Entropy of Hermite polynomials with application to the harmonic oscillator

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Dedicated to Jean Meinguet

Abstract

We analyse the entropy of Hermite polynomials and orthogonal polynomials for the Freud weights $w(x) = \exp(-|x|^m)$ on $\mathbb{R}$ and show how these entropies are related to information entropy of the one-dimensional harmonic oscillator. The physical interest in such entropies comes from a stronger version of the Heisenberg uncertainty principle, due to Bialynicki-Birula and Mycielski, which is expressed as a lower bound for the sum of the information entropies of a quantum-mechanical system in the position space and in the momentum space.

1 The information entropies of the harmonic oscillator

The Schrödinger equation in D dimensions is given by

$$\left(-\frac{1}{2}\nabla^2 + V\right)\psi = E\psi,$$

where the potential V and the wave function ψ are functions of $x = (x_1, \ldots, x_D)$ and E is the energy. The wave function ψ is normalized in such a way that $\rho(x) = |\psi(x)|^2$ is a probability density in position space. If $\hat{\psi}(p)$ is the Fourier transform of the wave function ψ, then by the Plancherel formula $\gamma(p) = |\hat{\psi}(p)|^2$ is also a probability density, but now in the momentum space. The information entropy for the quantum-mechanical system with potential V is then given by

$$S_\rho = -\int_{\mathbb{R}^D} \rho(x) \log \rho(x)\, dx$$

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in the position space, and

$$S_\gamma = -\int_{\mathbb{R}^D} \gamma(p) \log \gamma(p)\, dp$$

in the momentum space. These two entropies have allowed Bialynicki-Birula and Mycielski [4] to find a new and stronger version of the Heisenberg uncertainty principle. For a quantum mechanical system in D dimensions this new uncertainty relation is

$$S_\rho + S_\gamma \geq D(1 + \log \pi), \tag{1.1}$$

which expresses in a quantitative way that it is impossible to get precise information in both position and momentum space. High values of S_ρ are associated with low values of S_γ, and vice versa.

In this paper we restrict ourselves to the one-dimensional harmonic oscillator, in which case $D = 1$ and the potential has the form

$$V(x) = \frac{\lambda^2}{2} x^2.$$

The possible energy levels are

$$E_n = n + \frac{1}{2}, \qquad n = 0, 1, 2, \ldots,$$

and the wave functions are Hermite functions $e^{-\lambda x^2/2} H_n(x\sqrt{\lambda})$, where H_n ($n = 0, 1, 2, \ldots$) are Hermite polynomials [20]. This gives

$$\rho_n(x) = \frac{\sqrt{\lambda/\pi}}{2^n n!} e^{-\lambda x^2} H_n^2(x\sqrt{\lambda}),$$

and for Hermite functions, the Fourier transform is again a Hermite function, hence

$$\gamma_n(p) = \frac{1}{\sqrt{\lambda\pi} 2^n n!} e^{-p^2/\lambda} H_n^2(p/\sqrt{\lambda}).$$

The entropy in the position space then becomes

$$\begin{aligned} S_{\rho_n} = \log \sqrt{\pi/\lambda} + \log 2^n n! &+ \frac{1}{\sqrt{\pi} 2^n n!} \int_{-\infty}^{\infty} y^2 H_n^2(y) e^{-y^2}\, dy \\ &- \frac{1}{\sqrt{\pi} 2^n n!} \int_{-\infty}^{\infty} H_n^2(y) \log H_n^2(y)\ e^{-y^2}\, dy, \end{aligned} \tag{1.2}$$

and the entropy in the momentum space is

$$\begin{aligned} S_{\gamma_n} = \log \sqrt{\pi\lambda} + \log 2^n n! &+ \frac{1}{\sqrt{\pi} 2^n n!} \int_{-\infty}^{\infty} y^2 H_n^2(y) e^{-y^2}\, dy \\ &- \frac{1}{\sqrt{\pi} 2^n n!} \int_{-\infty}^{\infty} H_n^2(y) \log H_n^2(y)\ e^{-y^2}\, dy. \end{aligned} \tag{1.3}$$

