On the Chern-type problem in Kähler geometry

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Abstract

The purpose of this paper is to investigate the Chern-type problem on Kähler geometry. That is, we study some properties concerning the distribution of the value of the squared norm of the second fundamental form on a complex submanifold of a complex projective space.

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Key words: Kähler manifold, Chern-type problem, second fundamental form, holomorphic sectional curvature, totally real bisectional curvature, totally geodesic.

1 Introduction

The theory of Kähler submanifolds is one of fruitful fields in Riemannian geometry and we have many studies [1], [2], [7], [8] and [10] etc. One of them is the complex geometric version of Chern's problem concerning the distribution of the value of the squared norm h_2 of the second fundamental form on M. In his paper [11], Tanno tackled this problem and verified the following theorem.

Theorem A. *Let $M = M^n$ be an n-dimensional compact Kähler submanifold of an $(n+p)$-dimensional Kähler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. Then M is totally geodesic, $h_2 = c(n+2)/6$ or $h_2(x) > c(n+2)/6$ at a point x in M.*

In this paper, we assert the following theorem.

Theorem. *Let $M = M^n$ be an $n(\geq 3)$-dimensional complete complex submanifold of an $(n+p)$-dimensional Kähler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. If the squared norm h_2 of the second fundamental form on M satisfies*

$$h_2 < \frac{c}{12(n^2-1)}(n^2-4),$$

then M is totally geodesic.

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2 Kähler manifolds

This section is concerned with reviewing basic formulas on Kähler manifolds. Let M be a complex $n(\geq 2)$-dimensional Kähler manifold equipped with Kähler metric tensor g and almost complex structure J. We can choose a local field

$$\{E_\alpha\} = \{E_j, E_{j^*}\} = \{E_1, \cdots, E_n, E_{1^*}, \cdots, E_{n^*}\}$$

of orthonormal frames on a neighborhood of M, where $E_{j^*} = JE_j$ and $j^* = n + j$. Here and in the sequel, the Latin small indices $i, j, \cdots$ run from 1 to n and the small Greek indices $\alpha, \beta, \cdots$ run from 1 to $2n = n^*$. We set

$$U_j = \frac{1}{\sqrt{2}}(E_j - iE_{j^*}), \quad \overline{U}_j = \frac{1}{\sqrt{2}}(E_j + iE_{j^*}),$$

where i denotes the imaginary unit. Then $\{U_j\}$ constitutes a local field of unitary frames on the neighborhood of M. With respect to the Kähler metric, we have

$$g(U_j, \overline{U}_k) = \delta_{jk}.$$

Now let $\{\omega_j\}$ be the canonical form with respect to the local field $\{U_j\}$ of unitary frames on the neighborhood of M. Then $\{\omega_j\} = \{\omega_1, \cdots, \omega_n\}$ consists of complex valued 1-forms of type (1,0) on M such that $\omega_j(U_k) = \delta_{jk}$ and $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent. The Kähler metric g of M can be expressed as

$$g = 2\sum_j \omega_j \otimes \bar{\omega}_j.$$

Associated with the frame field $\{U_j\}$, there exist complex-valued 1-forms ω_{jk}, which are usually called *complex connection forms* on M such that they satisfy the structure equations of M

$$\begin{aligned} &d\omega_i + \sum_k \omega_{ik} \wedge \omega_k = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \\ &d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ &\Omega_{ij} = \sum_k K_{\bar{i}jk\bar{l}}\, \omega_k \wedge \bar{\omega}_l, \end{aligned}$$

where Ω_{ij} (resp. $K_{\bar{i}jk\bar{l}}$) the curvature form (resp. the components of the Riemannian curvature tensor R) of M. ¿From the structure equations, the components of the curvature tensor satisfy

$$\begin{aligned} &K_{\bar{i}jk\bar{l}} = \overline{K}_{\bar{j}il\bar{k}}, \\ &K_{\bar{i}jk\bar{l}} = K_{\bar{i}kj\bar{l}} = K_{\bar{l}jk\bar{i}} = K_{\bar{l}kj\bar{i}}. \end{aligned}$$

