Two Sandwich Theorems for Linear Operators and the Moment Problem

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Abstract

We give a direct proof for theorem 2 [13] (which is equivalent to theorem 1 [15]). Then we apply theorems 1 [15] and 4 [15] to some concrete spaces of sequences or functions which have a Schauder basis. The polynomials $x_j(t) = t^j$, $j \in N$ considered in the classical moment problem, are replaced by the elements of the Schauder basis.

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1 Introduction

Before stating the abstract moment problem, we recall some definitions. A subset X_+ of a real vector space X is said to be a convex cone if $X_+ + X_+ \subset X_+$ and $\alpha X_+ \subset X_+$ for any $\alpha \in [0, \infty)$. An ordered vector space is a vector space X endowed with an order relation defined by a convex cone $X_+ \subset X$ in the following way: $x_1 \leq x_2$ iff $x_2 - x_1 \in X_+$. X_+ is said to be the positive cone of X. A vector lattice is an ordered vector space Y such that for any $y_1,\ y_2 \in Y$, there exists the supremum $\sup\{y_1,\ y_2\} =: y_1 \vee y_2$. An order complete vector lattice is a vector lattice in which any family $\{y_j : j \in J\} \subset Y$ bounded from above has a supremum $\sup\{y_j : j \in J\} =: \vee_{j \in J}\, y_j \in Y$. For a deep study of ordered vector spaces see [6] or [20].

The abstract moment problem may be stated in the following way. One give two ordered vector spaces X, Y and two families of elements $\{x_j : j \in J\} \subset X$, $\{y_j : j \in J\} \subset Y$. One also gives a convex operator $p : X \to Y$. The problem is to find necessary and sufficient conditions on y_j (which are called *moments* since they generalize the classical moments), for the existence of a linear operator $f \in L(X, Y)$, with the following properties:

(1.1) $$f(x_j) = y_j \quad \forall j \in J \quad \text{(the moment conditions)},$$

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(1.2) $$f(X_+) \subset Y_+ \quad \text{(positivity)},$$

(1.3) $$f(x) \leq p(x) \quad \forall x \in X \quad \text{(the generalization of the continuity)}.$$

In the classical moment problem we have $Y = R$, X is a space of functions on an interval $I \subset R$ containing the polynomials $x_j(t) = t^j$, $j \in N$, $t \in I$ and $p : X \to R$ is a seminorm on X. X_+ is usually a convex cone such that p is monotone $(0 \leq x_1 \leq x_2 \Rightarrow p(x_1) \leq p(x_2))$. In the moment problem we are interested in the existence, unicity and construction of the solution $f \in L(X, Y)$. The main purpose of the present work is to characterize the existence of f. If we note $X_0 := Sp\{x_j : j \in J\}$ and if we suppose that

$$f_0 : X_0 \to Y, \ f_0 \left(\sum_{j \in F} \lambda_j x_j \right) := \sum_{j \in F} \lambda_j y_j$$

($F \subset J$ being a finite subset), is well defined, then the problem of the existence of the solution is in fact the problem of extending f_0 to a linear operator $f \in L(X, Y)$ which has the properties (1.2) and (1.3). When Y is an order complete vector lattice, the abstract moment problem is solved by theorem $1'$, which is equivalent to theorem 1, both of them being stated and proved below (here we give a direct proof for theorem 1). The Hahn-Banach theorem is a particular case of theorem 1. It may be obtained taking in theorem 1 $X_+ := \{0\}$, when the order relation on X is the equality relation. The theorem of H. Bauer (see [3] or [20]) may be easily obtained from theorem 1. The corollary 2 [9, p.336] may be also got using theorem $1'$. In [16] we applied theorem $1'$ to some calssical spaces X of functions (we considered $X = C^1([0, b]),\ C^2([0, b]),\ C^1([0, b_1] \times [0, b_2]),\ L^1([0, b_1] \times [0, b_2]),\ BV([a, b])$). On the other hand, in [17] we proved theorem 2 of the present work and we applied it to some spaces of functions and measures. In section 3 of the present work we prove some applications of the two general theorems of section 2.

