Perturbation theorems on Morse theory for continuous functions

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Abstract

We use the extension of Morse theory for continuous functions on metric spaces in order to prove a stability property of lower critical values for such a function. More exactly, if X is a metric space, $f, g : X \longrightarrow \mathbb{R}$ are continuous and f has an isolated lower critical value, in suitable hypothesis g has a lower critical value. This property is still true if X is a G-metric space, where G is a compact Lie group, and f, g are continuous and invariant.

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1 Introduction

Morse theory for continuous functions on metric spaces was introduced in [16] and developed in [5]-[13] and independently in [17]- [18].

We are interested in the stability under perturbation of isolated critical values in this setting. More exactly, we want to know if two "close" continuous functions on a metric space have "close" critical values.

For C^2- functions on complete Riemann manifolds, this problem was analyzed in [19] and in [2] in equivariant context. We also mention [1]. The case of Finsler manifolds appears in [21] respectively in [3]. The same problem is studied in [15] and [14] for continuous functions, in a complete different approach.

2 Preliminaries

In this paper X is a metric space endowed with the metric d. If $x \in X$ and $r > 0$, then $B_r(x)$ denotes the open ball in X of center x and radius r.

Let $f : X \longrightarrow \mathbf{R}$ be a continuous function.

Definition 2.1 The *weak slope* of f at x, denoted by $|df|(x)$, is the supremum of all $\sigma \in [0, \infty)$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(x) \times [0, \delta] \longrightarrow X$ which satisfies the properties

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$$d(\mathcal{H}(y,t),y) \le t$$
$$f(\mathcal{H}(y,t)) \le f(y) - \sigma t$$

for all $y \in B_\delta(x)$ and $t \in [0,\delta]$.

This extended real number gives a generalization for the norm of Fréchet derivative. If X is an open set in a normed space and $f \in C^1(X,\mathbf{R})$, then $|df|(x) = \|f'(x)\|$, $\forall x \in X$. If X is a C^1-Finsler manifold and $f \in C^1(X,\mathbf{R})$, then $|df|(x) = \|f'(x)\|$, $\forall x \in X$ (see [16], [5]), so in this case the notion of lower critical point agrees with the classical one.

The function $x \mapsto |df|(x)$ is lower semicontinuous.

In general, if $f_0, f_1 : X \longrightarrow \mathbf{R}$ are continuous, it is not possible to compare $|d(f_0+f_1)|(x)$ with $|df_0|(x)$ or $|df_1|(x)$. By using [16], it is easy to see that for $f_0 : X \longrightarrow \mathbf{R}$ continuous and $f_1 \in C^1(X,\mathbf{R})$ the following inequalities hold:

$$-\|(df_1)(x)\| \le |d(f_0+f_1)|(x) - |df_0|(x) \le \|(df_1)(x)\|, \ \forall x \in X.$$

Definition 2.2 We call a point $x \in X$ a *lower critical point* of f if $|df|(x) = 0$. A real number c is called a *lower critical value* of f if $\exists\, x \in X$ such that $|df|(x) = 0$ and $f(x) = c$.

It is clear from the definition that if $x \in X$ is a local minimum point of f, then x is a lower critical point of f.

In the following, we use the notation $K(f) = \{x \in X |\ |df|(x) = 0\}$ for the lower critical set of f and $B(f) = f(K(f))$. If c is a real number, then $K_c(f) = K(f) \cap f^{-1}(c)$ is the lower critical set of level c of f and $f^c = \{x \in X |\ f(x) < c\}$ denotes the set of sublevel c of f.

Definition 2.3 We say that f satisfies the *Palais-Smale condition* on $A \subset X$, denoted by (PS), if for any sequence (x_n) in A such that $(f(x_n))$ is bounded and $|df|(x_n) \longrightarrow 0$, there exists a subsequence (x_{n_k}) converging to some $x \in A$.

The lower semicontinuity of $|df|$ implies the fact that a limit point of a subsequence (x_{n_k}) as in previous definition is a lower critical point of f.