For a local field $\{E_\alpha\} = \{E_j, E_{j^*}\} = \{E_1, \cdots, E_n, E_{1^*}, \cdots, E_{n^*}\}$ of orthonormal frame on a neighborhood of M, we denote by $R_{\alpha\beta\gamma\delta}$ the components of the Riemannian curvature tensor R. Then we have

$$K_{\bar{i}jk\bar{l}} = -\{(R_{ijkl} + R_{i^*jk^*l}) + i(R_{i^*jkl} - R_{ijk^*l})\}.$$

Relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows :

$$S = \sum_{i,j}(S_{i\bar{j}}\omega_i \otimes \bar{\omega}_j + S_{\bar{i}j}\bar{\omega}_i \otimes \omega_j),$$

where $S_{i\bar{j}} = \sum_k K_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{\bar{i}j}$. The scalar curvature r of M is also given by

$$r = 2\sum_j S_{j\bar{j}}.$$

An n-dimensional Kähler manifold M is said to be *Einstein*, if the Ricci tensor S satisfies the condition

$$S_{i\bar{j}} = \frac{r}{2n}\delta_{ij}.$$

The components $K_{\bar{i}jk\bar{l}m}$ and $K_{\bar{i}jk\bar{l}\bar{m}}$ (resp. $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (resp. the Ricci tensor S) are given by

$$\sum_m (K_{\bar{i}jk\bar{l}m}\omega_m + K_{\bar{i}jk\bar{l}\bar{m}}\bar{\omega}_m) = dK_{\bar{i}jk\bar{l}} - \sum_m (K_{\bar{m}jk\bar{l}}\bar{\omega}_{mi} + K_{\bar{i}mk\bar{l}}\omega_{mj} + K_{\bar{i}jm\bar{l}}\omega_{mk} + K_{\bar{i}jk\bar{m}}\bar{\omega}_{ml}),$$

$$\sum_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}).$$

The second Bianchi identity is given as follows :

$$K_{\bar{i}jk\bar{l}m} = K_{\bar{i}jm\bar{l}k}.$$

And hence we have

$$S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_m K_{\bar{j}ik\bar{m}m}.$$

A Kähler manifold of constant holomorphic sectional curvature is called a *complex space form*. The components $K_{\bar{i}jk\bar{l}}$ of the Riemannian curvature tensor R of an n-dimensional complex space form of constant holomorphic sectional curvature c is given by

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}).$$

3 Complex submanifolds

This section is reviewed complex submanifolds of a Kähler manifold. First of all, the basic formulas for the theory of complex submanifolds are prepared.

Let $M' = M^{n+p}$ be an $(n+p)$-dimensional Kähler manifold with Kähler structure (g', J'). Let M be an n-dimensional complex submanifold of M' and g the induced Kähler metric tensor on M from g'. We can choose a local field

$$\{U_A\} = \{U_i, U_x\} = \{U_1, \cdots, U_{n+p}\}$$

of unitary frames on a neighborhood of M' in such a way that, restricted to M, $U_1, \cdots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated :

$$\begin{aligned} &A, B, \cdots = 1, \cdots, n, n+1, \cdots, n+p, \\ &i, j, \cdots = 1, \cdots, n, \\ &x, y, \cdots = n+1, \cdots, n+p. \end{aligned}$$

With respect to the frame field, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame fields. Then the Kähler metric tensor g' of M' is given by

$$g' = 2\sum_A \omega_A \otimes \bar{\omega}_A.$$

The canonical forms ω_A, the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$\begin{aligned} &d\omega_A + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_C = 0, \quad \omega_{AB} + \bar{\omega}_{BA} = 0, \\ &d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \\ &\Omega'_{AB} = \sum_{C,D} K'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \end{aligned} \tag{3.1}$$

where Ω'_{AB} (resp. $K'_{\bar{A}BC\bar{D}}$) denotes the curvature form (resp. the components of the Riemannian curvature tensor R') of M'. Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.2}$$

and the induced Kähler metric tensor g of M is given by

$$g = 2\sum_j \omega_j \otimes \bar{\omega}_j.$$

Then $\{U_j\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_j\}$ is a local dual frame filed due to $\{U_j\}$, which consists of complex-valued 1-forms of type (1,0) on M. Moreover, $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent, and $\{\omega_j\}$ is the canonical forms on M. It follows from (3.2) and Cartan's lemma that the exterior derivatives of (3.2) give rise to

$$\omega_{xi} = \sum_j h^x_{ij} \omega_j, \qquad h^x_{ij} = h^x_{ji}. \tag{3.3}$$