The problem thus consists of computing integrals of the form

$$\int_{-\infty}^{\infty} y^2 H_n^2(y) e^{-y^2}\, dy$$

and entropy integrals of the Hermite polynomials

$$\int_{-\infty}^{\infty} H_n^2(y) e^{-y^2} \log H_n^2(y)\, dy.$$

We will consider a more general situation involving integrals for orthogonal polynomials $p_n(x)$ $(n = 0, 1, 2 \ldots)$ for the weight function $w(x) = e^{-|x|^m}$ on $\mathbb{R}$. This weight function is known as a Freud weight and the case $m = 2$ corresponds to Hermite polynomials. Freud studied these weights and the corresponding orthogonal polynomials in detail for $m = 2, 4, 6$ and formulated conjectures regarding the asymptotic behaviour of the largest zero of the orthogonal polynomials and the behaviour of the coefficients in the three-term recurrence relation for the orthonormal polynomials. If $x_{n,n}$ is the largest zero of p_n, then Freud conjectured that

$$\lim_{n\to\infty} n^{-1/m} x_{n,n} = \left(\frac{2}{\lambda_m}\right)^{1/m}, \tag{1.4}$$

where

$$\lambda_m = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} = \frac{m}{\pi} \int_{-1}^{1} \frac{|x|^m}{\sqrt{1-x^2}}\, dx.$$

If the three-term recurrence relation for the orthonormal polynomials is

$$x p_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x),$$

and the orthonormal polynomials are given by $p_n(x) = \gamma_n x^n + \cdots$, with $\gamma_n > 0$, then $a_n = \gamma_{n-1}/\gamma_n$ and Freud conjectured that

$$\lim_{n\to\infty} n^{-1/m} a_n = \lim_{n\to\infty} n^{-1/m} \frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2}\left(\frac{2}{\lambda_m}\right)^{1/m}. \tag{1.5}$$

The conjecture regarding the largest zero behaviour was proved by Rakhmanov [18], the conjecture regarding the behaviour of the recurrence coefficients was proved by Alphonse Magnus [12] for even integer values of m, and by Lubinsky, Mhaskar, and Saff [10] for arbitrary $m > 0$. For more on these Freud conjectures we refer to Nevai [16] or to [21, §4.2]. For orthogonal polynomials with respect to Freud weights the quantities to be computed are

$$\int_{-\infty}^{\infty} |x|^m p_n^2(x) e^{-|x|^m}\, dx \tag{1.6}$$

and

$$\int_{-\infty}^{\infty} p_n^2(x) e^{-|x|^m} \log p_n^2(x)\, dx. \tag{1.7}$$

Usually we will take the orthonormal polynomials in these integrals. For small n these integrals give information related to the ground state and a few excited states, the asymptotic behaviour as $n \to \infty$ gives information about the so called Rydberg states.

The analytic computation of the integral in (1.6) is easy. By symmetry we have

$$\int_{-\infty}^{\infty} |x|^m p_n^2(x) e^{-|x|^m}\,dx = 2\int_0^\infty x^m p_n^2(x) e^{-x^m}\,dx,$$

and integration by parts gives

$$\begin{aligned}\int_{-\infty}^{\infty} |x|^m p_n^2(x) e^{-|x|^m}\,dx &= -\frac{2}{m}\int_0^\infty x p_n^2(x)(e^{-x^m})'\,dx \\ &= \frac{2}{m}\int_0^\infty [p_n^2(x) + 2xp_n(x)p_n'(x)] e^{-x^m}\,dx.\end{aligned}$$

By symmetry again this is

$$\int_{-\infty}^{\infty} |x|^m p_n^2(x) e^{-|x|^m}\,dx = \frac{1}{m}\int_{-\infty}^\infty [p_n^2(x) + 2xp_n(x)p_n'(x)] e^{-|x|^m}\,dx$$

