On the maximal spacelike submanifolds of a pseudo- Riemannian quasi constant curvature manifold

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Abstract. In this paper, we first give a definition of pseudo-Riemannian quasi constant curvature manifold and then generalize T.Ishihara's results.

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1 Introduction

Let $N_P^{n+p}(c)$ be an $(n+p)$-dimensional pseudo-Riemannian manifold of constant curvature c, whose index is p. Let M^n be an n-dimensional complete spacelike submanifold isometrically immersed in $N_P^{n+p}(c)$. Noting that the codimension is equal to the index. Its curvature tensor satisfies

$$R_{ABCD} = c\varepsilon_A\varepsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

T.Ishihara [7] proved:

Theorem A. *Let M^n be a complete maximal spacelike submanifold in $N_P^{n+p}(c)$. Then either M^n is totally geodesic $(c \geq 0)$ or $0 \leq S \leq -npc(c < 0)$, where S is the square of the length of the second fundamental form of M^n.* Here, similar to the definition of the quasi constant curvature manifold defined by [2], we give the following definition:

Definition. *An $(n+p)$-dimensional pseudo-Riemannian manifold N_p^{n+p} with index p is said to be a pseudo-Reimannian quasi constant curvature manifold, if its curvature tensor satisfies*

$$(1.1.1) \qquad \begin{aligned} K_{ABCD} =& a\varepsilon_A\varepsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) + b\varepsilon_A\varepsilon_B(\delta_{AC}\upsilon_B\upsilon_D \\ &- \delta_{AD}\upsilon_B\upsilon_C + \delta_{BD}\upsilon_A\upsilon_C - \delta_{BC}\upsilon_A\upsilon_D), \end{aligned}$$

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where a, b are real functions and v_A is the component of a unit vector field which is called the generator of the manifold.

Remark 1. When $b \equiv 0$, $N_p^{n+p} = N_p^{n+p}(a)$. From now on, we make use of the following convention on the range of the indices:

$$1 \leq A, B, \dots, \leq n+p;\ 1 \leq i, j, \dots, \leq n;\ n+1 \leq \alpha, \beta, \dots, \leq n+p.$$

In this paper, we study the case that the ambient space is a pseudo-Riemannian quasi constant curvature manifold N_p^{n+p} and generalize Theorem A. We obtain:

Theorem 1. *Let M^n be an n-dimensional complete maximal spacelike submanifold in an $(n+p)$-dimensional pseudo-Riemannian quasi constant curvature manifold N_p^{n+p}, whose index is p. We suppose a, b are constant.* (1)*: If the generator is orthogonal to M^n, then M^n is totally geodesic $(a \geq 0)$ or $0 \leq S \leq -nap(a < 0)$.* (2)*:If the generator is parallel to M^n, then M^n is totally geodesic $(na + b - n|b| \geq 0)$ or $0 \leq S \leq -p(na + b - n|b|)(na + b - n|b| < 0)$.*

Remark 2. When $b = 0$, from Theorem 1, we can obtain Theorem A immediately.

Theorem 2. *Let M^n be an n-dimensional maximal spacelike submanifold with parallel second fundamental form in an $(n+p)$-dimensional pseudo-Riemannian quasi constant curvature manifold N_p^{n+p}. We suppose that a, b are constant.* (1)*: If $a < 0$ and the generator is orthogonal to M^n, then M^n is totally geodesic or $S \geq -na/[1 + \frac{1}{2}sgn(p-1)]$.* (2) *: If $na + b - n|b| < 0$ and the generator is parallel to M^n, then M^n is totally geodesic or $S \geq -(na + b - n|b|)/[1 + \frac{1}{2}sgn(p-1)]$.*

In particular, taking $b = 0$ in Theorem 2 and using the results in [7] and [4] we can obtain easily:

Corollary. *Let M^n be an n-dimensional maximal spacelike submanifold with parallel second fundamental form in $N_p^{n+p}(a)(a < 0)$, then M^n is totally geodesic or $S \geq -na/[1 + \frac{1}{2}sgn(p-1)]$.*