The quadratic form

$$\alpha = \sum_{i,j,x} h^x_{ij} \omega_i \otimes \omega_j \otimes U_x$$

with values in the normal bundle on M in M' is called the *second fundamental form* on the submanifold M. ¿From the structure equations for M', it follows that the structure equations for M are similarly given by

$$
\begin{aligned}
&d\omega_i + \sum_k \omega_{ik} \wedge \omega_k = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \\
&d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_k = \Omega_{ij}, \\
&\Omega_{ij} = \sum_{k,l} K_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l.
\end{aligned} \tag{3.4}
$$

For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (3.1), (3.3) and (3.4) that

$$K_{\bar{i}jk\bar{l}} = K'_{\bar{i}jk\bar{l}} - \sum_x h^x_{jk} \bar{h}^x_{il}. \tag{3.5}$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r on M are given by

$$S_{i\bar{j}} = \sum_k K'_{\bar{k}ki\bar{j}} - h_{i\bar{j}}{}^2, \tag{3.6}$$

$$r = 2\Big(\sum_{j,k} K'_{\bar{k}kj\bar{j}} - h_2\Big), \tag{3.7}$$

where $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{m,x} h^x_{im} \bar{h}^x_{mj}$ and $h_2 = \sum_j h_{j\bar{j}}{}^2$.

Now the components h^x_{ijk} and $h^x_{ij\bar{k}}$ of the covariant derivative of the second fundamental form on M are given by

$$
\begin{aligned}
&\sum_k (h^x_{ijk} \omega_k + h^x_{ij\bar{k}} \bar{\omega}_k) \\
&\qquad = dh^x_{ij} - \sum_k (h^x_{jk} \omega_{ki} + h^x_{ik} \omega_{kj}) + \sum_y h^y_{ij} \omega_{xy}.
\end{aligned}
$$

Then, substituting dh^x_{ij} in this definition into the exterior derivative

$$d\omega_{xi} = \sum_j (dh^x_{ij} \wedge \omega_j + h^x_{ij} d\omega_j)$$

of (3.3) and using (3.1) $\sim$ (3.4) and (3.6), we have

$$h^x_{ijk} = h^x_{ikj}, \qquad h^x_{ij\bar{k}} = -K'_{\bar{x}ij\bar{k}}.$$

In particular, let the ambient space $M' = M^{n+p}(c)$ be an $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature c. Then, by (3.5) $\sim$ (3.7), we get

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x h^x_{jk} \bar{h}^x_{il}, \tag{3.8}$$

$$S_{i\bar{j}} = \frac{c}{2}(n+1)\delta_{ij} - h_{i\bar{j}}{}^2, \tag{3.9}$$

$$r = cn(n+1) - 2h_2, \tag{3.10}$$

Finally, let $M' = M^{n+p}$ be an $(n+p)$-dimensional Kähler manifold and let M be an n-dimensional complex submanifold of M'. Then the Laplacian Δh_2 of the squared norm h_2 of the second fundamental form α on M is given by Aiyama, Kwon and Nakagawa [1] as follows :

$$\Delta h_2 = 2\|\nabla\alpha\|_2 + c(n+2)h_2 - 4h_4 - 2\mathrm{Tr}\ A^2, \tag{3.11}$$

where $h_4 = \sum_{i,j} h_{i\bar{j}}{}^2 h_{j\bar{i}}{}^2$ and A is a Hermitian matrix of order p with entry $A^x_y = \sum_{i,j} h^x_{ij}\bar{h}^y_{ij}$.