Observe that we can write $xp_n'(x) = np_n(x) + \pi_{n-1}(x)$, where π_{n-1} is a polynomial of degree at most $n-1$. Then by orthonormality we finally get

$$\int_{-\infty}^{\infty} |x|^m p_n^2(x) e^{-|x|^m}\,dx = \frac{1+2n}{m}. \tag{1.8}$$

For orthonormal Hermite polynomials $p_n(x) = (2^n n!\sqrt{\pi})^{-1/2} H_n(x)$ we choose $m=2$ and find

$$\int_{-\infty}^{\infty} y^2 H_n^2(y) e^{-y^2}\,dy = (n+\frac{1}{2})2^n n!\sqrt{\pi}.$$

Observe that for the ground state $n=0$ this already gives (since $H_0 = 1$)

$$S_{\rho_0} = \log\sqrt{\pi/\lambda} + \frac{1}{2}, \quad S_{\gamma_0} = \log\sqrt{\pi\lambda} + \frac{1}{2},$$

so that $S_{\rho_0} + S_{\gamma_0} = 1 + \log\pi$, and thus the Bialynicki-Birula-Mycielski inequality (1.1) is sharp in this case.

The evaluation of the entropy integral (1.7) is much more difficult and requires more knowledge of Hermite polynomials and orthogonal polynomials with Freud weights.

2 A heuristic approach using logarithmic potentials

If we denote the zeros of p_n by $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$, then $p_n(x) = \gamma_n \prod_{j=1}^n (x - x_{j,n})$. The entropy integral (1.7) then becomes

$$\int_{-\infty}^{\infty} p_n^2(x) e^{-|x|^m} \log p_n^2(x)\,dx = \log\gamma_n^2 + 2\sum_{j=1}^n \int_{-\infty}^\infty p_n^2(x) e^{-|x|^m} \log|x - x_{j,n}|\,dx.$$

If we denote by

$$U(z;\mu) = \int \log\frac{1}{|z-x|}\,d\mu(x)$$

the (logarithmic) potential of a probability measure μ, and by μ_n the measure with density $p_n^2(x)e^{-|x|^m}$, then μ_n is a probability measure and

$$U(z;\mu_n) = -\int_{-\infty}^{\infty} p_n^2(x)e^{-|x|^m} \log|z-x|\,dx,$$

so that the entropy integral is also given by

$$\int_{-\infty}^{\infty} p_n^2(x)e^{-|x|^m} \log p_n^2(x)\,dx = \log\gamma_n^2 - 2\sum_{j=1}^{n} U(x_{j,n};\mu_n).$$

This connection with the potential of μ_n indicates that we can get an idea of the asymptotic behaviour of the entropy integral in (1.7) if we know the asymptotic behaviour of γ_n, the weak convergence of the measures μ_n and the asymptotic distribution of the zeros $x_{j,n}$ $(1 \le j \le n)$. This information is available in the literature. For the asymptotic behaviour of γ_n we have the result that

$$\lim_{n\to\infty} n^{1/m}\gamma_n^{1/n} = 2(\frac{e\lambda_m}{2})^{1/m},$$

which follows from (and is weaker than) the Freud conjecture (1.5). For the behaviour of the measures μ_n we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f\left((\frac{\lambda_m}{2n})^{1/m}x\right) p_n^2(x)e^{-|x|^m}\,dx = \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} f(x)\,d\mu_e(x),$$

where μ_e is the equilibrium measure of the interval $[-1,1]$ and f is an arbitrary continuous function of at most polynomial growth at $\pm\infty$ [6]. Finally, for the distribution of the zeros we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f\left((\frac{\lambda_m}{2n})^{1/m}x_{j,n}\right) = \int_{-1}^{1} f(x)v_m(x)\,dx,$$

where f is a continuous function of at most polynomial growth at $\pm\infty$ and v_m is the Ullman weight

$$v_m(x) = \frac{m}{\pi}\int_{|x|}^{1}\frac{t^{m-1}}{\sqrt{t^2-x^2}}\,dt, \qquad x\in[-1,1].$$