2 General theorems

Theorem 1. *Let X be an ordered vector space, let Y be an order complete vector lattice and let $p : X \to Y$ be a convex operator. Let $X_0 \subset X$ be a vector subspace and let $f_0 \in L(X_0, Y)$ be a linear operator. The following statements are equivalent:*

(a) there exists a linear and positive extension $f \in L(X, Y)$ of f_0 such that $f(x) \leq p(x) \quad \forall x \in X$;

(b) $f_0(x') \leq p(x) \quad \forall (x', x) \in X_0 \times X$ with $x' \leq x$.

This theorem was published in [13], without proof. An indirect proof was published in [14], where we deduced it from a more general result. Here we give a direct proof.

Proof of theorem 1.
(a) $\Rightarrow$ (b) is almost obvious ($f_0(x') = f(x') \le f(x) \le p(x) \quad \forall (x', x) \in X_0 \times X$ with $x' \le x$, since $f(x') \le f(x)$ by the positivity and linearity of f).
(b) $\Rightarrow$ (a) Let $\mathcal{S} := \{(S, f_s) : X_0 \subset S \subset X,\ S$ is a vector subspace of $X,\ f_s \in L(S, Y),\ f_s/X_0 = f_0,\quad f_s(x) \ge 0 \quad \forall x \in S \cap X_+$ and $(x', x) \in S \times X,\ x' \le x$, imply $f_s(x') \le p(x)\}$.

We consider the following natural order relation on $\mathcal{S} : (S_1, f_{s_1}) < (S_2, f_{s_2})$ iff $S_1 \subset S_2$ and $f_{s_2}/S_1 = f_{s_1}$. It is easy to see that $\mathcal{S}$ is inductively ordered with respect to this order relation. Let $(M, f_M) \in \mathcal{S}$ be a maximal element of $\mathcal{S}$, which exists by Zorn's lemma. To finish the proof, it is sufficient to prove that $M = X$. Supposing the contrary, let $\tilde{x} \in X \setminus M$. We construct an element $(\tilde{M}, f_{\tilde{M}}) \in \mathcal{S}$, where $\tilde{M} := M \oplus Sp\{\tilde{x}\},\ f_{\tilde{M}} : \tilde{M} \to Y$ being a linear extension of f_M. This will contradict the maximality of (M, f_M) in $\mathcal{S}$. We have to choose $\tilde{y} \in Y$ such that defining $f_{\tilde{M}} : \tilde{M} \to Y$ by $f_{\tilde{M}}(m + \lambda\tilde{x}) := f_M(m) + \lambda\tilde{y}$, to have $(\tilde{M}, f_{\tilde{M}}) \in \mathcal{S}$. So, we must show that

(2.1) $\qquad m_1 + \alpha\tilde{x} \in X_+,\ m_1 \in M,\ \alpha \in R,$ imply $f_M(m_1) + \alpha\tilde{y} \ge 0$ in Y,

(2.2) $m_2 + \beta\tilde{x} \le x,\ m_2 \in M,\ \beta \in R,\ x \in X,$ imply $f_M(m_2) + \beta\tilde{y} \le p(x)$ in Y.

For $\alpha = 0$, (2.1) is true since $(M, f_M) \in \mathcal{S}$. For $\beta = 0$, (2.2) is accomplished by the same reason. For $\alpha \ne 0$ (2.1) is equivalent to (2.1.1) and (2.1.2) taken together, where:

(2.1.1)
$$m_1 + \lambda_1\tilde{x} \in X_+,\ m_1 \in M,\ \lambda_1 > 0 \Rightarrow$$
$$\Rightarrow f_M(m_1) + \lambda_1\tilde{y} \ge 0,\ \text{i.e. } \tilde{y} \ge -f_M(m_1)/\lambda_1,$$

(2.1.2)
$$\tilde{m}_1 + \mu_1\tilde{x} \in X_+,\ \tilde{m}_1 \in M,\ \mu_1 < 0 \Rightarrow$$
$$f_M(\tilde{m}_1) + \mu_1\tilde{y} \ge 0,\ \text{i.e. } \tilde{y} \le -f_M(\tilde{m}_1)/\mu_1.$$

Hence (2.1) is equivalent to (2.1'), where:

(2.1')
$$y_1 := -f_M(m_1)/\lambda_1 \le \tilde{y} \le -f_M(\tilde{m}_1)/\mu_1 =: \tilde{y}_1,$$
$$\lambda_1 > 0,\ \mu_1 < 0,\ m_1 + \lambda_1\tilde{x} \in X_+,\ \tilde{m}_1 + \mu_1\tilde{x} \in X_+.$$