4 Proof of Theorem

First, we are concerned with the totally real bisectional curvature of a Kähler manifold. Let (M, g) be an n-dimensional Kähler manifold with almost complex structure J. In their paper [3], Bishop and Goldberg introduced the notion for totally real bisectional curvature $B(X, Y)$ on a Kähler manifold.

A plane section P in the tangent space T_pM at any point p in M is said to be *totally real* or *anti-holomorphic* if P is orthogonal to JP. For an orthonormal basis $\{X, Y\}$ of the totally real plane section P, any vectors X, JX, Y and JY are mutually orthogonal. It implies that for orthogonal vectors X and Y in P, it is totally real if and only if two holomorphic plane sections spanned by X, JX and Y, JY are orthogonal. Houh [5] showed that an $n(\geq 3)$-dimensional Kähler manifold has constant totally real bisectional curvature c if and only if it has constant holomorphic sectional curvature $2c$. On the other hand, Goldberg and Kobayashi [4] introduced the notion of holomorphic bisectional curvature $H(X, Y)$ which is determined by two holomorphic planes $\mathrm{Span}\{X, JX\}$ and $\mathrm{Span}\{Y, JY\}$, and asserted that the complex projective space $CP^n(c)$ is the only compact Kähler manifold with positive holomorphic bisectional curvature and constant scalar curvature. If we compare the notion of $B(X, Y)$ with the holomorphic bisectional curvature $H(X, Y)$ and the holomorphic sectional curvature $H(X)$, then the holomorphic bisectional curvature $H(X, Y)$ turns out to be totally real bisectional curvature $B(X, Y)$ (resp. holomorphic sectional curvature $H(X)$), when two holomorphic planes $\mathrm{Span}\{X, JX\}$ and $\mathrm{Span}\{Y, JY\}$ are orthogonal to each other (resp. coincides with each other). From this, it follows that the positiveness of $B(X, Y)$ is weaker than the positiveness of $H(X, Y)$, because $H(X, Y) > 0$ implies that both of $B(X, Y)$ and $H(X)$ are positive but we do know whether or not $B(X, Y) > 0$ implies $H(X, Y) > 0$.

Definition 4.1. For a totally real plane section P spanned by orthonormal vectors X and Y, the *totally real bisectional curvature* $B(X, Y)$ is defined by

$$B(X, Y) = g(R(X, JX)JY, Y). \tag{4.12}$$

Then, using the first Bianchi identity to (4.12) and the fundamental properties of the Riemannian curvature tensor of Kähler manifolds, we get

$$B(X,Y) = g(R(X,Y)Y,X) + g(R(X,JY)JY,X) \tag{4.13}$$
$$= K(X,Y) + K(X,JY),$$

where $K(X,Y)$ is the sectional curvature of the plane spanned by X and Y.

In the rest of this section, we suppose that X and Y are orthonormal vectors in a non-degenerate totally real plane section. If we put

$$X' = \frac{1}{\sqrt{2}}(X+Y), \quad Y' = \frac{1}{\sqrt{2}}(X-Y),$$

then it is easily seen that

$$g(X',X') = g(Y',Y') = 1, \qquad g(X',Y') = 0.$$

Thus we get

$$\begin{aligned} B(X',Y') &= g(R(X',JX')JY',Y') \\ &= \frac{1}{4}\{H(X) + H(Y) + 2B(X,Y) - 4K(X,JY)\}, \end{aligned}$$

where $H(X) = K(X,JX)$ means the holomorphic sectional curvature of the holomorphic plane spanned by X and JX. Hence we have

