Slant lightlike submanifolds of indefinite Hermitian manifolds

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Abstract. In this paper, we introduce a new class, called slant lightlike submanifolds, of an indefinite Hermitian manifold. We provide a non-trivial example and obtain necessary and sufficient conditions for the existence of a slant lightlike submanifold. As well, we give an example of minimal slant lightlike submanifolds of $\mathbf{R}_2^8$ and prove some characterization theorems.

M.S.C. 2000: 53C15, 53C40, 53C50.
Key words: Degenerate metric, slant lightlike submanifold, Hermitian manifold.

1. Introduction

A $2k$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$ of constant index q, $0 < q < 2k$, is called an indefinite almost Hermitian manifold if there exists a tensor filed $\bar{J}$ of type (1,1) on $\bar{M}$ such that $\bar{J}^2 = -I$ and

$$\bar{g}(X, Y) = \bar{g}(\bar{J}X, \bar{J}Y), \forall X, Y \in \Gamma(T\bar{M}), \tag{0.1}$$

where I denotes the identity transformation of $T_p\bar{M}$. Moreover, $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, i.e, ([1])

$$(\bar{\nabla}_X \bar{J})Y = 0, \quad \forall X, Y \in \Gamma(T\bar{M}), \tag{0.2}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. As a generalization of complex and totally real submanifolds of almost Hermitian manifolds, Chen [3] defined a slant submanifold (M, g) of an almost Hermitian manifold $(\bar{M}, \bar{g}, \bar{J})$ as a real submanifold such that the angle between $\bar{J}X$ and T_xM is constant for every vector $X \in T_xM$ and $x \in M$. In 1996, Duggal-Bejancu presented the theory of lightlike submanifolds in [5]. However, the concept of slant lightlike submanifolds has not been studied as yet.

The objective of this paper is to introduce the notion of slant lightlike submanifolds of an indefinite Hermitian manifolds. We study the existence problem and establish

Balkan Journal of Geometry and Its Applications, Vol.13, No.1, 2008, pp. 107-119.

an interplay between slant lightlike submanifolds and Cauchy Riemann (CR)-lightlike submanifolds [5, chapter 6].

Section 2 includes basic information on the lightlike geometry as needed in this paper. In section 3, we introduce the concept of slant lightlike submanifolds and give a non-trivial example. We prove a characterization theorem for the existence of slant lightlike submanifolds and show that co-isotropic CR-lightlike submanifolds are slant lightlike submanifolds. Finally, in section 4, we consider minimal slant lightlike submanifolds, give an example and present two characterization theorems.

2. Preliminaries

We follow [5] for the notation and formulas used in this paper. A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a lightlike submanifold if the metric g induced from $\bar{g}$ is degenerate and the radical distribution $Rad(TM)$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, i.e., $TM = Rad\,(TM) \perp S(TM)$.

Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [5, page 144]. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then,

$$\begin{aligned} tr(TM) &= ltr(TM) \perp S(TM^{\perp}), \\ T\bar{M}|_M &= S(TM) \perp [Rad(TM) \oplus ltr(TM)] \perp S(TM^{\perp}). \end{aligned}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/Rad\,TM$ [8]. Following result is important to this paper.

Proposition 2.1 [5]. *The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM), S(TM^{\perp})$ and $ltr(TM)$.*

Throughout this paper, we will discuss the dependence (or otherwise) of the results on induced object(s) and refer [5] for their transformation equations.

Following are four subcases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp})$.

Case 1: r - lightlike if $r < min\{m, n\}$;

Case 2: Co - isotropic if $r = n < m$; $S(TM^{\perp}) = \{0\}$;

Case 3: Isotropic if $r = m < n$; $S(TM) = \{0\}$;

Case 4: Totally lightlike if $r = m = n$; $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten formulas are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM), \tag{2.1}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \forall X \in \Gamma(TM), V \in \Gamma(tr(TM)), \tag{2.2}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(ltr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $ltr(TM)$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X,N), \tag{2.4}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X,W), \quad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$. Denote the projection of TM on $S(TM)$ by $\bar{P}$. Then, by using (2.1), (2.3)-(2.5) and taking account that $\bar{\nabla}$ is a metric connection we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.6}$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(N, A_W X). \tag{2.7}$$

We set

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{2.8}$$

$$\nabla_X \xi = -A^*_\xi X + {\nabla^*}^t_X \xi, \tag{2.9}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. By using above equations we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \tag{2.10}$$

$$\bar{g}(h^*(X, \bar{P}Y), N) = g(A_N X, \bar{P}Y), \tag{2.11}$$

$$\bar{g}(h^l(X,\xi),\xi) = 0 \quad , \quad A^*_\xi \xi = 0. \tag{2.12}$$

In general, the induced connection ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.3) we get

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y). \tag{2.13}$$

However, it is important to note that $\nabla^\star$ is a metric connection on $S(TM)$. From now on, we briefly denote $(M, g, S(TM), S(TM^\perp))$ by M in this paper.