In particular, when the equality holds, M^n is the product of hyperbolic spheres or $n = p = 2$, $M^2 = H^2(\sqrt{-a})$ is a hyperbolic Veronese surface in $H_2^4(\sqrt{-\frac{a}{3}})$, where

$$H^2(\sqrt{-a}) = \{x \in R_1^3, \langle x, x\rangle = x_1^2 + x_2^2 - x_3^2 = a, a < 0\},$$

$$H_2^4(\sqrt{-\frac{a}{3}}) = \{x \in R_3^5, \langle x, x\rangle = x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 = \frac{a}{3}, a < 0\}.$$

2 Local Formulas

Let N_p^{n+p} be an $(n+p)$-dimensional pseudo-Riemannian quasi constant curvature manifold, whose index is p. Let M^n be an n-dimensional Riemannian manifold isometrically immersed in N_p^{n+p}. As the pseudo-Riemannian metric of N_p^{n+p} induces the Riemannian metric of M^n, the immersion is called spacelike. We choose a local field of orthogonal frames $e_1, \dots, e_{n+p}$ in N_p^{n+p}, such that $e_1, \dots, e_n$ are tangent to M^n. Let ω_A be the dual frames so that the pseudo-Riemannian metric of N_p^{n+p} is given by $dS^2_{N_p^{n+p}} = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1, \varepsilon_\alpha = -1$. Then the structure equations of N_p^{n+p} are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{CD} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D,$$

where K_{ABCD} satisfies (1.1.1).

Restricting these forms to M^n, then

$$\omega_\alpha = 0,\ \omega_{i\alpha} = \sum_j h^\alpha_{ij} \omega_j,\ h^\alpha_{ij} = h^\alpha_{ji},$$

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l,$$

(2.2.1) $$R_{ijkl} = K_{ijkl} - \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}),$$

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta,$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2} \sum_{ij} R_{\alpha\beta ij} \omega_i \wedge \omega_j,$$

(2.2.2) $$R_{\alpha\beta ij} = K_{\alpha\beta ij} + \sum_k (h^\alpha_{ki} h^\beta_{kj} - h^\alpha_{kj} h^\beta_{ki}).$$

We denote by $H = \frac{1}{n} \sum_{i\alpha} h^\alpha_{ii} e_\alpha$ the mean curvature vector of M^n. Then M^n is maximal if its mean curvature vector vanishes identically. Denote by $h = \sum_{ij\alpha} h^\alpha_{ij} \omega_i \omega_j e_\alpha$ the second fundamental form of the immersion and by $S = \sum_{ij\alpha} (h^\alpha_{ij})^2$ the square of the length of h. h^α_{ijk} and h^α_{ijkl} are defined by

$$\sum_k h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ik} \omega_{kj} + \sum_k h^\alpha_{kj} \omega_{ki} - \sum_\beta h^\beta_{ij} \omega_{\beta\alpha}$$

and

$$\sum_l h^\alpha_{ijkl} \omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ijl} \omega_{lk} + \sum_l h^\alpha_{ilk} \omega_{lj} + \sum_l h^\alpha_{ljk} \omega_{li} - \sum_\beta h^\beta_{ijk} \omega_{\beta\alpha}$$

respectively,
Where

(2.2.3) $$h^\alpha_{ijk} - h^\alpha_{ikj} = K_{\alpha ikj} = -K_{\alpha ijk},$$

and

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_m h^{\alpha}_{im} R_{mjkl} + \sum_m h^{\alpha}_{mj} R_{mikl} + \sum_\beta h^{\beta}_{ij} R_{\alpha\beta kl}. \tag{2.2.4}$$

Noting M^n is maximal, from (2.2.1) we have

$$R_{ik} = (n-1)a\delta_{ij} + b[\sum_i \upsilon_i^2 \delta_{ik} + (n-2)\upsilon_i \upsilon_k] + \sum_{j\alpha} h^{\alpha}_{ij} h^{\alpha}_{jk}. \tag{2.2.5}$$

From (2.2.5), we see that the scalar curvature of M^n satisfies

$$R = an(n-1) + b(n-1) + S = (n-1)(na+b) + S. \tag{2.2.6}$$

From (2.2.6), we obtain

Proposition. *Let M^n be an n-dimensional maximal spacelike submanifold in N_p^{n+p}. If*

$$R \leq (n-1)(na+b),$$

then M^n is totally geodesic.