Similarly, (2.2) is equivalent to (2.2'):

(2.2') $\qquad \tilde{y}_2 := (1/\mu_2)[p(\tilde{x}') - f_M(\tilde{m}_2)] \le \tilde{y} \le (1/\lambda_2)[p(x') - f_M(m_2)] =: y_2,$

where
$$\lambda_2 > 0,\ \mu_2 < 0,\ m_2 + \lambda_2\tilde{x} \le x',\ \tilde{m}_2 + \mu_2\tilde{x} \le \tilde{x}'.$$

To find an $\tilde{y} \in Y$ which fulfills (2.1′) and (2.2′), we must prove the following four inequalities:

(2.3)
$$y_1 \le \tilde{y}_1,\ y_1 \le y_2,\ \tilde{y}_2 \le \tilde{y}_1,\ \tilde{y}_2 \le y_2.$$

Supposing that (2.3) are proved, we may choose $\tilde{y}$ such that

$$y_1 \vee \tilde{y}_2 \leq \tilde{y} \leq \tilde{y}_1 \wedge y_2.$$

The proof of the inequalities (2.3) is not difficult. Here we prove that $y_1 \leq y_2$. Let $\lambda_1 > 0,\ \lambda_2 > 0,\ m_1, m_2 \in M$ such that $m_1+\lambda_1\tilde{x} \geq 0$ and $m_2+\lambda_2\tilde{x} \leq x' \in X$. Then we get:

$$-(1/\lambda_1)m_1 \leq \tilde{x} \leq (1/\lambda_2)(x' - m_2),$$

which imply

$$\lambda_2[-(1/\lambda_1)m_1 + (1/\lambda_2)m_2] \leq x'. \tag{2.4}$$

On the other hand, $(M, f_M) \in \mathcal{S}$ and (2.4) imply:

$$\lambda_2[-(1/\lambda_1)f_M(m_1) + (1/\lambda_2)f_M(m_2)] \leq p(x'),$$

which may be rewritten as follows:

$$-f_M(m_1)\lambda_1 \leq (1/\lambda_2)[p(x') - f_M(m_2)],$$

i.e.

$$y_1 \leq y_2.$$

Theorem 1′. (Theorem 1 [15]). *Let X, Y, p be as in theorem 1 stated above. Let $\{x_j : j \in J\} \subset X,\ \{y_j : j \in J\} \subset Y$. The following statements are equivalent:*

(a) there exists $f \in L(X,Y)$ such that $f(x) \geq 0\ \forall x \in X_+,\ f(x_j) = y_j\ \forall j \in J$ and $f(x) \leq p(x)\ \forall x \in X$;

(b) for any finite subset $F \subset J$ and any $\{\lambda_j : j \in F\} \subset R$, the relation $\sum_{j\in F} \lambda_j x_j \leq x$ in X implies $\sum_{j\in F} \lambda_j y_j \leq p(x)$ in Y.

Theorem 1′ is a rewritting of theorem 1 (we take in theorem 1 $X_0 := Sp\{x_j : j \in J\}$, etc).

Theorem 2. (Theorem 4 [15] and 2.1. [17]). *Let $X, Y, \{x_j : j \in J\},\ \{y_j : j \in J\}$ be as in theorem 1′ and let $f_1, f_2 \in L(X,Y)$. Let us consider the following statements:*

(a) there exists $f \in L(X,Y)$ such that $f(x_j) = y_j,\ \forall j \in J$ and $f_1(z) \leq f(z) \leq f_2(z) \quad \forall z \in X_+$;

(b) for any finite subset $F \subset J$ and any $\{\lambda_j : J \in F\} \subset R$, we have:

$$\begin{gathered}\textstyle\sum_{j\in F} \lambda_j x_j = z_2 - z_1 \quad \textit{with} \\ z_1, z_2 \in X_+ \Rightarrow \textstyle\sum_{j\in F} \lambda_j y_j \leq f_2(z_2) - f_1(z_1);\end{gathered} \tag{2.5}$$

If X is a vector lattice, we also consider the statement (b′):

(b′) $f_1(z) \leq f_2(z)\ \forall z \in X_+$ and for any finite subset $F \subset J$ and any $\{\lambda_j : j \in F\} \subset R$, we have

$$\sum_{j\in F} \lambda_j y_j \leq f_2\left(\left(\sum_{j\in F} \lambda_j x_j\right)^+\right) - f_1\left(\left(\sum_{j\in F} \lambda_j x_j\right)^-\right); \tag{2.6}$$

(for each $x \in X$, we note $x^+ := x \vee 0,\ x^- := (-x) \vee 0$ and we have $x = x^+ - x^-\ \forall x \in X$);

(c) if $x_j \in X_+\ \forall j \in J$, then $f_1(x_j) \leq y_j \leq f_2(x_j)\ \forall j \in J$.