Definition 2.1 [5] Let $\bar{M}$ be an indefinite Hermitian manifold and M be a real lightlike submanifold of $\bar{M}$. Then M is called CR-lightlike submanifold if the following conditions are fulfilled:

(A) $\bar{J}RadTM$ is a distribution on M such that

$$RadTM \cap \bar{J}RadTM = \{0\}.$$

(B) There exist vector bundles $S(TM)$, $S(TM^\perp)$, $ltr(TM)$, D_0 and D' over M such that

$$S(TM) = \{\bar{J}RadTM \oplus D'\} \perp D_o; \bar{J}(D_o) = D_o, \ \bar{J}(D') = L_1 \perp L_2$$

where D_o is non-degenerate distribution on M, L_1 and L_2 are vector subbundles of $ltr(TM)$ and $S(TM^\perp)$.

3. Slant lightlike submanifolds

We start with the following lemmas which will be useful for later.

Lemma 3.1. *Let M be an $r-$ lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2q$. Suppose that $\bar{J}RadTM$ is a distribution on M such that $RadTM \cap \bar{J}RadTM = \{0\}$. Then $\bar{J}ltr(TM)$ is subbundle of the screen distribution $S(TM)$ and $\bar{J}RadTM \cap \bar{J}ltr(TM) = \{0\}$*

Proof. Since by hypothesis $\bar{J}RadTM$ is a distribution on M such that $\bar{J}RadTM \cap RadTM = \{0\}$, we have $\bar{J}RadTM \subset S(TM)$. Now we claim that $ltr(TM)$ is not invariant with respect to $\bar{J}$. Let us suppose that $ltr(TM)$ is invariant with respect to $\bar{J}$. Choose $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltr(TM))$ such that $\bar{g}(N, \xi) = 1$. Then from (0.1) we have $1 = \bar{g}(\xi, N) = \bar{g}(\bar{J}\xi, \bar{J}N) = 0$ due to $\bar{J}\xi \in \Gamma(S(TM))$ and $\bar{J}N \in \Gamma(ltr(TM))$. This is a contradiction, so $ltr(TM)$ is not invariant with respect to $\bar{J}$. Also $\bar{J}N$ does not belong to $S(TM^{\perp})$, since $S(TM^{\perp})$ is orthogonal to $S(TM)$, $\bar{g}(\bar{J}N, \bar{J}\xi)$ must be zero, but from (0.1) we have $\bar{g}(\bar{J}N, \bar{J}\xi) = \bar{g}(N, \xi) \neq 0$ for some $\xi \in \Gamma(RadTM)$, this is again a contradiction. Thus we conclude $\bar{J}ltr(TM)$ is a distribution on M. Moreover,$\bar{J}N$ does not belong to $RadTM$. Indeed, if $\bar{J}N \in \Gamma(RadTM)$, we would have $\bar{J}^2N = -N \in \Gamma(\bar{J}RadTM)$, but this is impossible. Similarly, $\bar{J}N$ does not belong to $\bar{J}RadTM$. Thus we conclude that $\bar{J}ltr(TM) \subset S(TM)$ and $\bar{J}RadTM \cap \bar{J}ltr(TM) = \{0\}$. □

Remark 1. Lemma 3.1 shows that behavior of the lightlike transversal bundle $ltr(TM)$ is exactly same as the radical distribution $RadTM$, Thus, for this case L_1 has to be $ltr(TM)$ in the definition 2.1 of a CR-lightlike submanifold.

Lemma 3.2. *Under the hypothesis of Lemma 3.1, if $r = q$, then any complementary distribution to $\bar{J}(RadTM) \oplus \bar{J}ltr(TM)$ in $S(TM)$ is Riemannian.*

Proof. Let $dim(\bar{M}) = m + n$ and $dim(M) = m$. Lemma 3.1 implies that $\bar{J}ltr(TM) \oplus \bar{J}RadTM \subset S(TM)$. We denote the complementary distribution to $\bar{J}ltr(TM) \oplus \bar{J}RadTM$ in $S(TM)$ by D'. Then we have a local quasi orthonormal field of frames on $\bar{M}$ along M

$$\{\xi_i,\, N_i,\, \bar{J}\xi_i,\, \bar{J}N_i, X_\alpha,\, W_a\}, i \in \{1, ..., r\},\; \alpha \in \{3r+1, ..., m\},\; a \in \{r+1, ..., n\},$$

