Coalgebra deformations of bialgebras by Harrison cocycles, copairings of Hopf algebras and double crosscoproducts

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Abstract

We study how the comultiplication on a Hopf algebra can be modified in such a way that the new comultiplication together with the original multiplication and a suitable antipode gives a new Hopf algebra. To this end, we have to study Harrison type cocycles, and it turns out that there is a relation with the Yang-Baxter equation. The construction is applied to deform the coalgebra structure on the tensor product of two bialgebras using a copairing. This new bialgebra can be viewed as a double crosscoproduct. It is also shown that a crossed coproduct over an inner comeasuring is isomorphic to a twisted coproduct.

Introduction

Let H be a bialgebra over a field k. If $\sigma: H \otimes H \to k$ is a Sweedler cocycle, then we can define a new multiplication on H as follows:

$$a.b = \sum \sigma(a_{(1)} \otimes b_{(1)}) a_{(2)} b_{(2)} \sigma^{-1}(a_{(3)} \otimes b_{(3)})$$

In [7], Doi shows that H, with this new multiplication and the original comultiplication is a bialgebra. This construction was further investigated by Doi and Takeuchi

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in [8], where it was applied to alterate the algebra structure on $A \otimes H$, where A and H are both bialgebras, and where a skew pairing (a Hopf pairing in the sense of [2]) is given. They also show that the new Hopf algebra obtained after this algebra deformation is a double crossproduct in the sense of Majid([10]).

In this note, we discuss the dual situation. How can we change the comultiplication on H such that the newly obtained coalgebra, together with the original multiplication (and a suitable antipode in case H is a Hopf algebra) is a new bialgebra? It will turn out that we need Harrison cocycles rather than Sweedler cocycles. These are invertible elements from $H \otimes H$ satisfying the cocycle condition (CH). In the case where H is commutative, these cocycles have been considered before, we cite [14] to justify the name Harrison cocycle. We are able to show the following: if $R = \sum R^1 \otimes R^2$ is a Harrison cocycle with inverse $U = \sum U^1 \otimes U^2$, then the comultiplication rule

$$\Delta_{(R)}(h) = \sum R^1 h_{(1)} U^1 \otimes R^2 h_{(2)} U^2$$

makes H into a Hopf algebra. Moreover, if H is cocommutative, then we can define a triangular structure on the new Hopf algebra $H_{(R)}$. Our results are dual to those obtained by Doi (cf. [7, Theorem 1.6]), in fact, if H is finite dimensional, then they are equivalent.

In the second part of Section 2, we investigate the relationship between Harrison cocycles and quasitriangular Hopf algebras: we can show that an invertible $R \in H \otimes H$ that satisfies (QT1) and (QT3) is a Harrison cocycle if and only if it is a solution of the Yang-Baxter equation, cf. Proposition 2.5. More specifically, if (H, R) is quasitriangular, then R is a Harrison cocycle. In Theorem 2.7, we present a sufficient condition for the Hopf algebras $H_{(R)}$ and $H_{(W)}$ for two different cocycles R and W to be isomorphic.

In Section 3, we introduce copairings (B, H, N) of bialgebras, and we apply the coalgebra deformation from Section 2 to $B \otimes H$. We then obtain a new bialgebra, denoted by $B \bowtie^N H$. Some interesting applications for quasitriangular bialgebras are given. Most of them are dual to results from [8].

In [16, Proposition 11], Radford gives a characterization of the dual of the Drinfel'd double suggesting a general definition of a "product" of comodule algebras, dual to the definition of the product of module coalgebras that was called "Double Crossproduct" in [10]. However he gives no definition or details about it. In Section 4, we develop the construction of this new "product", and we call it the double crosscoproduct. This completes the picture of different kinds of products of Hopf algebras (see [10, 11]). As an application, we show that the bialgebra $B \bowtie^N H$ constructed in the second section is a double crosscoproduct of B and H.

If R is a Harrison cocycle, then one can change the comultiplication in another way, by putting ${}_R\Delta(h) = R\Delta(h)$. This makes H into an H-module coalgebra. A generalization of this construction is the following: if C is a coalgebra, $\nu : C \to C \otimes H$ is a comeasuring, and $\alpha : C \to H \otimes H$ is a map satisfying the three cocycle conditions (CC), (NC) and (TC), then we can consider the crossed coproduct $C \bowtie_\alpha H$. Our main result (Theorem 5.1) is now that a crossed coproduct over an inner comeasuring is isomorphic to a twisted coproduct, that is, a crossed coproduct over a trivial comeasuring.

1 Preliminaries

Let k be a field. Unless specified otherwise, all vector spaces, algebras, coalgebras, bialgebras and Hopf algebras that we consider are over k. $\otimes$ and Hom will mean $\otimes_k$ and Hom_k. For a coalgebra C, we will use Sweedler's Σ-notation, that is, $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, $(I \otimes \Delta)\Delta(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc. We will also use the Sweedler notation for left and right C-comodules: $\rho_M(m) = \sum m_{[0]} \otimes m_{[1]}$ for any m in a right C-comodule M, and $\rho_N(n) = \sum n_{[-1]} \otimes n_{[0]}$ for any n in a left C-comodule N.
If V and W are two vector spaces, $\tau : V \otimes W \to W \otimes V$ will denote the switch map, that is, $\tau(v \otimes w) = w \otimes v$ for all $v \in V$ and $w \in W$.
Let H be a Hopf algebra. For an element $R \in H \otimes H$, we use the notation $R = \sum R^1 \otimes R^2$. We will then also write

$$R^{12} = \sum R^1 \otimes R^2 \otimes 1, \quad R^{23} = \sum 1 \otimes R^1 \otimes R^2$$