3 Proof of Theorems

In order to prove our Theorems, we need the following:

Lemma 1. [3, 6] *Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function which is bounded from above on M^n. Then for all $\varepsilon > 0$, there exists a point x in M^n such that, at x*

$$| \bigtriangledown f| < \varepsilon, \quad \triangle f > -\varepsilon, \quad f(x) < inf f + \varepsilon.$$

Lemma 2. *Let M^n be an n-dimensional maximal spacelike submanifold in N_p^{n+p}. Then the Ricci curvature of M^n satisfies*

$$R_{ik} \geq [(n-1)a - |b|]\delta_{ik} - (n-2)|b|.$$

Proof of Theorem 1: In the first, we have

$$\begin{aligned} \sum_i \upsilon_i^2 \delta_{ik} + (n-2)\upsilon_i \upsilon_k &\leq \sum_A \upsilon_A^2 \delta_{ik} + \frac{1}{2}(n-2)(\sum_A \upsilon_A^2 + \sum_A \upsilon_A^2) \\ &= \delta_{ik} + n - 2, \end{aligned} \tag{3.3.1}$$

and then, for fixed α, we choose $e_1, \dots, e_n$ such that

$$h^{\alpha}_{ij} = h^{\alpha}_{ii}\delta_{ij}.$$

Thus

$$\sum_j h^\alpha_{ij}h^\alpha_{jk} = h^\alpha_{ii}h^\alpha_{kk}\delta_{ik} \geq 0,$$

and so

$$\sum_{j\alpha} h^\alpha_{ij}h^\alpha_{jk} \geq 0. \tag{3.3.2}$$

Combining (3.3.1), (3.3.2) and (2.2.5), we obtain Lemma 2.

Lemma 3. [1, 5] *Let $H_i, i \geq 2$ be symmetric $(n\times n)$-matrixes, $S = \sum_i trH_i^2$. Then*

$$-\sum_{ij} tr(H_iH_j - H_jH_i)^2 + \sum_i (trH_i^2)^2 \leq (1 + \frac{1}{2}sgn(p-1))S^2.$$

Let $S_{\alpha\beta} = \sum_{ij} h^\alpha_{ij}h^\beta_{ij}$, then $(S_{\alpha\beta})$ can be assumed to be diagonal for a suitable choice of $e_{n+1}, \dots, e_{n+p}, i.e., S_{\alpha\beta} = S_\alpha\delta_{\alpha\beta}, S_\alpha = \sum_{ij}(h^\alpha_{ij})^2$. Since M^n is maximal, when a, b are constant, from (2.2.1)-(2.2.4), we can get

(3.3.3)

$$\begin{aligned}
\frac{1}{2}\triangle S &= \sum_{ijk\alpha}(h^\alpha_{ijk})^2 + \sum_{ij\alpha} h^\alpha_{ij}\triangle h^\alpha_{ij} \\
&= \sum_{ijk\alpha}(h^\alpha_{ijk})^2 + \sum_{ijkm\alpha} h^\alpha_{ij}h^\alpha_{mk}R_{mijk} + \sum_{ijkm\alpha} h^\alpha_{ij}h^\alpha_{im}R_{mkjk} + \sum_{ijk\alpha\beta} h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk} \\
&+ \sum_{ijk\alpha} h^\alpha_{ij}\nabla_j K_{\alpha kki} + \sum_{ijk\alpha} h^\alpha_{ij}\nabla_k K_{\alpha ikj} \\
&= \sum_{ijk\alpha}(h^\alpha_{ijk})^2 + naS + \sum_\alpha S_\alpha^2 - \sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 + bS\sum_k \upsilon_k^2 \\
&+ nb\sum_{ijm\alpha} h^\alpha_{ij}h^\alpha_{mi}\upsilon_m\upsilon_j - n\sum_{ij\alpha} h^\alpha_{ij}\nabla_j(b\upsilon_\alpha\upsilon_i).
\end{aligned}$$