$$4B(X',Y') - 2B(X,Y) = H(X) + H(Y) - 4K(X,JY). \tag{4.14}$$

If we put

$$X'' = \frac{1}{\sqrt{2}}(X+JY), \quad Y'' = \frac{1}{\sqrt{2}}(JX+Y),$$

then we get

$$g(X'',X'') = g(Y'',Y'') = 1, \qquad g(X'',Y'') = 0.$$

Using the similar method as in (4.14), we have

$$4B(X'',Y'') - 2B(X,Y) = H(X) + H(Y) - 4K(X,Y). \tag{4.15}$$

Summing up (4.14) and (4.15) and taking account of (4.13), we obtain

$$2B(X',Y') + 2B(X'',Y'') = H(X) + H(Y). \tag{4.16}$$

Now we calculate here the totally real bisectional curvatures of a Kähler manifold. Let $M = M^n$ be an $n(\geq 3)$-dimensional complex submanifold of an $(n+p)$-dimensional Kähler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature c. Assume that the totally real bisectional curvatures on M is bounded from below (resp. above) by a constant a (resp. b), and let $a(M)$ and $b(M)$ be the infimum and the supremum of the set $B(M)$ of the totally real bisectional curvatures on M, respectively. By definition, we see

$$a \leq a(M) \text{ (resp. } b \geq b(M)).$$

¿From (4.16), we have

$$H(X) + H(Y) \geq 4a \text{ (resp. } \leq 4b). \tag{4.17}$$

For an orthonormal frame field $\{E_1, \cdots, E_n\}$ on a neighborhood of M, the holomorphic sectional curvature $H(E_j)$ of the holomorphic plane spanned by E_j can be expressed as

$$H(E_j) = g(R(E_j, JE_j)JE_j, E_j) = R_{jj^*j^*j} = K_{\bar{j}j j\bar{j}}.$$

On the other hand, it is easily seen that the plane sections Span$\{E_j, JE_j\}$, and Span$\{E_k, JE_k\}$, $j \neq k$, are orthogonal and the totally real bisectional curvature $B(E_j, E_k)$ is given by

$$B(E_j, E_k) = g(R(E_j, JE_j)JE_k, E_k) = K_{\bar{j}jk\bar{k}}, \quad j \neq k.$$

¿From the inequality (4.17) for $X = E_j$ and $Y = E_k$, we have

$$K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}} \geq 4a \text{ (resp. } \leq 4b), \quad j \neq k. \tag{4.18}$$

Thus we have

$$\sum_{j<k}(K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \geq 2an(n-1) \text{ (resp. } \leq 2bn(n-1)), \tag{4.19}$$

which implies that

$$\sum_j K_{\bar{j}jj\bar{j}} \geq 2an \text{ (resp. } \leq 2bn), \tag{4.20}$$

where the equality holds if and only if

$$K_{\bar{j}jj\bar{j}} = 2a \text{ (resp. } = 2b)$$

for any index j.

Since the scalar curvature r is given by

$$r = 2\sum_{j,k} K_{\bar{j}jk\bar{k}} = 2\Big(\sum_j K_{\bar{j}jj\bar{j}} + \sum_{j\neq k} K_{\bar{j}jk\bar{k}}\Big),$$

we have by (4.19)

$$r \geq 2\sum_j K_{\bar{j}jj\bar{j}} + 2an(n-1) \Big(\text{resp. } \leq 2\sum_j K_{\bar{j}jj\bar{j}} + 2bn(n-1)\Big),$$

from which it follows that

$$\sum_j K_{\bar{j}jj\bar{j}} \leq \frac{r}{2} - an(n-1) \Big(\text{resp. } \geq \frac{r}{2} - bn(n-1)\Big), \tag{4.21}$$

where the equality holds if and only if

$$K_{\bar{j}jk\bar{k}} = a \text{ (resp. } = b)$$

for any distinct indices j and k. In this case, M is locally congruent to $M^n(a)$ (resp. $M^n(b)$) due to Houh [5]. Also (4.18) gives us

$$\sum_{k(\neq j)} (K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \geq 4a(n-1) \text{ (resp. } \leq 4b(n-1))$$

for each j, so that

$$(n-2)K_{\bar{j}jj\bar{j}} + \sum_{k} K_{\bar{k}kk\bar{k}} \geq 4a(n-1) \text{ (resp. } \leq 4b(n-1)).$$

¿From this inequality together with (4.21), it follows that

$$(n-2)K_{\bar{j}jj\bar{j}} \geq a(n-1)(n+4) - \frac{r}{2} \quad \left(\text{resp. } \leq b(n-1)(n+4) - \frac{r}{2}\right) \tag{4.22}$$

for any index j, so that the holomorphic sectional curvature $K_{\bar{j}jj\bar{j}}$ is bounded from below (resp. above) for $n \geq 3$. Moreover, the equality holds for some index j if and only if M is locally congruent to $M^n(2a)$ (resp. $M^n(2b)$).

