On the convergence of multivariate Padé approximants

Annie Cuyt [*], Doron Lubinsky

Dedicated to Jean Meinguet
at the occasion of his 65th birthday

Abstract

While the concept of Padé approximant is essentially several centuries old, its multivariate version dates only from the early seventies. In the last century many univariate convergence results were proven, describing the approximation power for several function classes. It is not our aim to give a general review of the univariate case, but to discuss only these theorems that have a multivariate counterpart. The first section summarizes the theorems under discussion, in a univariate framework. The second and third section discuss the multivariate versions of these theorems, for different approaches to the multivariate Padé approximation problem.

1 Convergence of univariate Padé approximants.

Given a function $f(z)$, through its series expansion at a certain point in the complex plane, the Padé approximant $[n/m]^f$ of degree n in the numerator and m in the denominator for f is defined by

$$f(z) = \sum_{i=0}^{\infty} c_i z^i$$
$$p(z) = \sum_{i=0}^{n} a_i z^i$$

[*] Onderzoeksleider NFWO

$$q(z) = \sum_{i=0}^{m} b_i z^i$$
$$(fq - p)(z) = \sum_{i \geq n+m+1} d_i z^i$$

with $[n/m]^f$ equal to the irreducible form of p/q. Usually these Padé approximants are ordered in a table with double entry, the numerator degree n indexing the rows and the denominator degree m indexing the columns:

$$\begin{array}{ccccc} [0/0] & [0/1] & [0/2] & [0/3] & [0/4] \\ [1/0] & [1/1] & [1/2] & [1/3] & \dots \\ [2/0] & [2/1] & [2/2] & [2/3] & \dots \\ [3/0] & [3/1] & [3/2] & [3/3] & \dots \\ [4/0] & \vdots & \vdots & \vdots & \ddots \end{array}$$

When discussing convergence results of Padé approximants, one compares a sequence of approximants in the table with the given function f. The selection of an appropriate sequence is possible using information about f. If the given function has a fixed number of poles in a certain region, it makes sense to consider a sequence of Padé approximants with fixed denominator degree, in other words a column in the table. If the function has a countable number of singularities, it is wiser to consider a diagonal or ray in the table. We shall now list a number of famous theorems that have also been generalized to the multivariate case. In comparing the results we have to distinguish between 'uniform' convergence, which is an overall convergence with the Chebyshev-norm of the error tending to zero, and convergence in 'measure' or 'capacity', where one has convergence except for an area of disruption of which the location is usually unknown but of which the size can be made arbitrarily small. In this text we restrict ourselves to the notion of measure, to avoid the discussion of multivariate generalizations of the notion of capacity later on. If more general results hold however, we shall refer the reader to the literature. We denote

$$B(0,r) = \{z \in \mathbb{Z} : |z| < r\}$$
$$\overline{B}(0,r) = \{z \in \mathbb{Z} : |z| \leq r\}$$
$$B((0,0),r) = \{(x,y) \in \mathbb{Z}^2 : ||(x,y)|| < r\}$$
$$\overline{B}((0,0),r) = \{(x,y) \in \mathbb{Z}^2 : ||(x,y)|| \leq r\}$$
$$B((0,0);r_1,r_2) = \{(x,y) \in \mathbb{Z}^2 : |x| < r_1, |y| < r_2\}$$
$$\overline{B}((0,0);r_1,r_2) = \{(x,y) \in \mathbb{Z}^2 : |x| \leq r_1, |y| \leq r_2\}$$

and Λ_{2p} for the Lesbesgue-measure in $\mathbb{Z}^p$.

Montessus de Ballore [Mont05]. *Let the function $f(z)$ be meromorphic in $\overline{B}(0,r)$ with poles z_i in $B(0,r)$ of total multiplicity M. Then the sequence $\{[n/M]^f\}_{n\in\mathbb{N}}$ converges uniformly to f on compact subsets of $\overline{B}(0,r)\setminus\{z_i\}$ with z_i attracting zeros of the Padé denominator according to its multiplicity:*

$$\lim_{n\to\infty} ||[n/M]^f - f||_K = 0 \qquad \textit{compact } K \subset \overline{B}(0,r) \setminus \{z_1,\ldots,z_M\}$$