It is clear that

$$-\sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 \geq 0.$$

Putting

$$P\sigma_1 = \sum_\alpha S_\alpha = S, \quad p(p-1)\sigma_2 = 2\sum_{\alpha<\beta} S_\alpha S_\beta,$$

then we have

$$p^2(p-1)(\sigma_1^2 - \sigma_2) = \sum_{\alpha<\beta}(S_\alpha - S_\beta)^2. \tag{3.3.4}$$

Substituting (3.3.4) into (3.3.3), we get

(3.3.5)

$$\frac{1}{2}\triangle S = \sum_{ijk\alpha}(h^{\alpha}_{ijk})^2 + naS + \frac{1}{p}S^2 + \frac{1}{p}\sum_{\alpha<\beta}(S_\alpha - S_\beta)^2 - \sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2$$
$$+ bS\sum_k \upsilon_k^2 + nb\sum_{ijm\alpha} h^{\alpha}_{ij}h^{\alpha}_{mi}\upsilon_m\upsilon_j - n\sum_{ij\alpha} h^{\alpha}_{ij}\nabla_j(b\upsilon_\alpha\upsilon_i).$$

Now, we assume that the generator $\upsilon = \sum_A \upsilon_A e_A$ is orthogonal to M^n, then we see that $\upsilon_i = 0$ and (3.3.5) becomes

(3.3.6)

$$\frac{1}{2}\triangle S = \sum_{ijk\alpha}(h^{\alpha}_{ijk})^2 + naS + \frac{1}{p}S^2 + \frac{1}{p}\sum_{\alpha<\beta}(S_\alpha - S_\beta)^2 - \sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2$$
$$\geq naS + \frac{1}{p}S^2.$$

Let $f = \frac{1}{\sqrt{S+c}}$ for any positive constant c, then f is bounded c^∞-function on M^n. By calculation, we get

$$|\nabla f|^2 = \frac{1}{4}f^6|\nabla S|^2, \tag{3.3.7}$$

and

$$\triangle f = -\frac{1}{2}f^3\triangle S + \frac{3}{4}f^5|\nabla S|^2. \tag{3.3.8}$$

From (3.3.7) and (3.3.8), we get

$$f^4\triangle S = 6|\nabla f|^2 - 2f\triangle f. \tag{3.3.9}$$

Combining (3.3.6) and (3.3.9), we get

$$(naS + \frac{1}{p}S^2)f^4 \ \leq\ 3|\nabla f|^2 \ -\ f\triangle f. \tag{3.3.10}$$

When $\upsilon_i = 0$ and a is constant, from (2.2.5) we see that $R_{ik} \geq a(n-1)\delta_{ik}$. Thus, from Lemma 1 and (3.3.10) we will get at point x,

$$(naS + \frac{1}{p}S^2)f^4 \leq 3\varepsilon + \varepsilon(inff + \varepsilon).$$

So

$$\frac{naS + \frac{1}{p}S^2}{(S+c)^2} \ \leq\ 3\varepsilon + \varepsilon(inff + \varepsilon). \tag{3.3.11}$$

Since when $\varepsilon \to 0, f(x)$ goes to the infimum and $S(x)$ goes to the supremum. Thus letting $\varepsilon \to 0$, from (3.3.11) we get

$$(na + \frac{1}{p} supS)supS \leq 0. \tag{3.3.12}$$

(3.3.12) implies that when $a \geq 0, S \equiv 0$, i.e., M^n is totally geodesic; when $a < 0$, $S \leq -npa$. On the other hand, we assume that the generator $\upsilon = \sum_A \upsilon_A e_A$ is parallel to M^n, then we see that $\upsilon_\alpha = 0$ and $\sum_i \upsilon_i^2 = 1$. Since for fixed α, we can choose $e_1, \dots, e_n$ such that $h_{ij}^\alpha = h_{ii}^\alpha \delta_{ij}$, then

$$\sum_{ijm} h_{ij}^\alpha h_{im}^\alpha \upsilon_m \upsilon_j = \sum_i (h_{ii}^\alpha)^2 \upsilon_i^2 \leq \sum_{ij} (h_{ij}^\alpha)^2 \sum_i \upsilon_i^2 = \sum_{ij} (h_{ij}^\alpha)^2,$$

and so

$$\sum_{ijm\alpha} h_{ij}^\alpha h_{im}^\alpha \upsilon_m \upsilon_j \leq \sum_{ij\alpha} (h_{ij}^\alpha)^2 = S. \tag{3.3.13}$$