where $\{\xi_i\}$ and $\{N_i\}$ are lightlike basis of $RadTM$ and $ltr(TM)$, respectively and $\bar{J}\xi_i$, $\bar{J}N_i$, $\{X_\alpha\}$ and $\{W_a\}$ are orthonormal basis of $S(TM)$ and $S(TM^{\perp})$, respectively. From the basis $\{\xi_1, ..., \xi_r, \bar{J}\xi_1, ..., \bar{J}\xi_r, \bar{J}N_1, ..., \bar{J}N_r,\, N_1, ..., N_r\}$ of $ltr(TM) \oplus RadTM \oplus \bar{J}RadTM \oplus \bar{J}ltr(TM)$, we can construct an orthonormal basis $\{U_1, ..., U_{2r}, V_1, ..., V_{2r}\}$ as follows

$$\begin{array}{rlrl} U_1 = & \frac{1}{\sqrt{2}}(\xi_1 + N_1) & U_2 = & \frac{1}{\sqrt{2}}(\xi_1 - N_1) \\ U_3 = & \frac{1}{\sqrt{2}}(\xi_2 + N_2) & U_4 = & \frac{1}{\sqrt{2}}(\xi_2 - N_2) \\ & \cdots & & \cdots \\ & \cdots & & \cdots \\ U_{2r-1} = & \frac{1}{\sqrt{2}}(\xi_r + N_r) & U_{2r} = & \frac{1}{\sqrt{2}}(\xi_r - N_r) \\ V_1 = & \frac{1}{\sqrt{2}}(\bar{J}\xi_1 + \bar{J}N_1) & V_2 = & \frac{1}{\sqrt{2}}(\bar{J}\xi_1 - \bar{J}N_1) \\ V_3 = & \frac{1}{\sqrt{2}}(\bar{J}\xi_2 + \bar{J}N_2) & V_4 = & \frac{1}{\sqrt{2}}(\bar{J}\xi_2 - \bar{J}N_2) \\ & \cdots & & \cdots \\ & \cdots & & \cdots \\ V_{2r-1} = & \frac{1}{\sqrt{2}}(\bar{J}\xi_r + \bar{J}N_r) & V_{2r} = & \frac{1}{\sqrt{2}}(\bar{J}\xi_r - \bar{J}N_r). \end{array}$$

Hence, $Span\{\xi_i,\, N_i,\, \bar{J}\xi_i,\, \bar{J}N_i\}$ is a non-degenerate space of constant index $2r$. Thus we conclude that $RadTM \oplus \bar{J}RadTM \oplus ltr(TM) \oplus \bar{J}ltr(TM)$ is non-degenerate and of constant index $2r$ on $\bar{M}$. Since

$$\begin{aligned} index(T\bar{M}) \quad = \quad & index(RadTM \oplus ltr(TM)) + index(\bar{J}RadTM \oplus \bar{J}ltr(TM) \\ & + index(D' \perp S(TM^{\perp})) \end{aligned}$$

we have $2q = 2r + index(D' \perp S(TM^{\perp}))$. Thus, if $r = q$, then $D' \perp S(TM^{\perp})$ is Riemannian, i.e., $index(D' \perp (S(TM)^{\perp})) = 0$. Hence D' is Riemannian. □

Remark 2. As mentioned in the introduction, the purpose of this paper is to introduce the notion of slant lightlike submanifolds. To define this notion, one needs to consider angle between two vector fields. As we can see from section 2, a lightlike submanifold has two (radical and screen) distributions:. The radical distribution is totally lightlike and therefore it is not possible to define angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Although there are some definitions for angle between two vector fields in Lorentzian vector space (See: [9], Proposition 30, P:144), that is not appropriate for our goal, because a manifold with metric of Lorentz signature cannot admit an almost Hermitian structure (See: [7], Theorem VIII.3, P: 184). Thus one way to define slant notion is choose a Riemannian screen distribution on lightlike submanifold, for which we use Lemma 3.2.

Definition 3.1 Let M be a $q-$ lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2q$. Then we say that M is a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(A) $RadTM$ is a distribution on M such that

$$\bar{J}RadTM \cap RadTM = \{0\}. \tag{3.1}$$

(B) For each non-zero vector field tangent to D at $x \in \mathbf{U} \subset M$, the angle $\theta(X)$ between $\bar{J}X$ and the vector space D_x is constant, that is, it is independent of the choice of $x \in \mathbf{U} \subset M$ and $X \in D_x$, where D is complementary distribution to $\bar{J}RadTM \oplus \bar{J}ltr(TM)$ in the screen distribution $S(TM)$.

This constant angle $\theta(X)$ is called slant angle of the distribution D. A slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.
From the definition 3.1, we have the following decomposition:

$$\text{(3.2)} \qquad TM = RadTM \perp S(TM)$$
$$\text{(3.3)} \qquad = RadTM \perp (\bar{J}RadTM \oplus \bar{J}ltr(TM)) \perp D.$$

Proposition 3.1. *There exist no proper slant totally lightlike or isotropic submanifolds in indefinite Hermitian manifolds.*

Proof. We suppose that M is totally lightlike submanifold of $\bar{M}$. Then $TM = RadTM$, hence $D = \{0\}$. The other assertion follows similarly. □

Remark 3. As per Proposition 2.1, Definition 3.1 does not depend on $S(TM)$ and $S(TM^{\perp})$, but, it depends on the transformation equations (2.60) in [5, page 165], with respect to the screen second fundamental forms h^s. However, our conclusions of this paper do not change with respect to a change of h^s.