Zinn-Justin [Zinn71]. *Let the function $f(z)$ be meromorphic in $\overline{B}(0,r)$ with poles z_i in $B(0,r)$ of total multiplicity M. Then the sequence $\{n_k/m_k\}_{k\in\mathbb{N}}$ with $m_k \geq M$ and $\lim_{k\to\infty} n_k/m_k = \infty$, converges in $B(0,r)$ in measure to f:*

$$\forall\epsilon,\delta,\ \exists\kappa : |f(z) - [n_k/m_k]^f(z)| < \epsilon \textit{ for } k \geq \kappa \textit{ and } z \in B(0,r) \setminus \mathcal{E} \textit{ with } \Lambda_2(\mathcal{E}) < \delta$$

Karlsson and Wallin [KaWa77]. *Let the function $f(z)$ be meromorphic in $\overline{B}(0,r)$ with poles z_i in $B(0,r)$ of total multiplicity M. Then for $m \geq M$ there exist points $\zeta_1,\ldots,\zeta_{m-M}$ in $\mathbb{Z}$ and there exists a subsequence of $\{[n/m]^f\}_{n\in\mathbb{N}}$ that is uniformly convergent on compact subsets of $B(0,r) \setminus (\{z_1,\ldots,z_M\} \cup \{\zeta_1,\ldots,\zeta_{m-M}\})$.*

In short, when one is approximating a meromorphic function and one chooses the denominator degree of the approximant equal to the total number of poles within a distance of at most r, then one can expect uniform convergence of the Padé approximants in that region. If one chooses the denominator degree slightly too large, then one can only expect convergence in measure (and capacity [BaGr81]) or one can only expect a subsequence to converge uniformly. How information on the correct denominator degree can be obtained, is described in [GlKa94].

Nuttall–Pommerenke [Nutt70], [Pomm73]. *Let the function f be analytic in $\mathbb{Z}$ except for a countable number of isolated poles and essential singularities. Then the sequence $\{[n_k/m_k]^f\}_{k\in\mathbb{N}}$ with $\lambda < n_k/m_k < 1/\lambda$ for $0 < \lambda < 1$, converges to f in measure on compact sets:*

$$\forall\epsilon, r > 0 : \Lambda_2\left(\left\{z \in \overline{B}(0,r) : |f(z) - [n_k/m_k]^f(z)| \geq \epsilon\right\}\right) \longrightarrow_{k\to\infty} 0$$

This last theorem is a simpler version of the original one which proves convergence in capacity. Since the number of singularities of f is now countable, one has to let the denominator degree increase unboundedly, and hence column sequences make an inappropriate choice. The exceptional set that is excluded from the region of convergence is for instance caused by unwanted pole-zero combinations in the Padé approximant.

2 Homogeneous multivariate Padé approximants.

In order to discuss the multivariate case, we describe the bivariate situation because the higher-dimensional situation is only notationally more difficult. A first possibility to deal with functions $f(x,y)$ known by their bivariate series expansion at a certain point, is to group the terms in homogeneous expressions. For such series homogeneous Padé approximants $[n/m]_H^f$ were defined in [Cuyt79] by

$$f(x,y) = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} c_{ij} x^i y^j \right)$$
$$p(x,y) = \sum_{i+j=nm}^{nm+n} a_{ij} x^i y^j$$
$$q(x,y) = \sum_{i+j=nm}^{nm+m} b_{ij} x^i y^j$$
$$(fq-p)(x,y) = \sum_{k \geq nm+n+m+1} \left(\sum_{i+j=k} d_{ij} x^i y^j \right)$$

with $[n/m]_H^f$ given by the unique irreducible form of all p/q satisfying the above conditions. What these approximants look like can be seen from the following example. Consider

$$f(x,y) = 1 + \frac{x}{0.1-y} + \sin(xy)$$

and choose respectively $n=1, m=1$ and $n=1, m=2$. Then the numerator and denominator polynomials $p(x,y)$ and $q(x,y)$ respectively start with a homogeneous expression of degree 1 and 2. Fortunately if we compute the irreducible form

$$[1/1]_H = \frac{1+10x-10.1y}{1-10.1y}$$
$$[1/2]_H = \frac{x-1.01y+10x^2-20.2xy+10y^2}{x-1.01y-10.1xy+10y^2+2.01xy^2}$$

then the degree of the numerator and denominator polynomial and the introduced shift in those degrees are maximally reduced. The necessity of this 'shift' in the degrees in order to enjoy a number of nice properties among which the unicity of the irreducible form, is explained in detail in [Cuyt84]. Furthermore, a projection property that will play an important role in the sequel, was pointed out independently by Karlsson [Karl84] and by Chaffy [Chaff84]. We introduce for (u,v) in $\mathbb{Z}^2$ with $uv \neq 0$:

$$B_{(u,v)}(0,r) = \{z \in \mathbb{Z} : ||(uz,vz)|| < r\}$$
$$f_{(u,v)}(z) = f(uz,vz)$$