Then (b) $\Leftrightarrow$ (a) $\Rightarrow$ (c) and, if X is a vector lattice, we have (b′) $\Leftrightarrow$ (b) $\Leftrightarrow$ (a) $\Rightarrow$ (c).

3 Applications

Theorem 3. *Let X be a real separable Hilbert space and let $\{x_j : j \in N\}$ a fixed orthonormal basis in X. Let $X_+ := \{x \in X : < x, x_j >\geq 0 \ \forall j \in N\}$ and let $\{y_j : j \in N\} \subset X_+$ such that if we note $\rho_m := \sum_{j=0}^{\infty} < y_j, x_m >$ we must have $\sum_{m=0}^{\infty} \rho_m^2 < \infty$. Then there exists $f \in L(X, X)$, $f(X_+) \subset X_+$, $f(x_j) = y_j \ \forall j \in N$ and $f(x) \leq \| x \| \tilde{y}$, where*

$$\tilde{y} := \sum_{m=0}^{\infty} \rho_m \, x_m, \ x \in X.$$

Proof. We shall apply theorem $1'$, (b) $\Rightarrow$ (a). Let $n \in N$, let $\{\lambda_0, ..., \lambda_n\} \subset R$ and let $x \in X$ such that

$$\sum_{j=0}^{n} \lambda_j x_j \leq x = \sum_{m=0}^{\infty} < x, x_m > x_m$$

By the definition of X_+, this implies

$$\lambda_j \leq < x, x_j > \quad \forall j \in \{0, 1, ..., n\}. \tag{3.1}$$

On the other hand, $y_j \in X_+ \quad \forall j \in N$, is equivalent to

$$< y_j, x_m > \geq 0 \quad \forall (j, m) \in N^2. \tag{3.2}$$

So, from (3.1) and (3.2) we deduce

$$\sum_{j=0}^{n} \lambda_j y_j = \sum_{j=0}^{n} \lambda_j \left(\sum_{m=0}^{\infty} < y_j, x_m > x_m \right) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^{n} \lambda_j < y_j, x_m > \right) x_m \leq$$

$$\leq \sum_{m=0}^{\infty} \left(\sum_{j=0}^{n} < x, x_j >< y_j, x_m > \right) x_m \leq \|x\| \left[\sum_{m=0}^{\infty} \left(\sum_{j=0}^{n} < y_j, x_m > \right) x_m \right] \leq$$

$$\|x\| \left(\sum_{m=0}^{\infty} \rho_m x_m \right) = \|x\| \tilde{y} =: p(x),$$

where $\|x\| =< x, x >^{1/2}$. By theorem $1'$, (b) $\Rightarrow$ (a), the conclusion follows.

The theorem is proved.

We go on by two applications of theorem 2. We recall the following notations:

$$l^1 := \{(\alpha_0, ..., \alpha_n, ...) \in R^N : \sum_{j=0}^{\infty} | \alpha_j | < \infty\},$$

$$l^{\infty} := \{(\beta_0, ..., \beta_n, ...) \in R^N : sup_{j \in N} | \beta_j | < \infty\},$$

We consider the operator $f_2 : l^1 \to l^{\infty}$ defined by:

$$f_2((\alpha_0,...,\alpha_n,...)) = (\beta_0,...,\beta_n,...),$$

where

$$\beta_n := \sum_{k=0}^{n} \alpha_k.$$

In l^1 and l^∞ we consider the convex cone of sequences which have all their components positive. It is clear that $l^1 \subset l^\infty$ and $\forall x \in l^1_+$, we have $x \le f_2(x)$. It is also well known that l^∞ is an order complete vector lattice. So, we may apply theorem 2, (b′) ⇒ (a), for $X := l^1,\ Y := l^\infty,\ f_1(x) = x\ \forall x \in X$ and f_2 defined as above. We get the following result.