Example 1. Let $\bar{M} = (\mathbf{R}^8_2, \bar{g})$ be a semi-Riemannian manifold, where $\mathbf{R}^8_2$ is semi-Euclidean space of signature $(-,-,+,+,+,+,+,+)$ with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8\}.$$

Let M be a submanifold of $\mathbf{R}^8_2$ given by

$$X(u, v, \theta, t, s) = (u, v, sin\,\theta, cos\,\theta, -\theta\, sint, -\theta\, cost, u, s).$$

Then the tangent bundle TM is spanned by

$$\begin{aligned} Z_1 &= \partial x_1 + \partial x_7 \quad Z_2 = \partial x_2, \\ Z_3 &= cos\,\theta\, \partial x_3 - sin\,\theta\, \partial x_4 - sin\,t\, \partial x_5 - cos\,t\, \partial x_6, \\ Z_4 &= -\theta\, cos\,t\, \partial x_5 + \theta\, sin\,t\, \partial x_6, \quad Z_5 = \partial x_8. \end{aligned}$$

It follows that M is $1-$ lightlike submanifold of $\mathbf{R}^8_2$ with $RadTM = span\{Z_1\}$. Moreover we obtain $\bar{J}RadTM = span\{Z_2 + Z_5\}$ and therefore it is a distribution on M. Choose $D = span\{Z_3, Z_4\}$ which is Riemannian. Then M is a slant distribution with slant angle $\frac{\pi}{4}$, with the screen transversal bundle $S(TM^{\perp})$ spanned by

$$\begin{aligned} W_1 &= -cosec\,\theta\, \partial x_4 + sin\,t\, \partial x_5 + cos\,t\, \partial x_6 \\ W_2 &= (2sec\,\theta - cos\,\theta)\, \partial x_3 + sin\,\theta\, \partial x_4 + sin\,t\, \partial x_5 + cos\,t\, \partial x_6 \end{aligned}$$

which is also Riemannian. Finally, $ltr(TM)$ is spanned by

$$N = \frac{1}{2}(-\partial x_1 + \partial x_7).$$

Hence we have $\bar{J}N = -Z_2 + Z_5 \in \Gamma(S(TM)$ and $\bar{g}(\bar{J}N, \bar{J}Z_1) = 1$. Thus we conclude that M is a proper slant lightlike submanifold of $\mathbf{R}^8_2$.

Proposition 3.2. *Slant lightlike submanifolds do not include invariant and screen real lightlike submanifolds of an indefinite Hermitian manifold.*

Proof. Let M be a invariant or screen real lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$. Then, since $\bar{J}RadTM = RadTM$, the first condition of slant lightlike submanifold is not satisfied which proves our assertion. □

It is known that CR-lightlike submanifolds also do not include invariant and real lightlike submanifolds [5]. Thus we may expect some relations between CR-lightlike submanifold and slant lightlike submanifold. Indeed we have the following.

Proposition 3.3 . *Let M be a $q-$ lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of index $2q$. Then any coisotropic CR-lightlike submanifold is a slant lightlike submanifold with $\theta = 0$. In particular, a lightlike real hypersurface of an indefinite Hermitian manifold $\bar{M}$ of index 2 is a slant lightlike submanifold with $\theta = 0$. Moreover, any CR-lightlike submanifold of $\bar{M}$ with $D_o = \{0\}$ is a slant lightlike submanifold with $\theta = \frac{\pi}{2}$.*

Proof. Let M be a $q-$ lightlike CR-lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$. Then, by definition of CR-lightlike submanifold, $\bar{J}RadTM$ is a distribution on M such that $RadTM \cap \bar{J}RadTM = \{0\}$. If M is coisotropic, then $S(TM^{\perp}) = \{0\}$, thus $L_2 = 0$ Then the complementary distribution to $\bar{J}RadTM \oplus \bar{J}ltr(TM)$ is D_o. Lemma 3.2 implies that D_o is Riemannian. Since D_o is invariant with respect to $\bar{J}$, it follows that $\theta = 0$. Our second assertion is clear due to a lightlike real hypersurface of $\bar{M}$ is coisotropic. Now, if M is CR-lightlike submanifold with $D_o = \{0\}$, then the complementary distribution to $\bar{J}RadTM \oplus \bar{J}ltr(TM)$ is D'. Since D' is anti-invariant with respect to $\bar{J}$, it follows that $\theta = \frac{\pi}{2}$. Thus proof is complete. □

From Proposition 3.3, coisotropic CR-lightlike submanifolds, lightlike real hypersurfaces and CR-lightlike submanifolds with $D_o = \{0\}$ are some of the many more examples of slant lightlike submanifolds.