Then for any fixed (u,v) in $\mathbb{Z}^2$, projecting onto the set $\{(x,y)\in\mathbb{Z}^2 : x=uz, y=vz\}$ yields

$$[n/m]^{f_{(u,v)}}(z) = [n/m]^f_H(uz,vz)$$

Thanks to this projection property the following convergence results were obtained. We do not cite them in their most general form. Therefore the reader is referred to the original reference.

Cuyt and Lubinsky [CuLu95]. *Let the function $f(x,y)$ be meromorphic in the ball $B((0,0),r)$ in the sense that there exists a polynomial $s(x,y)$ of homogeneous degree M such that fs is holomorphic in $B((0,0),r)$. If we denote*

$$\begin{aligned} W =&\{(u,v) : ||(u,v)|| = 1 \textit{ and} \\ &\qquad f_{(u,v)} \textit{ less than } M \textit{ poles in } B_{(u,v)}(0,r)\} \\ \mathcal{S} =&\{(x,y) : s(x,y)=0\} \\ \mathcal{E} =&\{(uz,vz) : (u,v)\in W\} \end{aligned}$$

then the sequence $\{[n/M]^f_H\}_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $B((0,0),r)$ not intersecting $\mathcal{E}\cup\mathcal{S}$. Outside W each zero of $s_{(u,v)}(z)$ attracts zeros of the projected Padé denominator according to its multiplicity.

The set W denotes the set of exceptional directions, meaning that for (u,v) in W the univariate convergence theorem of de Montessus de Ballore applies to a column different from that for the vectors outside W: for all vectors (u,v) outside W one has to consider the M-th column. Note than one does not have convergence in $(0,0)$, the point at which the series development for f was given, because it is always contained in $\mathcal{E}\cup\mathcal{S}$!

Cuyt and Lubinsky [CuLu95]. *Let the function $f(x,y)$ be meromorphic in the ball $B((0,0),r)$ in the sense that there exists a polynomial $s(x,y)$ of homogeneous degree M such that fs is holomorphic in $B((0,0),r)$. Then for $m\geq M$ the sequence $\{[n/m]^f_H\}_{n\in\mathbb{N}}$ converges in $B((0,0),r)$ in measure to f.*

This theorem nicely generalizes the univariate result obtained by Zinn-Justin while the next theorem generalizes the univariate result of Karlsson and Wallin. Both deal with a denominator choice that is again slightly too large. For the next convergence result we assume that the sequence $\{[n/m]^f_H\}_{n\in\mathbb{N}}$ with fixed $m\geq M$ has an infinite number of elements $[n_h/m]^f_H$ for which $[n_h/m]^f_H$ is holomorphic at the origin. We denote this subsequence of holomorphic entries by $\{[n_h/m]^f_H\}_{h\in\mathbb{N}}$.

Cuyt [Cuyt85]. *Let the function $f(x,y)$ be meromorphic in the ball $B((0,0),r)$ in the sense that there exists a polynomial $s(x,y)$ of homogeneous degree M such that fs is holomorphic in that ball. Then for $m\geq M$ there exists an analytic set $\mathcal{T}\supset\mathcal{S}$ and there exists a subsequence of $\{[n_h/m]^f_H\}_{h\in\mathbb{N}}$ that converges uniformly to f on compact subsets of $B((0,0),r)\setminus\mathcal{T}$.*

Let us now turn to a generalization of the Nuttall–Pommerenke result, for homogeneous Padé approximants.

Cuyt, Driver and Lubinsky [CuDL94b]. *Let the function $f(x,y)$ be analytic in $\mathbb{Z}^2 \setminus \mathcal{G}$ where the analytic set $\mathcal{G} = \{(x,y) \in \mathbb{Z}^2 : g(x,y) = 0,\ g \text{ entire } \}$. Then the sequence $\{[n_k/m_k]_H^f\}_{k\in\mathbb{N}}$ with $\lambda < n_k/m_k < 1/\lambda$ for $0 < \lambda < 1$, converges on compact sets in measure to f.*

In [Gonc74] another kind of homogeneous approximants, which we shall denote by $[n/m]_{\tilde{H}}^f$, was defined where the so-called 'shift' in the numerator and denominator polynomial degrees is not introduced. For $f(x,y)$ given by the series development above, one computes

$$p(x,y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j$$
$$q(x,y) = \sum_{i+j=0}^{m} b_{ij} x^i y^j$$
$$(fq-p)(x,y) = \sum_{k \geq \lfloor \sqrt{2} n \rfloor + 1} \left(\sum_{i+j=k} d_{ij} x^i y^j \right)$$

For these approximants the following result holds but convergence in capacity has not been obtained.