Since the Ricci curvature $S_{j\bar{j}}$ is given by

$$S_{j\bar{j}} = K_{\bar{j}jj\bar{j}} + \sum_{j(\neq k)} K_{\bar{j}jk\bar{k}},$$

we have by the assumption

$$S_{j\bar{j}} \geq K_{\bar{j}jj\bar{j}} + a(n-1) \text{ (resp. } \leq K_{\bar{j}jj\bar{j}} + b(n-1)),$$

and hence by (4.22), we have

$$S_{j\bar{j}} \geq \frac{1}{2(n-2)}\{4a(n-1)(n+1) - r\} \quad \left(\text{resp. } \leq \frac{1}{2(n-2)}\{4b(n-1)(n+1) - r\}\right). \tag{4.23}$$

On the other hand, using (4.23), we get

$$r \geq 2S_{j\bar{j}} + \frac{1}{n-2}(n-1)\{4a(n-1)(n+1) - r\} \quad \left(\text{resp. } \leq 2S_{j\bar{j}} + \frac{1}{n-2}(n-1)\{4b(n-1)(n+1) - r\}\right),$$

and hence we have

$$S_{j\bar{j}} \leq \frac{1}{2(n-2)}\{(2n-3)r - 4a(n-1)^2(n+1)\} \quad \left(\text{resp. } \geq \frac{1}{2(n-2)}\{(2n-3)r - 4b(n-1)^2(n+1)\}\right). \tag{4.24}$$

In connection with Theorem A, we can verify the following theorem

Theorem 4.1. *Let $M = M^n$ be an $n(\geq 3)$-dimensional complete complex submanifold of an $(n+p)$-dimensional Kähler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(>0)$. If the squared norm h_2 of the second fundamental form on M satisfies*

$$h_2 < \frac{c}{12n(n^2-1)}(n^2-4),$$

then M is totally geodesic.

Proof. Since two matrices $H = (h_{j\bar{k}}{}^2)$ and $A = (A^x_y)$ are both positive Hermitian ones, the eigenvalues λ_j of H and the eigenvalues λ_x of A are non-negative real valued functions on M. Thus it is easily seen that

$$\begin{aligned} &\sum_j \lambda_j = \text{Tr } H = h_2, \quad \sum_x \lambda_x = \text{Tr } A = h_2, \\ &{h_2}^2 \geq h_4 = \sum_j {\lambda_j}^2 \geq \frac{1}{n}{h_2}^2, \\ &{h_2}^2 \geq \text{Tr } A^2 = \sum_x {\lambda_x}^2 \geq \frac{1}{p}{h_2}^2, \end{aligned} \tag{4.25}$$

where the second equality in the second relationship holds if and only if all eigenvalues of the matrix H are equal, and the second equality in the last relationship holds if and only if all eigenvalues of the matrix A are equal. It means that each equality holds if and only if the rank of matrices H and A are at most one. By (3.11), we have

$$\Delta h_2 \geq c(n+2)h_2 - 4h_4 - 2\text{Tr } A^2,$$

where the equality holds if and only if the second fundamental form α on M is parallel. Together the above inequality with the properties about eigenvalues (4.25), it follows that

$$\Delta h_2 \geq c(n+2)h_2 - 6{h_2}^2,$$

where the equality holds if and only if the second fundamental form on M is parallel and the rank of the matrices H and A are at most one. A non-negative function f is defined by h_2. Then the above inequality is reduced to

$$\Delta f \geq -6f^2 + c(n+2)f, \tag{4.26}$$

where the equality holds if and only if the second fundamental form on M is parallel and the rank of the matrices H and A are at most one. By (4.21), we have

$$\sum_j K_{\bar{j}jj\bar{j}} \leq \frac{r}{2} - n(n-1)a(M).$$

Hence we have by (4.20) and (3.10)