and so on.
Recall that a right H-comodule algebra is an algebra A which is also a right H-comodule, such that the structure map $\rho_A : A \to A \otimes H$ is an algebra map. A right H-module coalgebra is a coalgebra C that is also a right H-module such that the structure map $C \otimes H \longrightarrow C : c \otimes h \mapsto ch$ is a coalgebra map. A left H-comodule coalgebra is a coalgebra C that is also a left H-comodule such that the comodule structure map $\rho_C : C \to H \otimes C : c \mapsto \sum c_{[-1]} \otimes c_{[0]}$ is a coalgebra map. This means that

$$\sum c_{[-1]} \otimes \Delta(c_{[0]}) = \sum c_{(1)_{[-1]}} c_{(2)_{[-1]}} \otimes c_{(1)_{[0]}} \otimes c_{(2)_{[0]}}$$

and

$$\sum \varepsilon(c_{[0]}) c_{[-1]} = \varepsilon(c) 1_H$$

for all $c \in C$.
A quasitriangular Hopf algebra is a pair (H, R), where H is a Hopf algebra and $R \in H \otimes H$ such that the following 5 conditions are fulfilled:
(QT1) $\sum \Delta(R^1) \otimes R^2 = R^{13} R^{23}$
(QT2) $\sum \varepsilon(R^1) R^2 = 1$
(QT3) $\sum R^1 \otimes \Delta(R^2) = R^{13} R^{12}$
(QT4) $\sum R^1 \varepsilon(R^2) = 1$
(QT5) $\Delta^{\mathrm{cop}}(h) R = R \Delta(h)$, for all $h \in H$.
Recall from [15] (see also [13, p. 180]) that (H, R) is quasitriangular if and only if R is invertible and conditions (QT1), (QT3) and (QT5) hold. We call a quasitriangular Hopf algebra triangular if $R^{-1} = \tau(R)$.
Recall from [7] that a braided bialgebra is a bialgebra H together a convolution invertible bilinear form $\sigma : H \otimes H \to k$ satisfying the following conditions:
(BB1) $\sum \sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} = \sum y_{(1)} x_{(1)} \sigma(x_{(2)}, y_{(2)})$
(BB2) $\sigma(xy, z) = \sum \sigma(x, z_{(1)}) \sigma(y, z_{(2)})$
(BB3) $\sigma(x, yz) = \sum \sigma(x_{(1)}, z) \sigma(x_{(2)}, y)$ for all $x, y, z \in H$.

As a consequence of the above conditions we have that:
(BB4) $\sigma(x, 1_H) = \sigma(1_H, x) = \varepsilon(x)$ for all $x \in H$.

We say that a Hopf algebra H coacts weakly on a coalgebra C if there exists a k-linear map $C \longrightarrow H \otimes C : \ c \mapsto \sum c_{[-1]} \otimes c_{[0]}$ such that the following conditions hold:

(W1) $\sum c_{[-1]} \otimes \Delta(c_{[0]}) = \sum c_{(1)_{[-1]}} c_{(2)_{[-1]}} \otimes c_{(1)_{[0]}} \otimes c_{(2)_{[0]}}$

(W2) $\sum \varepsilon(c_{[0]}) c_{[-1]} = \varepsilon(c) 1_H$

(W3) $\sum \varepsilon(c_{[-1]}) c_{[0]} = c$

for all $c \in C$. Suppose that H coacts weakly on C, and let $\alpha : \ C \longrightarrow H \otimes H : \ c \mapsto \alpha_1(c) \otimes \alpha_2(c)$ be a k-linear map for which the following three conditions hold for all $c \in C$:

(CC) $$\sum c_{(1)_{[-1]}} \alpha_1(c_{(2)}) \otimes \alpha_1(c_{(1)_{[0]}}) \alpha_2(c_{(2)})_{(1)} \otimes \alpha_2(c_{(1)_{[0]}}) \alpha_2(c_{(2)})_{(2)}$$
$$= \sum \alpha_1(c_{(1)}) \alpha_1(c_{(2)})_{(1)} \otimes \alpha_2(c_{(1)}) \alpha_1(c_{(2)})_{(2)} \otimes \alpha_2(c_{(2)})$$
(*cocycle condition*)

(NC) $(I \otimes \varepsilon)\alpha = (\varepsilon \otimes I)\alpha = \eta_{H\otimes H} \circ \varepsilon_C$
(*normal cocycle condition*)

(TC) $$\sum c_{(1)_{[-2]}} \alpha_1(c_{(2)}) \otimes c_{(1)_{[-1]}} \alpha_2(c_{(2)}) \otimes c_{(1)_{[0]}}$$
$$= \sum \alpha_1(c_{(1)}) c_{(2)_{[-1]_{(1)}}} \otimes \alpha_2(c_{(1)}) c_{(2)_{[-1]_{(2)}}} \otimes c_{(2)_{[0]}}$$
(*twisted comodule condition*)