Gonchar [Gonc74]. *Let the function $f(x,y)$ be analytic in $\mathbb{Z}^2 \setminus \mathcal{G}$ where the analytic set $\mathcal{G} = \{(x,y) \in \mathbb{Z}^2 : g(x,y) = 0,\ g \text{ entire } \}$. Then the sequence $\{[n/n]_{\tilde{H}}^f\}_{n\in\mathbb{N}}$ converges on compact sets in measure to f.*

3 General order multivariate Padé approximants.

Several authors have followed a different approach for the definition of multivariate Padé approximants [Lutt74, KaWa77, Chis73, Levi76, CuVe83], in which the multivariate series development for f is also rewritten as a series indexed by only one index, but now in a different way. Instead of grouping terms in subexpressions one agrees on an order in which the terms are added to the series one by one. A multivariate Padé approximant is then defined as follows. For a given f and chosen $N \subset \mathbb{N}^2$ and $D \subset \mathbb{N}^2$, one computes

$$f(x,y) = \sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j$$
$$p(x,y) = \sum_{(i,j)\in N} a_{ij} x^i y^j \qquad \#N = n+1$$
$$q(x,y) = \sum_{(i,j)\in D} b_{ij} x^i y^j \qquad \#D = m+1$$
$$(fq-p)(x,y) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} d_{ij} x^i y^j \qquad N \subset E \qquad \#E = n+m+1$$

An irreducible form of p/q is again called a Padé approximant for f and it is denoted by $[N/D]_E^f$. Note that it need not be unique, as was already the case for the definition given in [Gonc74]. In order to compare these general order solutions to the one from the previous paragraph, we compute some approximants for the same function

$$f(x,y) = 1 + \frac{x}{0.1-y} + \sin(xy)$$

For

$$\begin{aligned} N_1 =& \{(0,0),(1,0),(0,1)\} \\ D_1 =& \{(0,0),(0,1)\} \\ E_1 =& N_1 \cup \{(1,1)\} \end{aligned}$$

one obtains

$$[N_1/D_1]_{E_1} = \frac{1+10x-10.1y}{1-10.1y}$$

and for

$$\begin{aligned} N_2 =& \{(0,0),(1,0),(0,1),(1,1),(2,0),(0,2)\} \\ D_1 =& \{(0,0),(1,0),(0,1)\} \\ E_2 =& N_2 \cup \{(2,1),(1,2)\} \end{aligned}$$

the solution is given by

$$[N_2/D_1]_{E_2} = \frac{1+10x-\frac{1000}{101}y+\frac{201}{101}xy}{1+\frac{1000}{101}y}$$

It is useful to compare these approximants respectively to the homogeneous approximants $[1/1]_H^f$ and $[2/1]_H^f$. Let us now turn to a discussion of the convergence results.

Cuyt [Cuyt90, Cuyt92]. *Let the function $f(x,y)$ be meromorphic in the polydisc $\overline{B}((0,0);r_1,r_2)$ in the sense that there exists a multivariate polynomial*

$$s(x,y) = \sum_{(i,j)\in M} s_{ij}x^i y^j$$

such that fs is holomorphic in that polydisc. Under suitable conditions for N and E which are detailed in [Cuyt90] and with

$$\mathcal{S} = \{(x,y) : s(x,y) = 0\}$$

the sequence $\{[N/M]_E\}_{\#N\to\infty, N\subset E}$ converges to f uniformly on compact subsets of $\overline{B}((0,0);r_1,r_2)\setminus\mathcal{S}$ with the general order Padé denominator $q(x,y)$ converging to $s(x,y)$.

