The Spectra and the Symmetric Structure on a Compact Symmetric Space

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Abstract

In this paper we study the influence of the spectra of different second order elliptic differential operator on fibre bundles over a compact Riemannian manifold (M, g).

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1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension n. Let V be a fibre bundle over M. We consider a second order elliptic differentiable operator of the cross section $C^{\infty}(V)$ of M. We consider the Laplace operator Δ, Laplace–Beltrami operator $\tilde{\Delta}$ and the one–parameter operators $D(\varepsilon) = \varepsilon\Delta + (1-\varepsilon)\tilde{\Delta}$, where $0 \leq \varepsilon \leq 1$. We study the influence of the spectra of D, ∇ and $D(\varepsilon)$ on the symmetric structure on (M, g).

The whole paper contains four paragraphs. Each of them is analyzed as follows.

The second paragraph has basic notions for some second order elliptic differential operator. We set some of the basic problems between the spectra and geometry.

The symmetric Riemannian manifold are studied in the third paragraph. We also study the influence of some spectra on the Riemannian symmetric structure on a compact manifold.

In the last paragraph we determine the symmetric manifolds which are determined by some spectra.

2 Second order elliptic differential operator

Let (M, g) be a compact Riemannian manifold of dimension n. From this we obtain the exterior algebra on M, that is

$$\wedge(M) = \overset{n}{\underset{k=1}{\oplus}} \wedge^k (M)$$

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where $\wedge^k(M)$ is the space of exterior k–forms $k = 0, 1, 2, ..., n$ and $\wedge^0(M) = C^\infty(M)$ the space of differential functions on M.

On these vector space $\wedge^k(M, R)$, $k = 0, 1, 2, ..., n$ we consider the Laplace operator Δ, that is

$$\Delta : \wedge^k(M, R) \to \wedge^k(M, R), \quad \Delta : \alpha \to \Delta\alpha, k = 0, 1, 2, ..., n.$$

Therefore we obtain $(n + 1)$ spectra denoted by

$$Sp^k(M, g, \Delta) = \{0 = ... < \lambda_1 = ...\lambda_1 < ...\infty \mid \Delta\alpha = \lambda\alpha, \ \lambda \in R\}.$$

We also have the Laplace–Beltrami operator

$$\tilde{\Delta} = \wedge^k(M, R) \to \wedge^k(M, R), \quad \tilde{\Delta} : \alpha \to \tilde{\Delta}\alpha, \ k = 1, ..., n - 1,$$

and hence we get $n - 1$ spectra, denoted by

$$Sp^k(M, g, \tilde{\Delta}) = \{0 = ... < \lambda_1 = ...\lambda_1 < ... < \infty \mid \tilde{\Delta}\alpha = \lambda\alpha, \lambda \in R\}.$$

From these operators we can construct one–parameter family $D(\varepsilon)$ of differentiable operators defined by

$$D(\varepsilon) = \varepsilon\Delta + (1 - \varepsilon)\tilde{\Delta} : \wedge^k(M, R) \to \wedge^k(M, R), \ D(\varepsilon) : \alpha \to D(\varepsilon)\alpha$$

$k = 1, 2, ..., n - 1$ and $0 \leq \varepsilon \leq 1$ and therefore we obtain n spectra denoted by

$$Sp^k(M, g, D(\varepsilon)) = \{0 = ... < \lambda_1 = ...\lambda_1 < ... < \infty \mid D(\varepsilon)\alpha = \lambda\alpha, \lambda \in R\}.$$

The above notions can be considered with different and more general views.

Let V be a vector bundle over a compact Riemannian manifold. The second order elliptic differential operator D_1 can be considered as a linear mapping of $C^\infty(V)$, that is

$$D_1 : C^\infty(V) \to C^\infty(V),$$

where $C^\infty(V)$ the set of cross sections of V. The leading symbol of D_1 is given by the metric tensor g.

The above cases are obtained by taking

$$V = \wedge^0(M, R) = C^\infty(M), \quad \text{or} \quad V = \wedge^1(M, R), \quad \text{or} \quad , ..., V = \wedge^n(M, R)$$

and the second order elliptic differential operator to be $\Delta, \tilde{\Delta}$ and $D(\varepsilon)$ respectively.

Now, we state three basic problems between spectra of second order elliptic differential operator and the geometry of a compact Riemannian manifold.