Substituting (3.3.13) into (3.3.5), we get

(3.3.14)
$$\begin{aligned} \frac{1}{2} \triangle S \geq & \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + naS + \frac{1}{p} S^2 + \frac{1}{p} \sum_{\alpha<\beta} (S_\alpha - S_\beta)^2 - \sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 \\ & + bS - n|b|S \\ & \geq naS + bS - n|b|S + \frac{1}{p} S^2. \end{aligned}$$

When a, b are constant, from Lemma 2 we see that the Ricci curvature of M^n is bounded from below. Using the same arguments as above, we can get

$$(na + b - n|b| + \frac{1}{p} supS)supS \leq 0. \tag{3.3.15}$$

(3.3.15) implies that when $na + b - n|b| \geq 0$, M^n is totally geodesic; when $na + b - n|b| < 0, 0 \leq S \leq -p(na + b - n|b|)$. This completes the proof of Theorem 1. □

Taking $b = 0$ in Theorem 1, we can obtain Theorem A immediately.

Proof of Theorem 2: When the second fundamental form of M^n is parallel, we have $h_{ijk}^\alpha = 0$ for all i, j, k, α and $S =$ constant. Therefore, when the generator υ is orthogonal to M^n. From (3.3.3) using Lemma 3, we get

$$0 \leq naS + [1 + \frac{1}{2} sgn(p-1)]S^2.$$

So when $a < 0$, which implies $S = 0$. Namely, M^n is totally geodesic or $S \geq -na/[1 + \frac{1}{2} sgn(p-1)]$. On the other hand, when the generator υ is parallel to M^n, combining (3.3.13), (3.3.3) and Lemma 3 we get

$$(3.3.16) \qquad 0 \leq naS + bS + n|b|S + [1 + \frac{1}{2}sgn(p-1)]S^2.$$

Thus, when $na + b + n|b| < 0$, (3.3.16) shows that M^n is totally geodesic or $S \geq -(na + b + n|b|)/[1 + \frac{1}{2}sign(p-1)]$. This completes the proof of Theorem 2. Taking $b = 0$ in Theorem 2, when $a < 0$, we see that M^n is not totally geodesic if $S \geq -na/[1 + \frac{1}{2}sgn(p-1)]$. In particular, when the equality holds, we see that $S = -na(p = 1)$ or $S = -\frac{3}{2}a$. Therefore, using the results in [7] and the Corollary in [4], we obtain the Corollary in the Introduction. □

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References

[1] A.M.Li and J.M.Li, *An intrinsic rigidety theorem for minimal submanifolds in a sphere*, Arch.Math, 58(1992), 582-594.

[2] B.Y.Chen and K.Yano,*Hyperfaces of a conformally flat space*, Tensor N.S.26(1972), 318-322.

[3] H.Omori, *Isometric immersions of Riemannian manifolds,* J.Math.Soc.Japan, 19(1976), 205-214.

[4] H.Sun,*On spacelike submanifold of a pseudo-Riemannian space form*, Note di Math, 15(1995), 215-224.

[5] S.S.Chern,M.do carmo and S.Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, In Functional Analysis and Related Fields, Springer-Verlag,New York, 1970: 60-75.

[6] S.T.Yau, *Harmornic function on complete Riemannian manifolds*, Comm.Pure Appl.Math,28(1975), 201-228.

[7] T.Ishihara, *Maximal spacelike submanifolds of a pseudo-riemannian space of curvature*,Michigan Math, J.35(1988), 345-352.

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