We now respectively state generalizations of the Zinn-Justin convergence theorem and the Nuttall–Pommerenke convergence in capacity. After each theorem we translate the conditions to the univariate case, so that it becomes clear why those conditions are natural generalizations of the ones in the univariate theorems. For index sets N_k, D_k, E_k and M we denote by

$$\begin{aligned} N_k * M =& \{(i,j) : i = i_1 + i_2, j = j_1 + j_2, (i_1, j_1) \in N_k, (i_2, j_2) \in M\} \\ i_{D_k} =& \max\{i : (i,j) \in D_k\} \\ j_{D_k} =& \max\{j : (i,j) \in D_k\} \\ \partial D_k =& \max\{i_{D_k}, j_{D_k}\} \\ \omega E_k =& \min\{i + j : (i,j) \in \mathbb{N}^2 \setminus E_k\} \end{aligned}$$

Cuyt, Driver and Lubinsky [CuDL94a]. *Let the function $f(x,y)$ be meromorphic in the polydisc $B((0,0); r_1, r_2)$ in the sense that there exists a multivariate polynomial*

$$s(x,y) = \sum_{(i,j)\in M} s_{ij} x^i y^j$$

such that fs is holomorphic in that polydisc. For N_k, D_k and E_k satisfying

$$\begin{aligned} &N_k * M \subset E_k \\ &\lim_{k\to\infty} \omega E_k / \partial D_k = \infty \end{aligned}$$

the sequence of approximants $\{[N_k/D_k]_{E_k}\}_{k\in\mathbb{N}}$ converges in $B((0,0); r_1, r_2)$ in measure to f.

In the univariate case the sets N_k, D_k and E_k equal

$$\begin{aligned} N_k =& \{0, \ldots, n_k\} \\ D_k =& \{0, \ldots, m_k\} \\ E_k =& \{0, \ldots, n_k + m_k\} \end{aligned}$$

Hence

$$\begin{aligned} \partial D_k =& m_k \\ \omega E_k =& n_k + m_k + 1 \end{aligned}$$

and the conditions in the above theorem amount to

$$\begin{aligned} &N_k * M \subset E_k \Longleftrightarrow n_k + M \le n_k + m_k \Longleftrightarrow m_k \ge M \\ &\lim_{k\to\infty} \frac{\omega E_k}{\partial D_k} = \infty \Longleftrightarrow \lim_{k\to\infty} \frac{n_k}{m_k} = \infty \end{aligned}$$

which are the standard univariate conditions.

Cuyt, Driver and Lubinsky [CuDL94a]. *Let $f(x,y)$ be such that for each ρ there exists a polynomial $s_\rho(x,y)$ such that $(fs_\rho)(x,y)$ is analytic in the polydisc $B((0,0);\rho,\rho)$. Let $\ell_k = \max\{\partial N_k, \partial D_k\}$ and*

$$C_{\lfloor k \rfloor} = \{(i,j) \in \mathbb{N}^2 : 0 \le i \le \lfloor k \rfloor, 0 \le j \le \lfloor k \rfloor\}$$

For N_k, D_k and E_k satisfying

$$\lim_{k\to\infty} \ell_k = \infty$$
$$N_k * C_{\lfloor \lambda \ell_k \rfloor} \subset E_k$$
$$D_k * C_{\lfloor \lambda \ell_k \rfloor} \subset E_k$$

with $0 < \lambda < 1$, the sequence of approximants $\{[N_k/D_k]_{E_k}\}_{k\in\mathbb{N}}$ converges on compact sets in measure to f.

In the univariate case these conditions translate to the following:

$$\ell_k = \max\{n_k, m_k\}$$
$$N_k * C_{\lfloor \lambda \ell_k \rfloor} \subset E_k \iff n_k + \lambda \ell_k \le n_k + m_k \Longrightarrow \lambda n_k \le m_k$$
$$D_k * C_{\lfloor \lambda \ell_k \rfloor} \subset E_k \iff m_k + \lambda \ell_k \le n_k + m_k \Longrightarrow \lambda m_k \le n_k$$

These last conditions amount to

$$\lambda \le n_k/m_k \le 1/\lambda$$

These last two theorems also hold if we replace the notion of measure by capacity as detailed in [CuDL94a]. **Summary.** We have reviewed these convergence results that

exist both in a univariate and multivariate context. In order to compare the results we have not stated them in their full generality. Readers interested in a particular theorem are referred to the original reference to obtain the most general formulation. Where appropriate we have indicated whether the resulting convergence in measure essentially also holds in capacity.

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Departement WIS–INF, Universiteit Antwerpen (UIA),
Universiteitsplein 1,
B–2610 Wilrijk-Antwerpen (Belgium),
email: cuythwins.uia.ac.be

Department of Mathematics,
Witwatersrand University, P.O. Wits 2050,
Johannesburg, South-Africa,
email: 036DSLcosmos.wits.ac.za