This result was proved by Rakhmanov [18] and also follows from the Freud conjecture (1.5) (see [21, p. 123]). Observe that for $m=2$ the Ullman density is

$$v_2(x) = \frac{2}{\pi}\sqrt{1-x^2}, \qquad x\in[-1,1],$$

which is also known as the semi-circle density and which is well known to describe the asymptotic distribution of eigenvalues of some random matrices (Wigner [23]). For $m\to\infty$ the Ullman density tends to the density of the measure μ_e

$$v_\infty(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, \qquad x\in[-1,1].$$

Observe that

$$U(z;\mu_n) = -\int_{-\infty}^{\infty} \log|z/c_n - x/c_n|\, p_n^2(x) e^{-|x|^m}\, dx - \log c_n = U(z/c_n;\hat{\mu}_n) - \log c_n,$$

where $\hat{\mu}_n$ is the rescaled measure with $\hat{\mu}_n(A) = \mu_n(c_n A)$. Then the entropy integral becomes

$$\frac{1}{n}\int_{-\infty}^{\infty} p_n^2(x) \log p_n^2(x)\ e^{-|x|^m}\, dx$$
$$= 2\left(\log \gamma_n^{1/n} - \frac{1}{n}\sum_{j=1}^{n} U\left((\frac{\lambda_m}{2n})^{1/m} x_{j,n};\hat{\mu}_n\right) - \log(\frac{\lambda_m}{2n})^{1/m}\right).$$

As $n \to \infty$ the right hand side is of the form

$$2\left(\log 2 + \log(\frac{e\lambda_m}{2})^{1/m} - \int_{-1}^{1} U(x;\mu_e) v_m(x)\, dx - \log(\frac{\lambda_m}{2})^{1/m}\right),$$

and since the potential of the equilibrium measure μ_e has the constant value $\log 2$ on $[-1,1]$, this quantity reduces to $2/m$. Hence, one expects to have

$$\lim_{n\to\infty} \frac{1}{n}\int_{-\infty}^{\infty} p_n^2(x)\log p_n^2(x)\ e^{-|x|^m}\, dx = \frac{2}{m}.$$

This is, however, a heuristic reasoning: the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U\left((\frac{\lambda_m}{2n})^{1/m} x_{j,n};\hat{\mu}_n\right) = \int_{-1}^{1} U(x;\mu_e) v_m(x)\, dx$$

does not follow immediately from the results given earlier. In fact the weak convergence of the measure $\hat{\mu}_n$ and the asymptotic distribution of the zeros only gives

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} U\left((\frac{\lambda_m}{2n})^{1/m} x_{j,n};\hat{\mu}_n\right) \geq \int_{-1}^{1} U(x;\mu_e) v_m(x)\, dx$$

(see [17, Thm. 2.1 on p. 168]), hence more work is needed to get the required result. Furthermore, this only shows that the entropy integral (1.7) behaves like $2n/m + o(n)$, and we would like know the term $o(n)$ in more detail.

3 Entropy of Hermite polynomials

Let us consider the entropy integral for orthonormal Hermite polynomials

$$\int_{-\infty}^{\infty} p_n^2(y)\log p_n^2(y)\ e^{-y^2}\, dy,$$

where $p_n(x) = (2^n n! \sqrt{\pi})^{-1/2} H_n(x)$. It is more convenient to consider

$$E_n = \int_{-\infty}^{\infty} p_n^2(y) e^{-y^2} \log[p_n^2(y) e^{-y^2}]\, dy,$$

since we already know exactly the integral

$$\int_{-\infty}^{\infty} p_n^2(y)e^{-y^2}\log e^{-y^2}\,dy$$

which is given in (1.8). It is known that the weighted polynomial $e^{-x^2/2}H_n(x)$ attains its extremum on the finite interval $[-\sqrt{2n+1},\sqrt{2n+1}]$ (see [20, Sonin's theorem in §7.31 combined with §7.6]). Moreover the successive relative maxima of $e^{-x^2/2}|H_n(x)|$ form an increasing sequence for $x \geq 0$ [20, Thm. 7.6.3 on p. 177], hence the maximum of $e^{-x^2}H_n^2(x)$ will be attained near the *turning point* $x_n = \sqrt{2n+1}$. We can use the asymptotic formula near this turning point [20, p. 201]