For any $X \in \Gamma(TM)$ we write

$$\bar{J}X = TX + FX \tag{3.4}$$

where TX is the tangential component of $\bar{J}X$ and FX is the transversal component of $\bar{J}X$. Similarly, for $V \in \Gamma(tr(TM))$ we write

$$\bar{J}V = BV + CV \tag{3.5}$$

where BV is the tangential component of $\bar{J}V$ and CV is the transversal component of $\bar{J}V$. Given a slant lightlike submanifold, we denote by P_1, P_2, Q_1 and Q_2 the projections on the distributions $RadTM$, $\bar{J}RadTM$, $\bar{J}ltr(TM)$ and D, respectively. Then we can write

$$X = P_1X + P_2X + Q_1X + Q_2X \tag{3.6}$$

for $X \in \Gamma(TM)$. By applying $\bar{J}$ to (3.6) we obtain

$$\bar{J}X = \bar{J}P_1X + \bar{J}P_2X + TQ_2X + FQ_1X + FQ_2X \tag{3.7}$$

for $X \in \Gamma(TM)$. By direct calculations we have

$$\bar{J}P_1X \in \Gamma(\bar{J}RadTM) \quad , \quad \bar{J}P_2X = TP_2X \in \Gamma(RadTM) \tag{3.8}$$

$$FP_1X = 0, FP_2X = 0 \quad , \quad TQ_2X \in \Gamma(D), FQ_1X \in \Gamma(ltr(TM)). \tag{3.9}$$

Moreover, (3.7), (3.8) and (3.9) imply

$$TX = TP_1X + TP_2X + TQ_2X. \tag{3.10}$$

Now we prove two characterization theorems for slant lightlike submanifolds.

Theorem 3.1. *Let M be a $q-$ lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2q$. Then M is slant lightlike submanifold if and only if the following conditions are satisfied:*

(1) *$\bar{J}ltr(TM)$ is a distribution on$M$.*

(2) *There exists a constant $\lambda \in [-1, 0]$ such that*

$$(Q_2T)^2X = \lambda X, \quad \forall X \in \Gamma(TM). \tag{3.11}$$

Moreover, in such case, $\lambda = -cos^2\theta$.

Proof. Let M be a $q-$ lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2q$. If M is slant lightlike submanifold of $\bar{M}$, then $\bar{J}RadTM$ is a distribution on $S(TM)$, thus from Lemma 3.1, it follows that $\bar{J}ltr(TM)$ is also a distribution on M and $\bar{J}ltr(TM) \subset S(TM)$. Thus (1) is satisfied. Moreover, the angle between $\bar{J}Q_2X$ and D_x is constant. Hence we have

$$\begin{aligned} cos\theta(Q_2X) &= \frac{\bar{g}(\bar{J}Q_2X, TQ_2X)}{\mid \bar{J}Q_2X \mid\mid TQ_2X \mid} = \frac{-\bar{g}(Q_2X, \bar{J}TQ_2X)}{\mid Q_2X \mid\mid TQ_2X \mid} \\ &= \frac{-\bar{g}(Q_2X, TQ_2TQ_2X)}{\mid Q_2X \mid\mid TQ_2X \mid}. \end{aligned} \tag{3.12}$$

On the other hand, we have

$$cos\theta(Q_2X) = \frac{\mid TQ_2X \mid}{\mid \bar{J}Q_2X \mid}. \tag{3.13}$$

Thus from (3.12) and (3.13) we get

$$cos^2\theta(Q_2X) = \frac{-\bar{g}(Q_2X, TQ_2TQ_2X)}{\mid Q_2X \mid^2}.$$

Since $\theta(Q_2X)$ is constant on D, we conclude that

$$(Q_2T)^2X = \lambda Q_2X, \, \lambda \in [-1, 0]$$

Furthermore, in this case $\lambda = -cos^2\,\theta(Q_2X)$.

Conversely, suppose that (1) and (2) are satisfied. Then (1) implies that $\bar{J}RadTM$ is a distribution on M. From Lemma 3.2, it follows that the complementary distribution to $\bar{J}RadTM \oplus \bar{J}ltr(TM)$ is a Riemannian distribution. The rest of proof is clear. □

Corollary 3.1. *Let M be a slant lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$. Then we have*

$$g(TQ_2X, TQ_2Y) = cos^2\theta g(Q_2X, Q_2Y) \tag{3.14}$$

and

$$g(FQ_2X, FQ_2Y) = sin^2\theta g(Q_2X, Q_2Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.15}$$