3 The Second Deformation Lemma

Deformation theorems for continuous functions on complete metric spaces was proved in [6] - [8]. In [9], the Second Deformation Lemma was refined and the exact statement is the following:

Theorem 3.1 *Let X be a metric space, $f : X \to \mathbf{R}$ a continuous function, $a \in \mathbf{R}$ and $b \in \mathbf{R} \cup \{+\infty\}$ with $a < b$. Assume that for any $u \in [a,b)$ the set $f^{-1}([a,u])$ is complete and f satisfies the (PS)-condition on $f^{-1}([a,u])$, f has no critical point x with $a < f(x) < b$ and either $K_a(f) = \emptyset$ or the connected components of $K_a(f)$ are single points.*

Then there exists a deformation $\eta : f^b \times [0,1] \longrightarrow f^b$ such that:
(i) $f(\eta(x,t)) \le f(x)$;
(ii) if $x \in K_a(f)$, *then* $\eta(x,t) = x$;
(iii) $\eta(f^b, 1) \subset f^a \cup K_a(f)$.

In particular, $f^a \cup K_a(f)$ is a deformation retract of f^b.

4 A stability property of lower critical values

In this section $H_q(B,A)$ denotes the q^{th} relative singular homology group of the pair (B,A) with real coefficients, where $A \subset B$ and q is a nonnegative integer (see [20]). $H_*(B,A)$ denotes the graded group $(H_q(B,A))_q$.

We begin with a lemma which will be useful in the proof of the main theorem.

Lemma 4.1 ([19]) *Let A, X, B, A', Y, B' be topological spaces such that $A \subset X \subset B \subset A' \subset Y \subset B'$. Assume that $H_n(B,A) = 0$ and $H_n(B',A') = 0$, for any n nonnegative integer. Then there exist an injective homomorphism $h : H_n(A',A) \longrightarrow H_n(Y,X)$.*

We recall that for a deformation retract A' of A we have, for any nonnegative integer n, $H_n(A,A') = 0$. Moreover, if $A'' \subset A' \subset A$ and A' is a deformation retract of A, then, for any nonnegative integer n, we have $H_n(A,A'') = H_n(A',A'')$. (See, for instance, [20].)

We state now the stability property of lower critical values.

Theorem 4.1 *Let X be a complete metric space and $f, g : X \longrightarrow \mathbf{R}$ two continuous functions such that $c \in \mathbf{R}$ is the only lower critical value of f in $[c-\varepsilon, c+\varepsilon]$, where $\varepsilon > 0$. Assume that for any u in $[c-\varepsilon, c+\varepsilon)$, f satisfies the (PS)-condition on $f^{-1}([c-\varepsilon, u])$ and g satisfies the (PS)-condition on $g^{-1}([c-\varepsilon, u])$. Assume, also, that there exist m such that $H_m(f^{c+\varepsilon}, f^{c-\varepsilon}) \neq 0$ and $\delta > 0$ which depends on ε such that*

$$|f(x) - g(x)| \leq \delta, \ \forall x \in X.$$

Then there exists a lower critical value of g in the interval $[c-(\varepsilon-\delta), c+(\varepsilon-\delta)]$.

Proof: We follow the idea of [19]. The above inequality implies the inclusions

$$f^{c-\varepsilon} \subset g^{c-(\varepsilon-\delta)} \subset f^{c-(\varepsilon-2\delta)} \subset f^{c+(\varepsilon-2\delta)} \subset g^{c+(\varepsilon-\delta)} \subset f^{c+\varepsilon}$$

with $\varepsilon - 2\delta > 0$.

Because $[c-\varepsilon, c+\varepsilon] \cap B(f) = \{c\}$, in accord with the Second Deformation Lemma for continuous functions, we conclude that $f^{c-\varepsilon}$ is a deformation retract of $f^{c-(\varepsilon-2\delta)}$ and $f^{c+(\varepsilon-2\delta)}$ is a deformation retract of $f^{c+\varepsilon}$. Then we obtain $H_n(f^{c-(\varepsilon-2\delta)}, f^{c-\varepsilon}) = 0$ and $H_n(f^{c+\varepsilon}, f^{c+(\varepsilon-2\delta)}) = 0$, for all n positive integer. Apply Lemma 4.1 and it follows that

$$h : H_n(f^{c+(\varepsilon-2\delta)}, f^{c-\varepsilon}) \longrightarrow H_n(g^{c+(\varepsilon-\delta)}, g^{c-(\varepsilon-\delta)})$$

is injective. Because $f^{c+(\varepsilon-2\delta)}$ is a deformation retract of $f^{c+\varepsilon}$, we obtain

$$H_n(f^{c+(\varepsilon-2\delta)}, f^{c-\varepsilon}) = H_n(f^{c+\varepsilon}, f^{c-\varepsilon})$$

for any n positive integer.