The *crossed coproduct* $C \rtimes_\alpha H$ of C, H and α is the k-vector space $C \otimes H$ with the following comultiplication and counit:

$$\Delta_\alpha(c \rtimes_\alpha h) = \sum \Big(c_{(1)} \rtimes_\alpha c_{(2)_{[-1]}} \alpha_1(c_{(3)}) h_{(1)} \Big) \otimes \Big(c_{(2)_{[0]}} \rtimes_\alpha \alpha_2(c_{(3)}) h_{(2)} \Big) \quad (1)$$
$$\varepsilon_\alpha(c \rtimes_\alpha h) = \varepsilon_C(c) \varepsilon_H(h) \quad (2)$$

for all $c \in C$ and $h \in H$. It may be verified that the comultiplication Δ_α is coassociative, and that ε_α is a counit map, we refer to [6] for details. If the cocycle α is trivial, that is, $\alpha(c) = \varepsilon(c) 1_H \otimes 1_H$, the condition (TC) tells us that C is a left H-comodule coalgebra, and in this case $C \rtimes_\alpha H$ is the usual smash coproduct $C \rtimes H$. If the weak coaction $C \to C \otimes H$ is trivial, that is, $\rho(c) = 1_H \otimes c$ for every $c \in C$, then we call the crossed coproduct a *twisted coproduct*. We then write $C_\alpha[H] = C \rtimes_\alpha H$. The comultiplication formula for a twisted coproduct is less complicated: it is easy to see that equation (1) takes the following form:

$$\Delta_{C_\alpha[H]}(c \rtimes_\alpha h) = \sum \Big(c_{(1)} \rtimes_\alpha \alpha_1(c_{(3)}) h_{(1)} \Big) \otimes \Big(c_{(2)} \rtimes_\alpha \alpha_2(c_{(3)}) h_{(2)} \Big) \quad (3)$$

Observe that if the cocycle α and the weak coaction are both trivial, then crossed coproduct simplifies to the usual tensorproduct $C \otimes H$ of the coalgebras C and H.

2 Modifying the comultiplication on a bialgebra

Let H be a bialgebra, and let $R \in H \otimes H$ be an invertible element. In the sequel, we will use the following notations: $R = \sum R^1 \otimes R^2 = \sum r^1 \otimes r^2 = r, \ R^{-1} = U = \sum U^1 \otimes U^2 = \sum u^1 \otimes u^2 = u$ and $B = \tau(R) = \sum R^2 \otimes R^1$.

Let $H_{(R)}$ be equal to H as a k-algebra, with comultiplication $\Delta_{(R)}$ given by

$$\Delta_{(R)}(h) = R \Delta(h) R^{-1} = \sum R^1 h_{(1)} U^1 \otimes R^2 h_{(2)} U^2 \quad (4)$$

for all $h \in H$. The starting point for this Section is the following question: *When is $H_{(R)}$ a bialgebra?* We call $R \in H \otimes H$ a *Harrison cocycle* if

(CH) $\sum R^1 r^1_{(1)} \otimes R^2 r^1_{(2)} \otimes r^2 = \sum R^1 \otimes r^1 R^2_{(1)} \otimes r^2 R^2_{(2)}$

Examples 2.1. 1. Let H be a Hopf algebra. Then any cleft H-coextension of k provides a Harrison cocycle for H.

Cleft coextensions have been introduced in [9]. In [5], where it is shown that for any right H-module coalgebra C, the H-coextension $C/\overline{C}$ (where $\overline{C} = C/CH^+$) is cleft if and only if C is isomorphic to a crossed coproduct $\overline{C} \rtimes_\alpha H$ with invertible cocycle $\alpha : \overline{C} \to H \otimes H$. If $\overline{C} = k$, then a cleft coextension C/k produces a cocycle $\alpha : k \to H \otimes H$, and $\alpha(1)$ is a Harrison cocycle.

2. If (H, R) is a quasitriangular Hopf algebra, then R is a Harrison cocycle of H (see Proposition 2.5).

Lemma 2.2. *Let H be a bialgebra, and $R \in H \otimes H$ an invertible cocycle, and let U be the inverse of R. Then*

1. $\sum \varepsilon(R^1)\varepsilon(R^2)\varepsilon(U^1)\varepsilon(U^2) = 1$;
2. $\sum R^1\varepsilon(R^2) = \sum \varepsilon(R^1)R^2 = \sum \varepsilon(R^1)\varepsilon(R^2)1_H$;
3. $\sum U^1_{(1)}u^1 \otimes U^1_{(2)}u^2 \otimes U^2 = \sum U^1 \otimes U^2_{(1)}u^1 \otimes U^2_{(2)}u^2$;
4. $\sum U^1\varepsilon(U^2) = \sum \varepsilon(U^1)U^2 = \sum \varepsilon(U^1)\varepsilon(U^2)1_H$;
5. *If H has an antipode S, then $\alpha = \sum R^1 S(R^2)$ is an invertible element of H and $\alpha^{-1} = \sum S(U^1)U^2$;*
6. *If H is cocommutative, and $B = \tau(R)$, then*
   - $U^{13}B^{23} = \sum r^1_{(2)}U^1 \otimes r^2 U^2_{(1)} \otimes r^1_{(1)}U^2_{(2)}$;
   - $U^{13}B^{12} = \sum r^2_{(2)}U^1_{(1)} \otimes r^1 U^1_{(2)} \otimes r^2_{(1)}U^2$.

Proof. 1) Apply $\varepsilon \otimes \varepsilon$ to $RU = 1 \otimes 1$.

2) First we apply $I \otimes \varepsilon \otimes I$ to both sides of (CH). We obtain

$$\sum R^1 \otimes \varepsilon(R^2)1_H \otimes 1_H = \sum 1_H \otimes \varepsilon(r^1)1_H \otimes r^2 \tag{5}$$

and it follows that $\sum R^1\varepsilon(R^2) = \sum \varepsilon(R^1)R^2$. Applying $I \otimes \varepsilon \otimes \varepsilon$ to (5) we obtain that $\sum R^1\varepsilon(R^2) = \varepsilon(R^1)\varepsilon(R^2)1_H$.

3) The left hand side of this equation is the inverse of the left hand side of (CH); the right hand side is the inverse of the right hand side of (CH).

4) follows from 3) in exactly the same way as 2) follows from (CH).