Proof. From (0.1) and (3.4) we have

$$g(TQ_2X, TQ_2Y) = -g(Q_2X, T^2Q_2Y), \quad \forall X, Y \in \Gamma(TM).$$

Then from Theorem 3.1, we obtain (3.14) and (3.15) follows from (3.14). □

Theorem 3.2. *Let M be a $q-$ lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2q$. Then M is slant lightlike submanifold if and only if the following conditions are satisfied:*

1. *$\bar{J}ltr(TM)$ is a distribution on M.*
2. *There exists a constant $\mu \in [-1, 0]$ such that*

$$BFQ_2X = \mu Q_2X, \quad \forall X \in \Gamma(TM).$$

In this case $\mu = -sin^2\theta$, where θ is the slant angle of M and Q_2 the projection on D which is complementary to $\bar{J}RadTM \oplus \bar{J}ltr(TM)$.

Proof. It is easy to see that $\bar{J}RadTM \cap \bar{J}ltr(TM) = \{0\}$ and $\bar{J}Rad(TM)$ is a subbundle of $S(TM)$. Moreover, the complementary distribution to $\bar{J}ltr(TM) \oplus \bar{J}RadTM$ in $S(TM)$ is Riemannian. Furthermore, from the proof of Lemma 3.2 $S(TM^{\perp})$ is also Riemannian. Thus condition (A) in the definition of slant lightlike submanifold is satisfied. On the other hand applying $\bar{J}$ to (3.7) and using (3.4) and (3.7) we obtain

$$-X = -P_1X - P_2X + T^2Q_2X + FTQ_2X + JFQ_1X + BFQ_2X + CFQ_2X.$$

Since $JFQ_1X = -Q_1X \in \Gamma(S(TM)$, taking the tangential parts we have

$$-X = -P_1X - P_2X + T^2Q_2X - Q_1X + BFQ_2X.$$

Then considering (3.6) we get

$$-Q_2X = T^2Q_2X + BFQ_2X. \tag{3.16}$$

Now, if M is slant lightlike then from Theorem 3.1 we have $T^2Q_2X = -cos^2\theta Q_2X$. hence we derive

$$BFQ_2X = -sin^2\theta\, Q_2X.$$

Conversely, suppose that $BFQ_2X = \mu Q_2X$, $\mu \in [-1, 0]$, then from (3.16) we obtain

$$T^2Q_2X = -(1 + \mu)Q_2X.$$

Put $-(1 + \mu) = \lambda$ so that $\lambda \in [-1, 0]$. Then proof follows from Theorem 3.1. □

4. Minimal slant lightlike submanifolds

A general notion of minimal lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ has been introduced by Bejan-Duggal in [2] as follows:

Definition 4.1. We say that a lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if:

(i) $h^s = 0$ on $Rad(TM)$ and

(ii) $trace\, h = 0$, where trace is written w.r.t. g restricted to $S(TM)$.

In the case 2, the condition (i) is trivial. Moreover, it has been shown in [2] that above definition is independent of $S(TM)$ and $S(TM^{\perp})$, but it depends on the choice of the transversal bundle $tr(TM)$. As in the semi-Riemannian case, any lightlike totally geodesic M is minimal.

Example 2. Let $\bar{M} = \mathbf{R}^8_2$ be a semi-Euclidean space of signature $(-,-,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial\, x_1, \partial\, x_2, \partial\, x_3, \partial\, x_4, \partial\, x_5, \partial\, x_6, \partial\, x_7, \partial\, x_8\}$. Consider a complex structure J_1 defined by

$$\begin{aligned} J_1(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) &= (-x_2, x_1, -x_4, x_3, -x_7 \cos\alpha - x_6 \sin\alpha, \\ &\quad -x_8 \cos\alpha + x_5 \sin\alpha, x_5 \cos\alpha + x_8 \sin\alpha, \\ &\quad x_6 \cos\alpha - x_7 \sin\alpha). \end{aligned}$$

for $\alpha \in (0, \frac{\pi}{2})$. Let M be a submanifold of $(\mathbf{R}^8_2, J_1)$ given by

$$\begin{aligned} x_1 &= u_1 \cosh\theta,\ x_2 = u_2 \cosh\theta,\ x_3 = -u_3 + u_1 \sinh\theta,\ x_4 = u_1 + u_3 \sinh\theta \\ x_5 &= \cos u_4 \cosh u_5,\ x_6 = \cos u_4 \sinh u_5,\ x_7 = \sin u_4 \sinh u_5 \\ x_8 &= \sin u_4 \cosh u_5 \end{aligned}$$