Theorem 4. *Let $X := l^1,\ Y := l^\infty$, let $x_j \in X$, $x_j := (0,...,0,1,0,...,0,...),\ j \in N,\ \{y_j : j \in N\} \subset X \subset Y,\ y_j = \sum_{m=0}^{\infty} \alpha_m^{(j)} x_m$. The following statements are equivalent:*

(a) there exists $f \in L(X,Y),\ f(x_j) = y_j\ \forall j \in N,\ x \le f(x) \le f_2(x)\ \forall x \in X_+$;

(b) for any $n \in N$ and any $\{\lambda_0,...,\lambda_n\} \subset R$, we have:

$$(b_1) \qquad \sum_{j=0}^{n} \lambda_j\, \alpha_m^{(j)} \le \lambda_0^+ + ... + \lambda_{m-1}^+ + \lambda_m, \quad \textit{if} \quad 0 \le m \le n,$$

and

$$(b_2) \qquad \sum_{j=0}^{n} \lambda_j\, \alpha_m^{(j)} \le \lambda_0^+ + ... + \lambda_p^+ + ... + \lambda_n^+, \quad \textit{if} \quad m \ge n+1, m \in N$$

(c) the following three conditions are fulfilled:

$$(c_1) \qquad \alpha_m^{(j)} = 0 \quad \forall j \in N, \quad \forall m \in N \textit{ such that } 0 \le m \le j-1,$$

$$(c_2) \qquad \alpha_j^{(j)} = 1 \quad \forall j \in N,$$

$$(c_3). \qquad \alpha_m^{(j)} \in [0,1] \quad \forall j \in N, \quad \forall m \ge j+1, \quad m \in N$$

Proof. To prove (a) ⇔ (b), we use (a) ⇔ (b′) of theorem 2. We have only to show that the relations (b_1) and (b_2) (together) are equivalent to (2.6). We have:

$$\sum_{j=0}^{n} \lambda_j\, y_j = \sum_{j=0}^{n} \lambda_j \left(\sum_{m=0}^{\infty} \alpha_m^{(j)} x_m \right) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^{n} \lambda_j \alpha_m^{(j)} \right) x_m$$

and so, (2.6) may be written as follows:

$$\sum_{m=0}^{\infty} \left(\sum_{j=0}^{n} \lambda_j \alpha_m^{(j)} \right) x_m \le f_2((\lambda_0^+,...,\lambda_k^+,...,\lambda_n^+,0,0,...) - (\lambda_0^-,...,\lambda_k^-,...,\lambda_n^-,0,0,...) =$$

$= (\lambda_0^+, \lambda_0^+ + \lambda_1^+, ..., \lambda_0^+ + ... + \lambda_n^+,\ \lambda_0^+ + ... + \lambda_n^+, 0, ...) - (\lambda_0^-, \lambda_1^-, ..., \lambda_n^-, 0, ...) =$
$= (\lambda_0, \lambda_0^+ + \lambda_1, ..., \lambda_0^+ + ... + \lambda_{n-1}^+ + \lambda_n,\ \lambda_0^+ + ... + \lambda_{n-1}^+ + \lambda_n^+,\ \lambda_0^+ + ... + \lambda_{n-1}^+ + \lambda_n^+, ...)$.

By the definition of Y_+, (2.6) is equivalent to the statement (b) of theorem 4.

(a) $\Rightarrow$ (c) is almost obvious. Indeed, since $x_j \in X_+$, we have:

$$x_j = (0, ..., 0, 1, 0, ...) \le y_j = (\alpha_0^{(j)}, ..., \alpha_j^{(j)}, ..., \alpha_m^{(j)}, ...) \le T_2(x_j) = (0, ..., 0, 1, 1, 1, ...)$$

which imply (c).

(c) $\Rightarrow$ (b) To prove (c) $\Rightarrow$ (b_1), let $n \in N$ and $m \in N$, $m \le n$.

From $(c_1), (c_2)$ and (c_3) we deduce:

$$\sum_{j=0}^{n} \lambda_j \alpha_m^{(j)} = \sum_{j=0}^{m-1} \lambda_j \alpha_m^{(j)} + \lambda_m \alpha_m^{(m)} + \sum_{j=m+1}^{n} \lambda_j \alpha_m^{(j)} =$$

$$\sum_{j=0}^{m-1} \lambda_j \alpha_m^{(j)} + \lambda_m \le \sum_{j=0}^{m-1} \lambda_j^+ + \lambda_m = \lambda_0^+ + ... + \lambda_{m-1}^+ + \lambda_m.$$

This proves (b_1). To finish the proof, we have to show that (c) $\Rightarrow$ (b_2).