We use the assumption $H_m(f^{c+\varepsilon}, f^{c-\varepsilon}) \neq 0$ and it follows that

$$H_m(g^{c+(\varepsilon-\delta)}, g^{c-(\varepsilon-\delta)}) \neq 0$$

and $[c-(\varepsilon-\delta), c+(\varepsilon-\delta)] \cap B(g) \neq \emptyset$. $\square$

Remark 4.1 It is obviously that in the hypothesis of Theorem 4.1, there exists at least a lower critical point of g.

Remark 4.2 We can easily prove the above theorem if f has p isolated lower critical values. We conclude that, in the corresponding hypothesis, g has at least p lower critical points.

Remark 4.3 The assumption $|f(x)-g(x)| \leq \delta = \frac{\varepsilon}{M}, \forall x \in X$, where $M > 2$, implies $f^{c-\varepsilon} \subset g^{c-(\varepsilon-\delta)} \subset g^{c-\frac{\varepsilon}{2}} \subset f^{c-(\frac{\varepsilon}{2}-\delta)} \subset f^{c+(\frac{\varepsilon}{2}-\delta)} \subset g^{c+\frac{\varepsilon}{2}} \subset g^{c+(\varepsilon-\delta)} \subset f^{c+\varepsilon}$. Then the shortest interval corresponding to g in the conclusion of Theorem 4.1 is $[c-\frac{\varepsilon}{2}, c+\frac{\varepsilon}{2}]$.

Remark 4.4 Homotopical stability of isolated critical points was studied in [10]. Recall that for $f : X \to \mathbf{R}$ continuous, $u \in X$, $c = f(u)$ and $U \subset X$ a neighborhood of u, the q^{th} critical group of f at u is

$$C_q(f;u) = H_q((f^c \cup \{u\}) \cap U, f^c \cap U)$$

and let $C_*(f;u) = \{C_q(f;u)\}_q$ (see [7]). Due to the excision property of singular homology, the definition of $C_*(f;u)$ does not depend on the particular choice of the neighborhood U.

This definition is justified by the following property:

Proposition 4.1 ([7]) *Let $f : X \to \mathbf{R}$ be continuous and $u \in X$. If $|df|(u) \neq 0$, then $C_*(f;u) = \{0\}_*$.*

This is equivalent with the fact that if $C_q(f;u)$ is nontrivial for some q, then u is a lower critical point of f.

If $f : X \to \mathbf{R}$, we set

$$\|f\|_\infty := \sup_X |f|$$

$$Lip(f) := \sup_{u \neq v} \frac{|f(u)-f(v)|}{d(u,v)}$$

$$\|f\|_{1,\infty} := \max\{\|f\|_\infty, Lip(f)\}$$

Theorem 4.2 ([10]) *Let X be a complete metric space, $f : X \to \mathbf{R}$ continuous, $Y \subset X$ open and $x_0 \in Y$. Assume that x_0 is the only lower critical point of f in $\overline{Y}$ and f satisfies the (PS) condition on $\overline{Y}$.*

Then there exists $\varepsilon > 0$ such that for every $g : X \to \mathbf{R}$ continuous which has an unique lower critical point $\overline{x}_0 \in \overline{Y}$ and satisfies the (PS) condition on $\overline{Y}$ and such that $\|g_{|Y} - f_{|Y}\|_{1,\infty} \leq \varepsilon$, we have $C_*(g;\overline{x}_0) = C_*(f;x_0)$.

5 The equivariant case

Let X be a metric space with the metric d. Assume that a compact Lie group G acts on X by isometric transformations. We will say that X is a metric G-space. Let $f : X \longrightarrow \mathbf{R}$ be a continuous invariant function.

Definition 5.1 The *equivariant weak slope* of f at x, denoted by $|d_G f|(x)$, is the supremum of all $\sigma \in [0, \infty)$ such that $\exists\, U$ an invariant neighborhood of x, $\exists\, \delta > 0$ and a continuous map $\mathcal{H} : U \times [0, \delta] \longrightarrow X$ which satisfies the properties

$$d(\mathcal{H}(y,t), y) \leq t$$

$$f(\mathcal{H}(y,t)) \leq f(y) - \sigma t$$

$$\mathcal{H}(\cdot, t) \text{is equivariant}$$

for all $y \in U$ and $t \in [0, \delta]$.