$$e^{-x^2/2}H_n(x) = \frac{3^{1/3}2^{n/2+1/4}\sqrt{n!}}{\pi^{3/4}n^{1/12}}[A(t)+O(n^{-2/3})],$$

where $x = x_n - 2^{-1/2}3^{-1/3}n^{-1/6}t$ and A is the Airy function, which is defined as the only bounded solution (up to a constant factor) of the differential equation $y'' + xy/3 = 0$, to find that

$$\max_{-\infty<x<\infty} e^{-x^2}p_n^2(x) = \max_{-x_n\leq x\leq x_n} e^{-x^2}p_n^2(x) = O(n^{-1/6}).$$

Therefore, we split up the integration as

$$\begin{aligned} E_n =& \int_{-x_n}^{x_n} p_n^2(y)e^{-y^2}\log[p_n^2(y)e^{-y^2}]\,dy + \int_{|x|>x_n} p_n^2(y)e^{-y^2}\log[p_n^2(y)e^{-y^2}]\,dy \\ :=& E_{1,n}+E_{2,n}. \end{aligned}$$

Here $E_{1,n}$ will be the dominating term and $E_{2,n}$ will be small compared to $E_{1,n}$. For $E_{1,n}$ we can use the Plancherel-Rotach asymptotics [20, Thm. 8.22.9]

$$e^{-x^2/2}H_n(x) = \frac{\sqrt{2^n n!}}{\sqrt{\sqrt{\pi n/2}\sin\theta}}\left[\sin\left(\frac{2n+1}{4}[\sin 2\theta - 2\theta] + \frac{3\pi}{4}\right) + O(1/n)\right],$$

which holds for $x = \sqrt{2n+1}\cos\theta$ and $\epsilon \leq \theta \leq \pi-\epsilon$, with $\epsilon > 0$. For the orthonormal polynomials this is

$$e^{-x^2}p_n^2(x) = \frac{2}{\pi\sqrt{2n}\sin\theta}\left[\sin^2\left(\frac{2n+1}{4}[\sin 2\theta - 2\theta] + \frac{3\pi}{4}\right) + O(1/n)\right],$$

for $x = \sqrt{2n+1}\cos\theta$ and $\epsilon \leq \theta \leq \pi - \epsilon$. With this we have

$$\begin{aligned} E_{1,n} = \frac{2}{\pi}\int_0^\pi \sin^2&\left((n+\frac{1}{2})\frac{\sin 2\theta - 2\theta}{2} + \frac{3\pi}{4}\right) \\ &\times \log\left[\frac{2}{\pi\sqrt{2n}}\frac{\sin^2\left((n+\frac{1}{2})[\sin 2\theta - 2\theta]/2 + \frac{3\pi}{4}\right)}{\sin\theta}\right] d\theta\;[1+O(1/n)]. \end{aligned}$$

The integral consists of three terms

$$E_1 = \frac{2}{\pi}\int_0^\pi \sin^2\left((n+\frac{1}{2})\frac{\sin 2\theta - 2\theta}{2} + \frac{3\pi}{4}\right) d\theta\;\left(\log\frac{2}{\pi\sqrt{2n}}\right),$$

$$E_2 = \frac{2}{\pi}\int_0^\pi \sin^2\left((n+\frac{1}{2})\frac{\sin 2\theta - 2\theta}{2} + \frac{3\pi}{4}\right) \times \log\sin^2\left((n+\frac{1}{2})\frac{\sin 2\theta - 2\theta}{2} + \frac{3\pi}{4}\right)\, d\theta,$$

and

$$E_3 = -\frac{2}{\pi}\int_0^\pi \sin^2\left((n+\frac{1}{2})\frac{\sin 2\theta - 2\theta}{2} + \frac{3\pi}{4}\right) \log\sin\theta\, d\theta.$$