5) We will show that

$$\sum R^1 S(R^2) S(U^1) U^2 = 1 \tag{6}$$

Left multiplication both sides of (CH) by U^{23}, and application of $m_H \circ (I \otimes S \otimes I)$ to both sides yields:

$$\sum R^1 r^1_{(1)} S(r^1_{(2)}) S(R^2) S(U^1) U^2 r^2 = \sum R^1 S(R^2_{(1)}) R^2_{(2)}$$

or (use 2))

$$\sum R^1S(R^2)S(U^1)U^2\varepsilon(r^1)r^2 = \sum R^1\varepsilon(R^2) = \sum \varepsilon(R^1)\varepsilon(R^2)1_H$$

If we multiply both sides by $\sum \varepsilon(u^1)\varepsilon(u^2)$, then we obtain (6). A similar computation, now using 3) instead of (CH), shows that $\sum S(U^1)U^2R^1S(R^2) = 1$.
6) The first formula is equivalent to (multiply both sides by R^{13}):

$$B^{23} = \sum R^1r^1_{(2)}U^1 \otimes r^2U^2_{(1)} \otimes R^2r^1_{(1)}U^2_{(2)}$$

We now easily compute that

$$\begin{aligned}
&\sum R^1r^1_{(2)}U^1 \otimes r^2U^2_{(1)} \otimes R^2r^1_{(1)}U^2_{(2)}\\
&= \sum(R^1r^1_{(2)} \otimes r^2 \otimes R^2r^1_{(1)})(U^1 \otimes U^2_{(1)} \otimes U^2_{(2)})\\
&= \sum(R^1r^1_{(1)} \otimes r^2 \otimes R^2r^1_{(2)})(U^1 \otimes U^2_{(2)} \otimes U^2_{(1)})\\
&\qquad\qquad (H \text{ is cocommutative})\\
&= \sum(R^1 \otimes r^2R^2_{(2)} \otimes r^1R^2_{(1)})(U^1 \otimes U^2_{(2)} \otimes U^2_{(1)})\\
&\qquad\qquad (\text{using(CH)})\\
&= \sum 1 \otimes r^2 \otimes r^1 = B^{23}
\end{aligned}$$

The second formula follows in a similar way. ■

The following result may be found in [17]. For completeness sake, we give a brief account of the proof.

Theorem 2.3. *Let H be a bialgebra, $R \in H \otimes H$ an invertible cocycle. Then:*

1. *$H_{(R)}$ is a bialgebra.*
2. *If H has an antipode S and $\alpha = \sum R^1S(R^2)$ then $H_{(R)}$ is a Hopf algebra with antipode S_R given by the formula*

$$S_R(h) = \alpha S(h)\alpha^{-1}$$

3. *If H is a cocommutative Hopf algebra and $\tilde{R} = \tau(R)R^{-1}$ then $(H_{(R)}, \tilde{R})$ is a triangular Hopf algebra.*

Proof. 1) Using 1), 2) and 4) in Lemma 2.2, we easily obtain that

$$\sum R^1h_{(1)}U^1\varepsilon(R^2h_{(2)}U^2) = \sum \varepsilon(R^1h_{(1)}U^1)R^2h_{(2)}U^2 = h$$

Using (CH) and 3) in Lemma 2.2, we find that $(I \otimes \Delta_{(R)})\Delta_{(R)}(h)$

$$\begin{aligned}
&= \sum\left(R^1 \otimes r^1R^2_{(1)} \otimes r^2R^2_{(2)}\right)\left(h_{(1)} \otimes h_{(2)} \otimes h_{(3)}\right)\left(U^1 \otimes U^2_{(1)}u^1 \otimes U^2_{(2)}u^2\right)\\
&= \sum\left(r^1R^1_{(1)} \otimes r^2R^2_{(1)} \otimes R^2\right)\left(h_{(1)} \otimes h_{(2)} \otimes h_{(3)}\right)\left(U^1_{(1)}u^1 \otimes U^1_{(2)}u^2 \otimes U^2\right)\\
&= (\Delta_{(R)} \otimes I)\Delta_{(R)}(h)
\end{aligned}$$

and it follows that $\Delta_{(R)}$ is coassociative. It is clear that $\Delta_{(R)}$ is an algebra map, hence $H_{(R)}$ is a bialgebra.
2) For all $h \in H$, we have that

$$\begin{aligned}
\sum S_R(R^1 h_{(1)} U^1) R^2 h_{(2)} U^2 &= \sum r^1 S(u^1 R^1 h_{(1)} U^1 r^2) u^2 R^2 h_{(2)} U^2 \\
&= \sum r^1 S(r^2) S(U^1) S(h_{(1)}) S(R^1) S(u^1) u^2 R^2 h_{(2)} U^2 \\
&= \sum r^1 S(r^2) S(U^1) S(h_{(1)}) S(u^1 R^1) u^2 R^2 h_{(2)} U^2 \\
&= \sum r^1 S(r^2) S(U^1) S(h_{(1)}) h_{(2)} U^2 \\
&= \sum r^1 S(r^2) S(U^1) U^2 \varepsilon(h) = \varepsilon(h)
\end{aligned}$$

where we used 5) of Lemma 2.2 in the last step. A similar computation shows that S_R is also the right convolution inverse of the identity.
3) Since H is cocommutative, $S^2 = I_H$, and therefore

$$S_R^2(h) = S_R(\alpha S(h) \alpha^{-1}) = \alpha S(\alpha^{-1}) S^2(h) S(\alpha) \alpha^{-1} = \beta h \beta^{-1}$$

where $\beta = \alpha S(\alpha^{-1})$. Therefore the antipode S_R is bijective. Write $B = \tau(R)$. A straightforward computation shows that

$$\widetilde{R}^{13} \widetilde{R}^{23} = \sum B^{13} (U^1 \otimes r^2 \otimes U^1 r^1) u^{23} = B^{13} U^{13} B^{23} U^{23}$$