Let $m, n \in N$, $m \ge n+1$. If $j \in N$, $j \le n \le m-1$, then $m \ge j+1$ and, by (c_3), $\alpha_m^{(j)} \in [0, 1]$. This implies:

$$\sum_{j=0}^{n} \lambda_j\ \alpha_m^{(j)} \le \sum_{j=0}^{n} \lambda_j^+,$$

i.e. (b_2). The theorem is proved.

We go on by an application of theorem 2 to a space of analitic functions.

Let $\rho > 0$. We denote by A_ρ the set of all complex functions, defined on the open disk $\mid z \mid < \rho$ of the complex plane, which can be represented as the sum of an absolutely convergent series

$$x(z) = \sum_{j=0}^{\infty} \alpha_j z^j,$$

the coefficients α_j being real numbers. Then $X = A_\rho$ is a real vector space which can be ordered by the convex cone

$$X_+ := \left\{ x \in A_\rho : x(z) = \sum_{j=0}^{\infty} \alpha_j\, z^j, \quad \alpha_j \ge 0 \quad \forall j \in N \right\}. \tag{3.3}$$

It is easy to see that X is an order complete vector lattice.

Theorem 5. *Let $X = A_\rho$ and let X_+ be the cone defined by (3.3). Let us denote $x_j(z) := z^j$, $j \in N$, $\mid z \mid < \rho$ and let us consider the function $g \in X$, $g(z) = 1 + z$. On the other hand, let $\{y_j : j \in N\} \subset X$ be a sequence in*

X, $y_j(z) = \sum_{m=0}^{\infty} \alpha_m^{(j)} z^m$, $j \in N$, $\mid z \mid < \rho$. Let us consider the following statements:

(a) there exists $f \in L(X,X)$ such that $f(x_j) = y_j \quad \forall j \in N$, $x \leq f(x) \leq xg \; \forall x \in X_+$;

(b) for any $n \in N$ and any $\{\lambda_0, \lambda_1, ..., \lambda_n\} \subset R$, we have:

$$(b_1) \qquad \sum_{j=0}^{n} \lambda_j \alpha_0^{(j)} \leq \lambda_0,$$

$$(b_2) \qquad \sum_{j=0}^{n} \lambda_j \alpha_m^{(j)} \leq \lambda_{m-1}^{+} + \lambda_m, \quad \forall m \in \{1, 2, ..., n\},$$

$$(b_3) \qquad \sum_{j=0}^{n} \lambda_j \alpha_{n+1}^{(j)} \leq \lambda_n^{+},$$

$$(b_4) \qquad \sum_{j=0}^{n} \lambda_j \alpha_m^{(j)} \leq 0 \quad \forall m \in \{n+2, n+3, ...\};$$

(c) the $\alpha_m^{(j)}$ fulfill the conditions:

$$(c_1) \qquad \alpha_m^{(j)} = 0 \quad j \in N, \quad \forall m \in N \backslash \{j, j+1\},$$

$$(c_2) \qquad \alpha_j^{(j)} = 1 \quad \forall j \in N,$$

$$(c_3) \qquad 0 \leq \alpha_{j+1}^{(j)} \leq 1 \quad \forall j \in N.$$

Then we have (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Proof. For (a) $\Leftrightarrow$ (b) we apply theorem 2, (a)$\Leftrightarrow$ (b$'$), for $Y = X$, $f_1(x) = x$, $f_2(x) = xg$, $\forall x \in X$. We check that the assertion (b) of theorem 5 is equivalent to the assertion (b$'$) of theorem 2. We remark that for any $x = \sum_{m=0}^{\infty} \alpha_m x_m \in X_+$, the relation $f_1(x) := x \leq f_2(x) := xg$ is true. Indeed, we have:

$$(xg)(z) = x(z)g(z) = \left(\sum_{m=0}^{\infty} \alpha_m z^m \right) (1+z) = \sum_{0}^{\infty} \alpha_m z^m +$$

$$+ \sum_{0}^{\infty} \alpha_m z^{m+1} = \sum_{m=0}^{\infty} \alpha_m z^m +$$

$$+ \sum_{m=1}^{\infty} \alpha_{m-1} z^m = \alpha_0 + \sum_{m=1}^{\infty} (\alpha_m + \alpha_{m-1}) z^m.$$