For the integral in E_1 we have

$$\frac{1}{\pi}\int_0^\pi \sin^2\left((n+\frac{1}{2})\frac{\sin 2\theta - 2\theta}{2} + \frac{3\pi}{4}\right) d\theta = \frac{1}{\pi}\int_0^\pi \sin^2\phi\, d\phi + O(1/n),$$

so that

$$E_1 = \left(\log\frac{2}{\pi\sqrt{2n}}\right)(1 + O(1/n)).$$

For E_2 and E_3 we can use a lemma from [2, Lemma 2.1] to evaluate these integrals asymptotically:

Lemma. *Let g be a continuous real function which is periodic with period π, $f \in L^1[0,\pi]$ and γ a measurable function, almost everywhere finite on $[0,\pi]$. Then*

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^\pi g(n\theta + \gamma(\theta))f(\theta)\, d\theta = \frac{1}{\pi}\int_0^\pi g(\theta)\, d\theta\, \frac{1}{\pi}\int_0^\pi f(\theta)\, d\theta.$$

Observe that $-[\sin 2\theta - 2\theta]/2$ is an increasing function, mapping $[0,\pi]$ to $[0,\pi]$. By a change of variables, we can then use this lemma to find

$$\lim_{n\to\infty} E_2 = \frac{2}{\pi}\int_0^\pi \sin^2\phi \log\sin^2\phi\, d\phi = 1 - 2\log 2,$$

and

$$\lim_{n\to\infty} E_3 = -\frac{2}{\pi}\int_0^\pi \sin^2\phi\, d\phi\, \frac{1}{\pi}\int_0^\pi \log\sin\theta\, d\theta = \log 2.$$

Together this gives

$$E_{1,n} = -\log\pi - \log\sqrt{2n} + 1 + o(1).$$

The remaining term $E_{2,n}$ can be shown to be of smaller order, so that finally

$$\int_{-\infty}^\infty p_n^2(y)e^{-y^2}\log[p_n^2(y)e^{-y^2}]\, dy = -\log\pi - \log\sqrt{2n} + 1 + o(1).$$

Together with (1.8) this gives

$$\int_{-\infty}^\infty p_n^2(y)e^{-y^2}\log p_n^2(y)\, dy = n + \frac{3}{2} - \log\pi - \log\sqrt{2n} + o(1).$$

4 Entropy of orthogonal polynomials with Freud weights

An asymptotic formula of Plancherel-Rotach type for orthonormal polynomials with Freud weight $e^{-|x|^m}$ was conjectured by Nevai in [16], and this conjecture was proved by Rakhmanov [19]. He showed that

$$p_n^2(x)e^{-|x|^m} = \frac{2}{\pi}\frac{\cos^2\Phi_n(x)}{\sqrt{x_n^2-x^2}}\left[1+o(1)\right],$$

for $|x| \le (1-\epsilon)x_n$, where $x_n = (2n/\lambda_m)^{1/m}$ and

$$\Phi_n(x) = \frac{2n+1}{2}\int_x^{x_n}\left(1-\frac{x^m}{t^m}\right)\frac{dt}{\sqrt{x_n^2-t^2}} - \frac{\pi}{4}, \qquad x\in[0,x_n].$$

With the scaling $x = x_n y$ this gives

$$\begin{aligned}\Phi_n(x_n y) =& \frac{2n+1}{2}\int_y^1\left(1-\frac{y^m}{t^m}\right)\frac{dt}{\sqrt{1-t^2}} - \frac{\pi}{4}\\ :=& \frac{2n+1}{2}\phi(y) - \frac{\pi}{4},\end{aligned}$$

so that

$$p_n^2(x_n y)e^{-2n|y|^m/\lambda_m} = \frac{2}{\pi x_n}\frac{\cos^2[(n+\frac{1}{2})\phi(y)-\pi/4]}{\sqrt{1-y^2}}\left[1+o(1)\right], \qquad |y|\le 1-\epsilon.$$