$u_1 \in (0, \frac{\pi}{2})$. Then TM is spanned by

$$\begin{aligned} Z_1 &= \cosh\theta\, \partial\, x_1 + \sinh\theta \partial\, x_3 + \partial\, x_4 \\ Z_2 &= \cosh\theta\, \partial\, x_2,\ Z_3 = -\partial\, x_3 + \sinh\theta\, \partial\, x_4 \\ Z_4 &= -\sin u_4 \cosh u_5\, \partial x_5 - \sin u_4 \sinh u_5\, \partial x_6 + \cos u_4 \sinh u_5\, \partial x_7 \\ &\quad + \cos u_4 \cosh u_5\, \partial x_8 \\ Z_5 &= \cos u_4 \sinh u_5\, \partial x_5 + \cos u_4 \cosh u_5\, \partial x_6 + \sin u_4 \cosh u_5\, \partial x_7 \\ &\quad + \sin u_4 \sinh u_5\, \partial x_8. \end{aligned}$$

Hence M is $1-$ lightlike with $RadTM = span\{Z_1\}$ and $J_1(RadTM)$ spanned by $J_1 Z_1 = Z_2 + Z_3$. Thus $J_1 RadTM$ is a distribution on M. Then it is easy to see that $D = \{Z_4, Z_5\}$ is a slant distribution with respect to J_1 with slant angle α. The screen transversal bundle $S(TM^{\perp})$ is spanned by

$$\begin{aligned} W_1 &= -\cosh u_5\, \partial\, x_5 + \sinh u_5\, \partial\, x_6 + \tan u_4 \sinh u_5 \partial\, x_7 \\ &\quad -\tan u_4 \cosh u_5\, \partial\, x_8 \\ W_2 &= -\tan u_4 \sinh u_5\, x_5 + \tan u_4 \cosh u_5\, \partial\, x_6 - \cosh u_5\, \partial\, x_7 \\ &\quad + \sinh u_5\, \partial\, x_8. \end{aligned}$$

On the other hand, the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = \tanh\theta\, sinh\,\theta\,\partial\, x_1 + sinh\,\theta\, x_3 + \partial\, x_4.$$

Hence $J_1 N = tanh^2\,\theta\, Z_2 + Z_3$. Thus we conclude that M is a slant lightlike submanifold of $(\mathbf{R}^8_2, J_1)$. Now by direct computations, using Gauss and Weingarten formulas, we have

$$h^l = 0, h^s(X, Z_1) = h^s(X, J_1 Z_1) = 0,\ h^s(X, J_1 N) = 0, \forall X \in \Gamma(TM),$$

$$h^s(Z_4, Z_4) = \frac{cos\, u_4}{sinh^2 u_5 + cosh^2 u_5} W_1,\ h^s(Z_5, Z_5) = \frac{-cos\, u_4}{sinh^2 u_5 + cosh^2 u_5} W_1.$$

Hence, the induced connection is a metric connection and M is not totally geodesic, but, it is a proper minimal slant lightlike submanifold of $(\mathbf{R}^8_2, J_1)$.

Remark 4. We note that the method established in Example 2 can be generalized. Namely, let M be a $1-$ lightlike submanifold of R^8_2, If an integral manifold M_θ of the distribution D complementary to the distribution $\bar{J}RadTM \oplus \bar{J}ltr(TM)$ in $S(TM)$ is an invariant submanifold of $\bar{M}$ with respect to J_o defined by

$$J_o(x_1, x_2, x_3, x_4) = (-x_3, -x_4, x_1, x_2).$$

Then M is a slant lightlike submanifold with respect to J_1. Thus, there are many examples of minimal slant lightlike submanifolds of semi-Euclidean space $\mathbf{R}^8_2$.

Next, we prove two characterization results for minimal slant lightlike submanifolds. First we give the following lemma which will be useful later.

Lemma 4.1. *Let M be a proper slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that $dim(D) = dim(S(TM^\perp)$. If $\{e_1, ..., e_m\}$ is a local orthonormal basis of $\Gamma(D)$, then $\{\csc\,\theta F e_1,\ ...,\ \csc\,\theta\, F e_m\}$ is a orthonormal basis of $S(TM^\perp)$.*

Proof. Since $e_1, ..., e_m$ is a local orthonormal basis of D and D is Riemannian, from Corollary 3.1, we obtain

$$\bar{g}(\csc\,\theta F e_i, \csc\,\theta F e_j) = \csc^2\,\theta\, sin^2\,\theta\, g(e_i, e_j) = \delta_{ij},$$

which proves the assertion. □

Theorem 4.1. *Let M be a proper slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is minimal if and only if*

$$traceA^*_{\xi_j}\mid_{S(TM)} = 0,\ traceA_{W_\alpha}\mid_{S(TM)} = 0$$

and

$$\bar{g}(D^l(X, W), Y) = 0, \forall X, Y \in \Gamma(RaDTM),$$

where $\{\xi_j\}_{j=1}^r$ is a basis of $RadTM$ and $\{W_\alpha\}_{\alpha=1}^m$ is a basis of $S(TM^\perp)$.