1 st *Let (M, g) be a compact Riemannian manifold of dimension n. Determine the spectra*

$$Sp^k(M, g, \Delta) \quad k = 0, 1, ..., n$$

$$Sp^k(M, g, \tilde{\Delta}), \quad Sp^k(M, g, D(\varepsilon)), \quad k = 1, ..., n - 1.$$

This problem can be put for any other second order elliptic differential operator or other differential operator.

2 nd *Let* (M, g) *be a compact Riemannian manifold of dimension* n*. Do some of the spectra* $Sp^k(M, g, D_1)$*, maybe with different* D_1*, determine some special structure on* (M, g) *?*

Such structures are Einstein, symmetric, contact structure etc.

This problem can be put as follows

Let (M, g) *and* (N, h) *be two compact Riemannian manifold, with*

$$Sp^k(M, g, D_1) = Sp^k(N, h, D_1)$$

for some k *and maybe with different operators* D_1*. If* (M, g) *is an Einstein, then is* (N, h) *an Einstein, or any other structure.*

3 rd *Let* (M, g) *be a compact Riemannian manifold of dimension* n*. Do some of the spectra* $Sp^k(M, g, D_1)$*, maybe with different* D_1*, determine the geometry on* (M, g) *?*

This problem can be put as follows

Let (M, g) *and* (N, h) *be two compact Riemannian manifolds of dimension* n*. If we have*

$$Sp^k(M, g, D_1) = Sp^k(n, h, D_1)$$

for some k*, and maybe with different operators* D_1*, then is* (M, g) *isometric onto* (N, h) *?*

3 Basic theorems for symmetric manifolds

Let (M, g) be a compact Riemannian symmetric manifold. We state basic theorems concerning of the compact Riemannian symmetric manifolds.

Theorem 3.1. (M_1, g_1), $(M_2, g_2), ..., (M_k, g_k)$ *be compact Riemannian manifolds. If their Riemannian product* $(M_1 \times ... \times M_k,\ g_1 \times ... \times g_k)$ *is a Riemannian symmetric manifold, then each of* (M_i, g_i), $i = 1, ..., k$ *is a Riemannian symmetric manifold.*

Theorem 3.2. *Let* (M, g) *be a compact irreducible Riemannian symmetric manifold. Then* M *is Einstein.*

Now, we state some theorems, which connect spectra and symmetric structure.

Theorem 3.3. ([1]) *Let* (M, g), (N, h) *be two Einstein manifolds, with* $Sp^k(M, g, \Delta) = Sp^k(N, h, \Delta)$, $k = 0, 1, 2$*. If* (M, g) *is locally symmetric so is* (N, h)*.*

We have improved the Theorem 3.3. The new theorem can be stated as follows

Theorem 3.4. ([3], [4]) *Let* $(M, g), (N, h)$ *be two compact Riemannian manifolds. We assume that* $Sp^k(M, g, \Delta) = Sp^k(N, h, \Delta)$ *for at least two values of* k*, where* $k = 0, 1, 2$*. Then if* (M, g) *is an Einstein so is* (N, h)*. If* (M, g) *is an Einstein and locally symmetric, so is* (N, h)*.*

If we assume that the manifold (M, g) is not Einstein then we have

Theorem 3.5. ([5]) *Let* (M, g) *and* (N, h) *be two orientable Riemannian manifolds. If* $Sp^k(M, g, \Delta) = Sp^k(N, h, \Delta)$ *for* $k = 0, 1, 2$ *and* $Sp^1(M, g, D(\varepsilon)) = Sp^1(N, h, D(\varepsilon))$ *for three distinct values of* $\varepsilon \neq 0$*, then if* (M, g) *is locally symmetric, so is* (N, h)*.*

From the above we conclude that the spectra of some second order elliptic differential operators determine the symmetric structure on a compact Riemannian manifold.