$$2na(M) \leq \frac{1}{2}\{cn(n+1) - 2h_2\} - n(n-1)a(M).$$

This yields that

$$(4.27)\qquad f=\sum_j \lambda_j = h_2 \leq \frac{1}{2}\{c-2a(M)\}n(n+1), \quad \lambda_j \geq 0,$$

where the first equality holds if and only if $K_{\bar{j}j j\bar{j}}=2a(M)$ and $K_{\bar{j}jk\bar{k}}=a(M)$ for any indices $j\neq k$. This means that $a(M)$ is bounded from above by definition, which implies that each eigenvalue λ_j is bounded. Since the Ricci curvature $S_{j\bar{j}}$ of M is given by (3.9) as

$$S_{j\bar{j}}=\frac{c}{2}(n+1)-\lambda_j,$$

it is also bounded. So, we can apply the generalized maximum principle due to Omori [9] and Yau [12] to the bounded function f, and we see that for any sequence $\{\varepsilon_m\}$ of positive numbers which converges to 0 as m tends to infinity, there exists a point sequence $\{p_m\}$ such that

$$\|\nabla f(p_m)\|<\varepsilon_m, \quad \Delta f(p_m)<\varepsilon_m, \quad \sup f-\varepsilon_m<f(p_m).$$

Thus, we have

$$(4.28)\qquad \lim_{m\to\infty}\Delta f(p_m)\leq \lim_{m\to\infty}\varepsilon_m=0, \quad \lim_{m\to\infty} f(p_m)=\sup f.$$

By (4.26) and (4.28), we see

$$\sup f\ \{\sup f-\frac{c}{6}(n+2)\}\geq 0,$$

which means that

$$\sup f=0 \quad \text{or} \quad \sup f\geq \frac{c}{6}(n+2).$$

If $\sup f=0$, then f vanishes identically on M because f is non-negative. Then M is totally geodesic.

Suppose that M is not totally geodesic. So, f satisfies

$$\sup f\geq \frac{c}{6}(n+2).$$

On the other hand, we have by (4.27)

$$\sup f\leq \frac{1}{2}\{c-2a(M)\}n(n+1).$$

Thus, we see that

$$a(M)\leq \frac{c}{6n(n+1)}(3n^2+2n-2).$$

We denote the right hand side of the above inequality by a_2, which is the constant depending only on the dimension n of M and the constant holomorphic sectional curvature c of the ambient space. Then, it is seen that the infimum $a(M)$ of the totally real bisectional curvatures of M satisfies $a(M)\leq a_2$ for the constant

$$a_2=\frac{c}{6n(n+1)}(3n^2+2n-2).$$

By (3.10), (4.22) and (4.24), we see

$$K_{\bar{j}jk\bar{k}} \geq \frac{1}{n-2}\{cn(n^2-1) - 2(n-1)h_2 - (2n^3-3n+2)b(M)\}$$

for any distinct indices j and k. By the definition of $a(M)$, we get

$$a(M) \geq \frac{1}{n-2}\{cn(n^2-1) - 2(n-1)h_2 - (2n^3-3n+2)b(M)\}.$$

On the other hand, by (3.8), it is seen that

$$K_{\bar{j}jk\bar{k}} = \frac{c}{2} - \sum_x h^x_{jk}\bar{h}^x_{jk} \leq \frac{c}{2}$$

for any distinct indices j and k, and hence it turns out to be $b(M) \leq c/2$, where the equality holds if and only if $h^x_{jk} = 0$ for any distinct indices j and k. Hence we have

$$h_2 \geq \frac{1}{4(n-1)}\{c - 2a(M)\}(n-2).$$

Since $a(M) \leq a_2$, we get

$$h_2 \geq \frac{c}{12n(n^2-1)}(n^2-4).$$

It completes the proof.

Remark 4.1. In Theorem 4.1, we shall remark M is not necessarily compact. Furthermore, on one hand, the theorem means that the zero point in the value distribution of h_2 is discrete. but on the other, Theorem A has no information about it.

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