On the other hand

$$\begin{aligned}
&(\Delta_{(R)} \otimes I)(\widetilde{R}) = \sum (r^1 R^2_{(1)} \otimes r^2 R^2_{(2)} \otimes R^1)(U^1_{(1)} u^1 \otimes U^1_{(2)} u^2 \otimes U^2) \\
&= \sum (R^2 r^1_{(2)} \otimes r_2 \otimes R^1 r^1_{(1)})(U^1 \otimes U^2_{(1)} u^1 \otimes U^2_{(2)} u^2) \\
&\qquad\qquad \text{(by (CH) and 3) from lemma 2.2)} \\
&= B^{13} (\sum r^1_{(2)} U^1 \otimes r^2 U^2_{(1)} \otimes r^1_{(1)} U^2_{(2)}) U^{23} \\
&= B^{13} U^{13} B^{23} U^{23} \qquad \text{(by 6) from lemma 2.2)}
\end{aligned}$$

and (QT1) holds. A similar argument shows that (QT3) holds. Using the fact that H is cocommutative, we obtain

$$\begin{aligned}
\widetilde{R} \Delta_{(R)}(h) \widetilde{R}^{-1} &= \tau(R) R^{-1} R \Delta(h) R^{-1} R \tau(R^{-1}) \\
&= \tau(R) \Delta(h) \tau(R^{-1}) = \tau(\Delta_{(R)}(h))
\end{aligned}$$

and (QT5) follows. Finally observe that $\widetilde{R}^{-1} = \tau(\widetilde{R})$. Indeed

$$\widetilde{R} \tau(\widetilde{R}) = \tau(R) R^{-1} R \tau(R^{-1}) = \tau(R) \tau(R^{-1}) = 1 \otimes 1$$

and this finishes our proof. ∎

Example 2.4. If (H, R) is a quasitriangular Hopf algebra, then $H_{(R)} = H^{\mathrm{cop}}$. Indeed, it follows from (QT5) that $R \Delta(x) R^{-1} = \Delta^{cop}(x)$.

In the following Proposition, we will present an important class of cocycles.

Proposition 2.5. *Let H be a Hopf algebra and $R \in H \otimes H$ satisfying (QT1) and (QT3). Then R is a Harrison cocycle if and only if R is a solution of the Yang-Baxter equation. In particular, if R satisfies (QT1), (QT3) and (QT5) (this is the case when (H, R) is quasitriangular), then R is a solution of the Yang-Baxter equation (cf. [16]), and therefore a Harrison cocycle.*

Proof. Multiply (QT1) on the left side by R^{12}. This gives

$$\sum R^1 r^1_{(1)} \otimes R^2 r^1_{(2)} \otimes r^2 = R^{12}R^{13}R^{23}$$

Multiplying (QT3) on the left side by R^{23}, we obtain

$$\sum R^1 \otimes r^1 R^2_{(1)} \otimes r^2 R^2_{(2)} = R^{23}R^{13}R^{12}$$

Equality of two left hand sides is therefore equivalent to equality of the right hand-side, so R is a solution of the Yang-Baxter equation if and only if R is a Harrison cocycle. ■

Now let $R \in H \otimes H$ be an invertible cocycle. We define a new comultiplication on H as follows. Let ${}_RH = H$, as a vector space, and define ${}_R\Delta : {}_RH \to {}_RH \otimes {}_RH$ by ${}_R\Delta(h) := R\Delta(h)$, for all $h \in H$. We say that the cocycle R is *normal* if $\sum \varepsilon(R^1)R^2 = 1_H$. From 2) of lemma 2.2 it then follows that $\sum R^1\varepsilon(R^2) = 1_H$.

Lemma 2.6. *Let H be a Hopf algebra and let $R \in H \otimes H$ be a normal cocycle. Then ${}_RH$ is a right H-module coalgebra with ${}_R\Delta(h) = R\Delta(h)$, and with H-action given by right multiplication.*

Proof. We only prove that ${}_RH = H$ is a coalgebra, as the rest is obvious. We have:

$$\begin{aligned} (I \otimes_R \Delta)_R\Delta(h) &= \sum (R^1 \otimes r^1 R^2_{(1)} \otimes r^2 R^2_{(2)})(h_{(1)} \otimes h_{(2)} \otimes h_{(3)}) \\ &= \sum (R^1 r^1_{(1)} h_{(1)} \otimes R^2 r^1_{(2)} h_{(2)} \otimes r^2 h_{(3)}) \\ &\qquad\qquad\qquad\qquad \text{(by (CH))} \\ &= \sum {}_R\Delta(r^1 h_{(1)}) \otimes r^2 h_{(2)} = ({}_R\Delta \otimes I)_R\Delta(h) \end{aligned}$$

i.e. ${}_R\Delta$ is coassociative. ■

Theorem 2.7. *Let H be a Hopf algebra, and consider two normal invertible cocycles $R, W \in H \otimes H$. The following statements are equivalent:*

1. *${}_RH \cong {}_WH$ as right H-module coalgebras;*
2. *there exists an invertible $x \in H$ such that*

$$W = (x^{-1} \otimes x^{-1})R\Delta(x).$$

In this case, the Hopf algebras $H_{(R)}$ and $H_{(W)}$ are isomorphic.

Proof. 1) $\Rightarrow$ 2). Let $\varphi : {}_W H \to {}_R H$ be an isomorphism of right H-module coalgebras, and take $x = \varphi(1)$. We have

$$1 = \varphi^{-1}(\varphi(1)) = \varphi^{-1}(x) = \varphi^{-1}(1)x$$

and

$$1 = \varphi(\varphi^{-1}(1)) = x\varphi^{-1}(1)$$

and therefore x is invertible. φ and φ^{-1} are H-linear, and so we have $\varphi(h) = xh$ and $\varphi^{-1}(h) = x^{-1}h$ for all $h \in H$.