Remark that $|d_G f|(x) \leq |df|(x)$. The function $x \mapsto |d_G f|$ is lower semicontinuous and invariant (see [5]).

Definition 5.2 A point $x \in X$ is called a *lower G-critical point* of f if $|d_G f|(x) = 0$. A real number c is called a *lower G-critical value* of f if $\exists\, x \in X$ such that $|d_G f|(x) = 0$ and $f(x) = c$. An orbit $\mathcal{O}$ is called lower critical if $|d_G f|(x) = 0$, for some $x \in \mathcal{O}$.

The lower G-critical set at level c of f will be denoted by $K_{c,G}(f)$.

Definition 5.3 We say that f satisfies the *G-Palais-Smale condition* on an invariant subset A of X, denoted by $(PS)_G$, if for any sequence (x_n) in A such that $(f(x_n))$ is bounded and $|d_G f|(x_n) \longrightarrow 0$, there exists a subsequence (x_{n_k}) converging to some $x \in A$.

The Second Deformation Lemma extends to equivariant setting:

Theorem 5.1 *Let X be a metric G-space, $f : X \longrightarrow \mathbf{R}$ a continuous invariant function, $a \in \mathbf{R}$ and $b \in \mathbf{R} \cup \{+\infty\}$ with $a < b$. Assume that for any $u \in [a, b)$ the set $f^{-1}([a, u])$ is complete, f satisfies the $(PS)_G$-condition on $f^{-1}([a, u])$, f has no G-critical point x with $a < f(x) < b$ and either $K_{a,G}(f) = \emptyset$ or the connected components of $K_{a,G}(f)$ are parts of a certain critical orbit.*

Then there exists a deformation $\eta_G : f^b \times [0, 1] \longrightarrow f^b$ such that:
(i) $f(\eta_G(x,t)) \leq f(x)$;
(ii) if $x \in K_{a,G}(f)$, then $\eta_G(x,t) = x$;
(iii) $\eta_G(f^b, 1) \subset f^a \cup K_{a,G}(f)$;
(iv) $\eta_G(\cdot, t)$ is equivariant, for all $t \in [0, 1]$.

In particular, $f^a \cup K_{a,G}(f)$ is an equivariant deformation retract of f^b.

In order to prove the previous theorem, it is sufficient to make an average by means of Haar measure (see [4], Theorem 0.3.1), defining

$$\eta_G(x,t) = \int_G \eta(gx, t)\, dg$$

where $\eta(x,t)$ is given by Theorem 3.1.

We consider a G-equivariant homology theory h_*^G, for example we take the Borel homology (see [4]).

Let EG be a contractible space on which G acts freely. It is known that EG exists for any topological group and is uniquely determined up to G-homotopy. A standard model is $EG = G * G * \ldots$ (Milnor's construction).

For a G-space we define the homotopy quotient

$$X_G = EG \times_G X := (EG \times X)/G,$$

where G acts diagonally on $EG \times X$.

If (X, X') is a G-pair, we define

$$H_*^G(X, X') := H_*(EG \times_G X, EG \times_G X').$$

At first it seems more natural to take the homology of the orbit space; it is possible but difficult to deal with because the projection $X \longrightarrow X/G$ is not a bundle in general. If the action of G on X is free, then X_G is homotopy equivalent to X/G, hence $H_*^G(X) = H_*(X/G)$.

We can give now the equivariant version of Theorem 4.1:

Theorem 5.2 *Let X be a complete metric G-space and let $f, g : X \to \mathbf{R}$ be continuous invariant functions such that $c \in \mathbf{R}$ is the only lower G-critical value of f in $[c-\varepsilon, c+\varepsilon]$, where $\varepsilon > 0$. Assume that for any u in $[c-\varepsilon, c+\varepsilon)$, f satisfies the $(PS)_G$-condition on $f^{-1}([c-\varepsilon, u])$ and g satisfies the $(PS)_G$-condition on $g^{-1}([c-\varepsilon, u])$. Assume that there exist m such that $H_m^G(f^{c+\varepsilon}, f^{c-\varepsilon}) \neq 0$ and $\delta > 0$ which depends on ε such that*

$$|f(x) - g(x)| \leq \delta, \forall x \in X.$$

Then there exists a lower G-critical value of g in the interval $[c - (\varepsilon - \delta, c + (\varepsilon - \delta)]$ and consequently g has at least a lower critical G-orbit.

It is sufficient to adapt step by step the proof of Theorem 4.1 to the equivariant setting.

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