we can choose the appropriate functional based on geometrical-physical meaning or other criteria.

Theorem 7 [23]. *The multiple integral*

$$I(t_0) = \int_{\Omega_{0,t_0}} X(x(t), u(t)) dt,$$

with X as continuous function, is equivalent to the curvilinear integral

$$J(t_0) = \int_{\Gamma_{0,t_0}} X^0_\beta(x(t), u(t)) dt^\beta,$$

where $\omega = X^0_\beta(x(t), u(t)) dt^\beta$ is a closed (completely integrable) Lagrangian 1-form and the functions X^0_β have partial derivatives of the form

$$\frac{\partial}{\partial t^\alpha}, \ \frac{\partial}{\partial t^\alpha \partial t^\beta} \ (\alpha < \beta), \ ..., \ \frac{\partial^{m-1}}{\partial t^1 ... \hat{\partial t^\alpha} ... \partial t^m},$$

where the symbol "^" posed over ∂t^α designates that ∂t^α is omitted.

5   Multitime maximum principle approach of variational calculus

It is well known that the single-time Pontryaguin's maximum principle is a generalization of the Lagrange problem in the single-time variational calculus and that these problems are equivalent when the control domain is open [1], [7], [10]. Does this property survive for *simplified multitime maximum principle*? The aim of this Section is to formulate an answer to this question.

In fact we show that the *simplified multitime maximum principle* motivates the *multitime Euler-Lagrange or Hamilton PDEs*. For that, suppose that the evolution system is reduced to a completely integrable system

$$\frac{\partial x^i}{\partial t^\alpha}(t) = u^i_\alpha(t), \ x(0) = x_0, \ t \in \Omega_{0,t_0} \subset R^m_+, \qquad\qquad (PDE)$$

and the functional is a path independent curvilinear integral

$$J(u(\cdot)) = \int_{\Gamma_{0,t_0}} X^0_\beta(x(t), u(t)) dt^\beta, \ u = (u^i_\alpha), \qquad\qquad (J)$$

where Γ_{0,t_0} is an arbitrary piecewise C^1 curve joining the points 0 and t_0, the *running cost* $\omega = X^0_\beta(x(t), u(t)) dt^\beta$ is a closed (completely integrable) *Lagrangian 1-form*.

The associated basic control problem leads necessarily to the multitime maximum principle. Therefore, to solve it we need the control Hamiltonian 1-form

$$H_\beta(x, p_0, p, u) = X^0_\beta(x, u) + p_i u^i_\beta$$

and the adjoint PDEs

$$\frac{\partial p_i}{\partial t^\beta}(t) = -\frac{\partial X_\beta^0}{\partial x^i}(x(t), u(t)). \qquad (ADJ)$$

Suppose the simplified multitime maximum principle is applicable (see the relation (10))

$$\frac{\partial H_\beta}{\partial u_\gamma^i} = \frac{\partial X_\beta^0}{\partial u_\gamma^i} + p_i\delta_\beta^\gamma = 0, p_i\delta_\beta^\gamma = -\frac{\partial X_\beta^0}{\partial u_\gamma^i}, \ u_\gamma^i = x_\gamma^i. \qquad (13)$$

Suppose the functions X_β^0 are dependent on x (a strong condition!). Then (ADJ) shows that

$$p_i(t) = p_i(0) - \int_{\Gamma_{0,t}} \frac{\partial X_\beta^0}{\partial x^i}(x(s), u(s))ds^\beta, \qquad (14)$$

where $\Gamma_{0,t}$ is an arbitrary piecewise C^1 curve joining the points $0, t \in \Omega_{0,t_0}$.

5.1 Multitime Euler-Lagrange PDEs

From the relations (13) and (14), it follows

$$-\frac{\partial X_\beta^0}{\partial x_\gamma^i}(x(t), u(t)) = \delta_\beta^\gamma p_i(0) - \delta_\beta^\gamma \int_{\Gamma_{0,t}} \frac{\partial X_\lambda^0}{\partial x^i}(x(s), u(s))ds^\lambda.$$

Suppose that X_β^0 are functions of class C^2. Applying the divergence operator $D_\gamma = \frac{\partial}{\partial t^\gamma}$ we find *the multitime Euler-Lagrange PDEs* $\frac{\partial X_\beta^0}{\partial x^i} - D_\gamma \frac{\partial X_\beta^0}{\partial x_\gamma^i} = 0$.

5.2 Conversion to multitime Hamilton PDEs (canonical variables)

Let $u(\cdot)$ be an optimal control, $x(\cdot)$ the optimal evolution m-sheet, and $p(\cdot)$ be the solution of (ADJ) which corresponds to $u(\cdot)$ and $x(\cdot)$. Suppose that the critical point condition admits a unique solution $u_\gamma^i(t) = u_\gamma^i(x(t), p(t)) = \frac{\partial x^i}{\partial t^\gamma}(t)$. Then, using a path independent curvilinear integral, we can write

$$x^i(t) = x^i(0) + \int_{\Gamma_{0,t}} u_\gamma^i(x(s), p(s))ds^\gamma.$$

The control Hamiltonian 1-form $H_\beta = X_\beta^0 + p_j u_\beta^j$ must satisfy $\frac{\partial H_\beta}{\partial u_\gamma^i} = 0$. This last relation, $p_i\delta_\beta^\gamma + \frac{\partial X_\beta^0}{\partial u_\gamma^i} = 0$, defines the costate p as a *moment*. On the other hand

$$\frac{\partial H_\beta}{\partial p_i} = \frac{\partial X_\beta^0}{\partial u_\gamma^j}\frac{\partial u_\gamma^j}{\partial p_i} + u_\beta^i + p_j\frac{\partial u_\beta^j}{\partial p_i} = u_\beta^i \text{ or } \frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H_\beta}{\partial p_i}(x(t), p(t), u(t)).$$

Now, the relation

$$-\frac{\partial H_\beta}{\partial x^i} = -\left(\frac{\partial X_\beta^0}{\partial x^i} + \frac{\partial X_\beta^0}{\partial u_\gamma^j}\frac{\partial u_\gamma^j}{\partial x^i}\right) - p_j\frac{\partial u_\beta^j}{\partial x^i}$$

and (ADJ) show

$$\frac{\partial p_i}{\partial t^\beta}(t) = -\frac{\partial H_\beta}{\partial x^i}(x(t), p(t), u(t)).$$

In this way we find the canonical variables x, p and the *multitime Hamilton PDEs*

$$\frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H_\beta}{\partial p_i}(x(t), p(t)), \ \ \frac{\partial p_i}{\partial t^\beta}(t) = -\frac{\partial H_\beta}{\partial x^i}(x(t), p(t)).$$

Remark. To make a computer aided study of PDE-constrained optimization problems we can perform symbolic computations via Maple (see also [5]).

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