Also φ^{-1} is an H-coalgebra map and therefore

$$({}_W\Delta \circ \varphi^{-1})(1) = ((\varphi^{-1} \otimes \varphi^{-1}) \circ {}_R\Delta)(1)$$

This is equivalent to

$$W\Delta(x^{-1}) = \sum \varphi^{-1}(R^1) \otimes \varphi^{-1}(R^2)$$

or

$$W\Delta(x^{-1}) = (x^{-1} \otimes x^{-1})R$$

or

$$W = (x^{-1} \otimes x^{-1})R\Delta(x)$$

2) $\Rightarrow$ 1). Suppose that $W = (x^{-1} \otimes x^{-1})R\Delta(x)$ for some invertible $x \in H$. We define $\varphi : {}_W H \to {}_R H$ by

$$\varphi(h) = xh$$

for all $h \in H$. It is clear that φ is right H-linear, and it is not difficult to see that φ is a coalgebra map. Indeed, for all $h \in H$, we have that

$$\begin{aligned}
\Big((\varphi \otimes \varphi) \circ {}_W\Delta\Big)(h) &= (\varphi \otimes \varphi)(W\Delta(h)) \\
&= (\varphi \otimes \varphi) \sum (x^{-1}R^1 x_{(1)} h_{(1)} \otimes x^{-1}R^2 x_{(2)} h_{(2)}) \\
&= \sum R^1 x_{(1)} h_{(1)} \otimes R^2 x_{(2)} h_{(2)} \\
&= \sum R\Delta(xh) = {}_R\Delta(xh) = ({}_R\Delta \circ \varphi)(h)
\end{aligned}$$

The inverse of φ is given by the formula $\varphi^{-1}(h) = x^{-1}h$, so φ is an isomorphism of right H-comodule algebras.

Let us finally show that $H_{(R)}$ and $H_{(W)}$ are isomorphic as Hopf algebras. Define $\psi : H_{(R)} \to H_{(W)}$ by $\psi(h) = x^{-1}hx$ for all $h \in H$. It is clear that ψ is an algebra map. It is also a coalgebra map: for all $h \in H$, we have

$$\begin{aligned}
(\Delta_{(W)} \circ \psi)(h) &= W\Delta(x^{-1}hx)W^{-1} \\
&= (x^{-1} \otimes x^{-1})R\Delta(x)\Delta(x^{-1})\Delta(h)\Delta(x)\Delta(x^{-1})R^{-1}(x \otimes x) \\
&= (x^{-1} \otimes x^{-1})R\Delta(h)R^{-1}(x \otimes x)
\end{aligned}$$

and

$$\begin{aligned}
((\psi \otimes \psi) \circ \Delta_{(R)})(h) &= \sum \psi(R^1 h_{(1)} U^1) \otimes \psi(R^2 h_{(2)} U^2) \\
&= \sum x^{-1}R^1 h_{(1)} U^1 x \otimes x^{-1}R^2 h_{(2)} U^2 x \\
&= (\Delta_{(W)} \circ \psi)(h)
\end{aligned}$$

ψ is bijective, its inverse is given by $\psi^{-1}(h) = xhx^{-1}$ for all $h \in H$. We therefore have that ψ is an isomorphism of bialgebras. Let us show that ψ respects the antipode. For all $h \in H$, we have

$$\begin{aligned} S_W(h) = \sum x^{-1}R^1x_{(1)}S(x^{-1}_{(1)}U^1xx^{-1}hxx^{-1}R^2x_{(2)})x^{-1}_{(2)}U^2x \\ = \sum x^{-1}R^1x_{(1)}S(x_{(2)})S(R^2)S(h)S(U^1)S(x^{-1}_{(1)})x^{-1}_{(2)}U^2x \\ = \sum x^{-1}R^1\varepsilon(x)S(U^1hR^2)\varepsilon(x^{-1})U^2x \\ = x^{-1}S_{(R)}(h)x = \psi(S_{(R)}(h)) \end{aligned}$$

and this finishes our proof. ∎

Remarks 2.8. 1) If x is an invertible element of H, then the inner automorphism $\varphi_x: H \to H: h \mapsto xhx^{-1}$ is not necesarry a Hopf algebra automorphism. The last statement of Theorem 2.7 shows us how we can modify the comultiplication on H such that φ_x becomes a Hopf algebra isomorphism.
2) In the case where H is commutative, the equivalence $a) \Leftrightarrow b)$ can also be obtained from a long exact sequence of cohomology groups, we refer to [4].

3 Bialgebra Copairings

Let B and H two bialgebras. We say that (B, H, N) is a *bialgebra copairing* if $N = \sum N^1 \otimes N^2 \in B \otimes H$ (we will also denote $N = n = \nu$) satisfies the following properties:

(CP1) $\sum N^1_{(1)} \otimes N^1_{(2)} \otimes N^2 = \sum N^1 \otimes n^1 \otimes N^2n^2$

(CP2) $\sum N^1 \otimes N^2_{(1)} \otimes N^2_{(2)} = \sum N^1n^1 \otimes n^2 \otimes N^2$

(CP3) $\sum \varepsilon(N^1)N^2 = 1_H$

(CP4) $\sum N^1\varepsilon(N^2) = 1_B$

Remark 3.1. If N is invertible, then (CP3) follows from (CP1), and (CP4) from (CP2). Indeed, if we denote $x = \sum \varepsilon(N^1)N^2$, then x is an invertible idempotent, so $x = 1$.