Proof. From Proposition 3.1 in [2], we have $h^l = 0$ on $RadTM$. Thus M is minimal if and only if

$$\sum_{i=1}^{r} h(\bar{J}\xi_i, \bar{J}\xi_i) + \sum_{i=1}^{r} h(\bar{J}N_i, \bar{J}N_i) + \sum_{k=1}^{m} h(e_k, e_k) = 0.$$

Using (2.10) and (2.6) we obtain

$$\begin{aligned} \sum_{i=1}^{r} h(\bar{J}\xi_i, \bar{J}\xi_i) &= \sum_{i=1}^{r} \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} \bar{J}\xi_i, \bar{J}\xi_i) N_j \\ &+ \frac{1}{m} \sum_{\alpha=1}^{m} g(A_{W_\alpha} \bar{J}\xi_i, \bar{J}\xi_i) W_\alpha. \end{aligned} \tag{4.1}$$

In similar way we obtain

$$\begin{aligned} \sum_{i=1}^{r} h(\bar{J}N_i, \bar{J}N_i) &= \sum_{i=1}^{r} \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} \bar{J}N_i, \bar{J}N_i) N_j \\ &+ \frac{1}{m} \sum_{\alpha=1}^{m} g(A_{W_\alpha} \bar{J}N_i, \bar{J}N_i) W_\alpha \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \sum_{k=1}^{m} h(e_k, e_k) &= \sum_{k=1}^{m} \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} e_k, e_k) N_j \\ &+ \frac{1}{m} \sum_{\alpha=1}^{m} g(A_{W_\alpha} e_k, e_k) W_\alpha. \end{aligned} \tag{4.3}$$

Thus our assertion follows from (4.1), (4.2) and (4.3). □

Theorem 4.2. *Let M be a proper slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ such that $dim(D) = dim(S(TM^{\perp})$. Then M is minimal if and only if*

$$traceA^*_{\xi_j} \mid_{S(TM)} = 0, \; traceA_{Fe_i} \mid_{S(TM)} = 0$$

and

$$\bar{g}(D^l(X, Fe_i), Y) = 0, \forall X, Y \in \Gamma(RadTM)$$

where $\{e_1, ..., e_m\}$ is a basis of D.

Proof. From Lemma 4.1, $\{\csc\theta Fe_1, ..., \csc\theta\, Fe_m\}$ is a orthonormal basis of $S(TM^{\perp})$. Thus we can write

$$h^s(X, X) = \sum_{i=1}^{m} A_i \csc\theta Fe_i, \forall X \in \Gamma(TM)$$

for some functions A_i, $i \in \{1, ..., m\}$. Hence we obtain

$$h^s(X, X) = \sum_{i=1}^{m} \csc\theta g(A_{Fe_i} X, X)$$

for $X \in \Gamma(\bar{J}RadM \oplus \bar{J}ltr(TM) \perp D)$.

Then the assertion of theorem comes from Theorem 4.1. □

Concluding remarks. (a). It is known that a proper slant submanifold of a Kaehler manifold is even dimensional, but this is not true for our definition of slant lightlike submanifold. For instance, see two examples of this paper.

(b) We notice that the second fundamental forms and their shape operators of a non-degenerate submanifold are related by means of the metric tensor field. Contrary to this we see from (2.3)-(2.7) that in case of lightlike submanifolds there are interrelations between these geometric objects and those of its screen distributions. Thus, the geometry of lightlike submanifolds depends on the triplet $\big(S(TM), S(TM^{\perp}), ltr(TM)\big)$. However, it is important to highlight that, as per Proposition 2.1 of this paper, our results are stable with respect to any change in the above triplet. Moreover, we have verified that the conclusions of all our results will not change with the change of any induced object on M.

References

[1] M. Barros and A. Romero, *Indefinite Kaehler manifolds*, Math. Ann., 261 (1982), 55-62.

[2] C. L. Bejan and K. L. Duggal, *Global lightlike manifolds and harmonicity*, Kodai Math J. (2005).

[3] B.Y.Chen, Slant Immersions, Bull. Austral. Math. Soc. 41 (1990), 135-147.

[4] B.Y.Chen,*Geometry of Slant Submanifolds*, (Katholieke Universiteit, Leuven, 1990).

[5] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Its Applications*, Kluwer Academic Publishers, Dortrecht, 364. (1996).

[6] K. L. Duggal and D. H. Jin, Totally umbilical lightlike submanifolds, Kodai Math. J., 26 9, (2003), 49-68.

[7] E. J. Flaherty, *Hermitian and Kählerian Geometry in Relativity*, Lecture Notes in Physics, 46, Springer-Verlag (1976).

[8] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Kluwer Academic Publishers, 366, 1996.

[9] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity,* Academic Press New York (1983).

[10] K.Yano and M.Kon, *CR-submanifolds of Kaehlerian and Sasakian Manifolds*, Birkhauser, (1983)

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