Observe that for $m=2$ we have

$$\phi(\cos\theta) = \frac{1}{2}(\sin 2\theta - 2\theta),$$

so that this is compatible with the Plancherel-Rotach formula for Hermite polynomials, taking into account that now we use $x_n = \sqrt{2n}$ rather than $x_n = \sqrt{2n+1}$. A similar analysis as in the previous section is now possible, but we will not give the details, which can be found in [22]. It suffices to say that with this analysis one again has

$$\lim_{n\to\infty}\frac{1}{n}\int_{-\infty}^{\infty} p_n^2(x)e^{-|x|^m}\,\log p_n^2(x)\,dx = \frac{2}{m},$$

as we obtained by heuristic reasoning in Section 2. A more accurate formula can be obtained in a similiar way as for the Hermite polynomials, using some of the results obtained by Rakhmanov [18], [19] and Lubinsky and Saff [9], [11], [15]. This has been done in [2], where it was proved that

$$\begin{aligned}&-\int_{-\infty}^{\infty} p_n^2(x)e^{-|x|^m}\log p_n^2(x)\,dx\\ &\qquad = \frac{2n+1}{m} - \frac{\log 2n}{m} + \frac{1}{m}\log\frac{\sqrt{\pi}\Gamma(m/2)}{2\Gamma((m+1)/2)} + 1 - \log\pi + o(1).\end{aligned}$$

Observe that this is compatible with the case $m=2$ for Hermite polynomials.

Of independent interest for the asymptotic behaviour of orthogonal polynomials with Freud weights, would be to find an asymptotic formula in the neighborhood of the turning point $x_n = (2n/\lambda_m)^{1/m}$, similar to the Airy type asymptotics for Hermite polynomials. Such an asymptotic formula is not known yet, but the existence of such a formula is made plausible by an asymptotic formula for the largest zeros of certain Freud polynomials. Máté, Nevai and Totik [13] [14] proved that for positive even integers m the largest zeros of the orthogonal polynomials $p_n(x)$ for the weight function $e^{-|x|^m}$ satisfy

$$n^{-1/m} x_{n-k+1,n} = \left(\frac{2}{\lambda_m}\right)^{1/m} \left(1 - \frac{i_k}{(6n^2m^2)^{1/3}}\right) + o(n^{-2/3}),$$

where $i_1 < i_2 < \cdots$ are the zeros of the Airy function.

5 Conclusion

The described analysis (more details are given in [2]) gives for the orthonormal Hermite polynomials $p_n(x) = (2^n n! \sqrt{\pi})^{-1/2} H_n(x)$ as $n \to \infty$

$$-\int_{-\infty}^{\infty} p_n^2(x) \log p_n^2(x)\, e^{-x^2}\, dx = -n + \log\sqrt{2n} - \frac{3}{2} + \log\pi + o(1).$$

In terms of the information entropies (1.2) and (1.3) this gives

$$S_{\rho_n} + S_{\gamma_n} = \log n - 2 + \log 2 + 2\log\pi + o(1).$$

This shows that the sum $S_{\rho_n} + S_{\gamma_n}$ grows like $\log n$ as $n \to \infty$.

The result described in this paper is a joint effort of J. S. Dehesa and R. J. Yáñez from Granada (Spain), A. I. Aptekarev and V. S. Buyarov from Moscow (Russia), and the the present author. Together we also examined the harmonic oscillator in $D > 1$ dimensions and the hydrogen atom in D dimensions. For these quantum mechanical systems the information entropy in position space and in momentum space can be expressed in terms of the entropy of Laguerre polynomials, and the angular part also requires the entropy of Gegenbauer polynomials. For these classical orthogonal polynomials we were able to obtain similar results as those described in the present paper. See [1]–[3], [5], [22], and [24] for these extensions.

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