Examples 3.2. 1. Suppose that (H, R) is a quasitriangular bialgebra. Then (H, H, R) is a bialgebra copairing.
2. Let H be a finite dimensional bialgebra with basis $(e_i)_{i=1,\cdots,n}$, and let $(e^*_i)_{i=1,\cdots,n}$ be the dual basis of H^*. Then $(H, H^*, \sum_{i=1,n} e_i \otimes e^*_i)$ is a bialgebra copairing.
3. Any skew pairing of bialgebras (B, H, σ) (in the sense of [8]) with B and H finite dimensional provides a bialgebra copairing $(B^*, H^*, \sigma^*(1))$.

Proposition 3.3. *Let (B, H, N) be a bialgebra copairing with invertible N. Let $A = B \otimes H$ and $R = \sum(1_B \otimes N^2) \otimes (N^1 \otimes 1_H) \in A \otimes A$. Then R is an invertible Harrison cocycle of $A \otimes A$. If $M = \sum M^1 \otimes M^2$ is the inverse of N, then then the inverse of R is $R^{-1} = \sum(1_B \otimes M^2) \otimes (M^1 \otimes 1_H)$.*

Proof. It is clear that $\sum(1_B \otimes M^2) \otimes (M^1 \otimes 1_H)$ is an inverse of R. Now

$$\begin{aligned}
&\sum R^1 r^1_{(1)} \otimes R^2 r^1_{(2)} \otimes r^2 \\
&= \sum(1_B \otimes N^2)(1_B \otimes n^2)_{(1)} \otimes (N^1 \otimes 1_H)(1_B \otimes n^2)_{(2)} \otimes (n^1 \otimes 1_H) \\
&= \sum(1_B \otimes N^2)(1_B \otimes n^2_{(1)}) \otimes (N^1 \otimes 1_H)(1_B \otimes n^2_{(2)}) \otimes (n^1 \otimes 1_H) \\
&= \sum 1_B \otimes N^2 n^2_{(1)} \otimes N^1 \otimes n^2_{(2)} \otimes n^1 \otimes 1_H \\
&= \sum 1_B \otimes N^2 \nu^2 \otimes N^1 \otimes n^2 \otimes n^1 \nu^1 \otimes 1_H \qquad \text{(using (CP2))}
\end{aligned}$$

and

$$\begin{aligned}
&\sum r^1 \otimes R^1 r^2_{(1)} \otimes R^2 r^2_{(2)} \\
&= \sum(1_B \otimes n^2) \otimes (1_B \otimes N^2)(n^1 \otimes 1_H)_{(1)} \otimes (N^1 \otimes 1_H)(n^1 \otimes 1_H)_{(2)} \\
&= \sum(1_B \otimes n^2) \otimes (1_B \otimes N^2)(n^1_{(1)} \otimes 1_H) \otimes (N^1 \otimes 1_H)(n^1_{(2)} \otimes 1_H) \\
&= \sum 1_B \otimes n^2 \otimes n^1_{(1)} \otimes N^2 \otimes N^1 n^1_{(2)} \otimes 1_H \\
&= \sum 1_B \otimes n^2 \nu^2 \otimes n^1 \otimes N^2 \otimes N^1 \nu^1 \otimes 1_H \qquad \text{(by (CP1))} \\
&= \sum R^1 r^1_{(1)} \otimes R^2 r^1_{(2)} \otimes r^2
\end{aligned}$$

and this means that R is a cocycle. ■

Remark 3.4. In particular, if H is a Hopf algebra and $N = \sum N^1 \otimes N^2$ is an invertible element of $H \otimes H$ which satisfies $(QT1)$ and $(QT3)$ then $R = \sum(1_H \otimes N^2)\otimes(N^1 \otimes 1_H) \in A \otimes A$ is an invertible Harrison cocycle of $A \otimes A$, where $A = H \otimes H$.

Consider now a bialgebra copairing (B, H, N) with invertible N. From the results of Section 2, it follows that we can change the coalgebra structure of $A = B \otimes H$ (with the tensor product bialgebra structure) using the element R constructed in proposition 3.3. We obtain a bialgebra $A_{(R)}$, and we denote this bialgebra by $B \bowtie^N H$. The coalgebra structure is given by (let $M = N^{-1}$):

$$\Delta(b \bowtie h) = \sum b_{(1)} \bowtie N^2 h_{(1)} M^2 \otimes N^1 b_{(2)} M^1 \bowtie h_{(2)}$$

and

$$\varepsilon(b \bowtie h) = \varepsilon_B(b)\varepsilon_H(h)$$

for any $b \in B$ and $h \in H$.

Proposition 3.5. *Let (B, H, N) be a bialgebra copairing with N invertible. Then the bialgebra $B \bowtie^N H$ is a universal object among bialgebras J having the property that there exist bialgebra maps $\alpha : J \to B$ and $\beta : J \to H$ such that*

$$\sum N^1 \alpha(x_{(1)}) \otimes N^2 \beta(x_{(2)}) = \sum \alpha(x_{(2)}) N^1 \otimes \beta(x_{(1)}) N^2 \qquad (7)$$

for any $x \in J$.