So, we have got:

$$xg = \alpha_0 x_0 + \sum_{m=1}^{\infty} (\alpha_m + \alpha_{m-1}) x_m \geq \alpha_0 x_0 + \sum_{m=1}^{\infty} \alpha_m x_m$$

since $\alpha_{m-1} \geq 0 \quad \forall m \in \{1, 2, ...\}$ by the definition of $X_+ \ni x$.
So, we have only to verify the equivalence (2.6) $\Leftrightarrow$ (b) of theorem 5. Let us write (2.6) in our particular case. Let $n \in N$, $\{\lambda_0, \lambda_1, ..., \lambda_n\} \subset R$. Then (2.6) may be written in the following way:

$$\begin{aligned} \textstyle\sum_{j=0}^{n} \lambda_j y_j \quad &= \quad \textstyle\sum_{j=0}^{n} \lambda_j \left(\sum_{m=0}^{\infty} \alpha_m^{(j)} x_m \right) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^{n} \lambda_j \alpha_m^{(j)} \right) x_m \leq \\ &\leq f_2(\left(\left(\textstyle\sum_{j=0}^{n} \lambda_j x_j \right)^+ \right) - f_1 \left(\left(\textstyle\sum_{j=0}^{n} \lambda_j x_j \right)^- \right) = \\ &= f_2 \left(\textstyle\sum_{j=0}^{n} \lambda_j^+ x_j \right) - f_1 \left(\textstyle\sum_{j=0}^{n} \lambda_j^- x_j \right) = \\ &= \left(\textstyle\sum_{m=0}^{n} \lambda_m^+ x_m \right) (1 + x_1) - \textstyle\sum_{m=0}^{n} \lambda_m^- x_m = \\ &= \textstyle\sum_{m=0}^{n} \lambda_m^+ x_m + \sum_{m=0}^{n} \lambda_m^+ x_{m+1} - \sum_{m=0}^{n} \lambda_m^- x_m = \\ &= \textstyle\sum_{m=0}^{n} \lambda_m x_m + \sum_{m=1}^{n+1} \lambda_{m-1}^+ x_m = \\ &= \textstyle\sum_{m=1}^{n} \left(\lambda_{m-1}^+ + \lambda_m \right) x_m + \lambda_0 x_0 + \lambda_n^+ x_{n+1}. \end{aligned}$$

By the definition of X_+, this is equivalent to (b) of theorem 5.

(a) $\Rightarrow$ (c) Since $x_j \in X_+$, we have from (a):

$$x_j = f_1(x_j) \leq \sum_{m=0}^{\infty} \alpha_m^{(j)} x_m = y_j = f(x_j) \leq f_2(x_j) = x_j(1 + x_1) = x_j + x_{j+1}.$$

By the definition of X_+, (c) follows.

(c) $\Rightarrow$ (b)

$$(b_1) \qquad \sum_{j=0}^{n} \lambda_j \alpha_0^{(j)} = \lambda_0 \alpha_0^{(0)} + \sum_{j=1}^{n} \lambda_j \alpha_0^{(j)} = \lambda_0,$$

by (c_1) and (c_2).

$$(b_2) \qquad m \in \{1, 2, ..., n\} \Rightarrow \sum_{j=0}^{n} \lambda_j \alpha_m^{(j)} = \lambda_m \alpha_m^{(m)} + \lambda_{m-1} \alpha_m^{(m-1)} =$$

$$= \lambda_m + \lambda_{m-1} \alpha_m^{(m-1)} \leq \lambda_m + \lambda_{m-1}^+$$

by $(c_1), (c_2)$ and (c_3).

$$(b_3) \qquad \sum_{j=0}^{n} \lambda_j \alpha_{n+1}^{(j)} = \lambda_n \alpha_{n+1}^{(n)} \leq \lambda_n^+, \quad \text{by } (c_1) \text{ and } (c_3).$$

$$(b_4) \qquad \sum_{j=0}^{n} \lambda_j \alpha_m^{(j)} = 0$$

if $m \geq n + 2$, by (c_1). The theorem is proved.

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