4 Symmetric manifolds determined by Spectra

Problem 4.1. *Let (M,g), (N,h) be two compact Riemannian manifolds. If (M,g) is Riemannian symmetric and $Sp^k(M,g,\Delta) = Sp^k(N,h,\Delta)$, $k = 0,1$, then is (M,g) isometric onto (N,h) ?*

Solution. The answer to the general case is very difficult. It can be solved only in special cases. We assume the Riemannian manifolds (M,g) and (N,h) are irreducible. Now we consider only the compact irreducible Riemannian symmetric manifolds which are:

(I) Compact simple Lie groups with the Riemannian metric coming from the Killing–Cartan form on its Lie algebra. These Lie groups are

Lie group G:	**Center Z(G):**	**dim G:**
$A_n = SU(n+1), \quad n \geq 1$	Z_{n+1}	$n(n+2)$
$B_n = SO(2n+1), \quad n \geq 2$	Z_2	$n(2n+1)$
$C_n = S_p(2n), \quad n \geq 3$	Z_2	$n(2n+1)$
$D_n = SO(2n), \quad n \geq 4$	$\begin{cases} Z_4 & \text{if } n = \text{odd} \\ Z_2 \oplus Z_2 & \text{if } n = \text{even} \end{cases}$	$n(2n+1)$
G_2	Z_1	14
F_4	Z_1	52
E_6	Z_1	78
E_7	Z_2	133
E_8	Z_3	248

(II) The irreducible compact symmetric Riemannian manifolds $M = \frac{G}{H}$ whose Riemannian metric on each of them comes from the restriction in m of the Killing–Cartan form on k, where m and k are the tangent space of M at origin and k the Lie algebra of G which are connected by the relation $k = t \oplus m$, and k the Lie algebra of G which are connected by the relation $k = t \oplus m$, where t is the Lie algebra of H. These are the following:

Symmetric manifolds:	**Rank:**	**Dimension:**
$SU(n)/SO(n)$	$n-1$	$\frac{1}{2}(n-1)(n+2)$
$SU(2n)/Sp(n)$	$n-1$	$n(n-1)$
$SU(p+q)/S(U_p \times U_q)$	$\min(p,q)$	$2pq$
$SO(p+q)/SO(p) \times SO(q)$	$\min(p,q)$	pq
$SO(n)/U(n)$	$[n/2]$	$n(n-1)$
$Sp(p+q)/Sp(p) \times Sp(q)$	$\min(p,q)$	$4pq$
$G_2/SU(2) \times SU(2)$	2	8
$F_4/SU(q)$	1	16
$F_4/Sp(3) \times SO(2)$	4	28
$E_6/Sp(4)$	6	42
$E_6/SU(6) \times SU(2)$	4	40
$E_6/SO(10) \times SO(2)$	2	32
E_6/F_4	2	26
$E_7/SU(8)$	7	70
$E_7/SO(12) \times SO(2)$	4	64
$E_7/E_6 \times SU(2)$	3	54
$E_8/SO(16)$	8	128
$E_8/E_7 \times SU(2)$	4	112

Now, we shall prove the following theorem

Theorem 4.2. *Let (E_7, g) be the exceptional Lie group with the Riemannian metric coming from the Killing–Cartan form on the Lie algebra e_7. Let (M, h) be a compact simply connected irreducible Riemannian manifold with property $Sp^k(E_7, g, \Delta) = Sp^k(M, h, \Delta)$ for $k = 0, 1$. Then (M, h) is isometric onto (E_7, g).*

Proof. It is known that (E_7, g) is compact simply connected Einstein symmetric manifold of dimension 133. From this and the conditions

$$Sp^0(E_7, g, \Delta) = Sp^0(M, h, \Delta), \quad Sp^1(E_7, g, \Delta) = Sp^1(M, h, \Delta) \tag{4.1}$$

we conclude that (M, h) is an Einstein locally symmetric manifold. Since (M, h) is compact simply connected, we conclude that (M, h) is a symmetric manifold. From the condition (4.1) we conclude that

$$\dim E_7 = \dim M = 133$$

Therefore the symmetric manifold (M, h) is E_7 or S^{133} because there are the only irreducible symmetric manifolds as we can see from the above list of irredicuble Riemannian symmetric manifolds with this dimension 133 which is a prime number. Because $Sp^0(M, h, \Delta) \neq Sp^0(S^{133}, g_0, \Delta)$, we conclude that (M, h) is isometric onto (E_7, g) q.e.d.

From the above we have the corollary

Corollary 4.3.. *The Spectra $Sp^k(E_7, g, \Delta)$, $k = 0, 1$, where E_7 is the exceptional Lie group with metric g coming from the Killing– Cartan form on the Lie algebra e_7 of E_7, determine the geometry of (E_7, g).*

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