Proof. First define $\alpha' : B \bowtie^N H \to B$ and $\beta' : B \bowtie^N H \to H$ by

$$\alpha'(b \bowtie h) = \varepsilon_H(h) b \quad \text{and} \quad \beta'(b \bowtie h) = \varepsilon_B(b) h$$

It is clear that α' and β' are bialgebra maps that satisfy (7). Indeed, the left hand side of 7 is

$$\begin{aligned}&\sum n^1 b_{(1)}\varepsilon_H(N^2 h_{(1)}M^2) \bowtie N^2 h_{(2)}\varepsilon_B(N^1 b_{(2)}M^1)\\ &= \sum n^1 b \bowtie n^2 h\end{aligned}$$

while the right hand side amounts to

$$\begin{aligned}&\sum N^1 b_{(2)}M^1\varepsilon_H(h_{(2)})n_1 \bowtie \varepsilon_B(b_{(1)})N^2 h_{(1)}M^2 n^2\\ &= \sum N^1 b M^1 n_1 \bowtie N^2 h M^2 n^2\\ &= \sum n^1 b \bowtie n^2 h\end{aligned}$$

Now suppose that $\alpha: J \to B$ and $\beta: J \to H$ satisfy (7). We define $f: J \to B \bowtie^N H$ by the following formula

$$f(x) = \sum \alpha(x_{(1)}) \otimes \beta(x_{(2)}) \tag{8}$$

for all $x \in J$. We then have that

$$f(xy) = \sum \alpha(x_{(1)}y_{(1)}) \otimes \beta(x_{(2)}y_{(2)}) = \sum \alpha(x_{(1)})\alpha(y_{(1)}) \otimes \beta(x_{(2)})\beta(y_{(2)}) = f(x)f(y)$$

and $f(1) = \alpha(1) \otimes \beta(1) = 1 \otimes 1$, so f is an algebra morphism.
To see that f is a coalgebra morphism, observe that:

$$\begin{aligned}&\Delta_{(R)}(f(x))\\ &= \sum \alpha(x_{(1)})_{(1)} \otimes N^2\beta(x_{(2)})_{(1)}M^2 \otimes N^1\alpha(x_{(1)})_{(2)}M^1 \otimes \beta(x_{(2)})_{(2)}\\ &= \sum \alpha(x_{(1)}) \otimes N^2\beta(x_{(3)})M^2 \otimes N^1\alpha(x_{(2)})M^1 \otimes \beta(x_{(4)})\\ &= \sum \alpha(x_{(1)}) \otimes \beta(x_{(2)})N^2M^2 \otimes \alpha(x_{(3)})N^1M^1 \otimes \beta(x_{(4)})\\ &\qquad\qquad\text{(by the condition in the hypothesis)}\\ &= \sum \alpha(x_{(1)}) \otimes \beta(x_{(2)}) \otimes \alpha(x_{(3)}) \otimes \beta(x_{(4)}) = \sum f(x_{(1)}) \otimes f(x_{(2)})\end{aligned}$$

and

$$\varepsilon(f(x)) = \sum \varepsilon_B(\alpha(x_{(1)}))\varepsilon_H(\beta(x_{(2)})) = \sum \varepsilon_J(x_{(1)})\varepsilon_J(x_{(2)}) = \varepsilon_J(x)$$

Finally, it is clear that $\alpha' \circ f = \alpha$ and $\beta' \circ f = \beta$, and this finishes the proof. ■

Proposition 3.6. *Let (B,H,N) be a bialgebra copairing with invertible N. Moreover, suppose that (H,σ) is a braided bialgebra. Then the map $j: H \to B \bowtie^N H$ defined by $j(h) = \sum \sigma(N^2, h_{(1)})N^1 \otimes h_{(2)}$, is an injective bialgebra morphism.*

Proof. Let us define $\alpha: H \to B$ and $\beta: H \to H$ by $\alpha(h) = \sum \sigma(N^2, h)N^1$ and $\beta = Id$.
Then $j(h) = \sum \alpha(h_{(1)}) \otimes \beta(h_{(2)})$, so according to the previous proposition, to prove that j is a bialgebra morphism is enough to show that α is a bialgebra morphism, and that the relation

$$\sum N^1\alpha(x_{(1)}) \otimes N^2 x_{(2)} = \sum \alpha(x_{(2)})N^1 \otimes x_{(1)}N^2 \tag{9}$$

holds for any $x \in H$. We then have that:

$$\begin{aligned} \alpha(xy) &= \sum \sigma(N^2, xy)N^1 \\ &= \sum \sigma(N^2_{(1)}, y)\sigma(N^2_{(2)}, x)N^1 && \text{(by (BB3))} \\ &= \sum \sigma(n^2, y)\sigma(N^2, x)N^1 n^1 \\ &= \sum \sigma(N^2, x)N^1 \sigma(n^2, y)n^1 = \alpha(x)\alpha(y) \end{aligned}$$

and

$$\alpha(1_H) = \sum \sigma(N^2, 1_H)N^1 = \sum \varepsilon(N^2)N^1 = 1_B$$

(by (BB4) and (CP4)). Hence α is an algebra map. On the other hand

$$\begin{aligned} \Delta(\alpha(x)) &= \sum \sigma(N^2, x)N^1_{(1)} \otimes N^1_{(2)} \\ &= \sum \sigma(N^2 n^2, x)N^1 \otimes n^1 && \text{(by (CP1))} \\ &= \sum \sigma(N^2, x_{(1)})\sigma(n^2, x_{(2)})N^1 \otimes n^1 && \text{(by (BB2))} \\ &= \sum \sigma(N^2, x_{(1)})N^1 \otimes \sigma(n^2, x_{(2)})n^1 \\ &= \sum \alpha(x_{(1)}) \otimes \alpha(x_{(2)}) \end{aligned}$$

and

$$\varepsilon(\alpha(x)) = \sum \sigma(N^2, x)\varepsilon(N^1) = \sum \sigma(\varepsilon(N^1)N^2, x) = \sigma(1_H, x) = \varepsilon(x)$$

(by (BB4) and (CP3)), and it follows that α is a bialgebra morphism. To prove that (9) holds, we proceed as follows.

$$\begin{aligned} \sum N^1\alpha(x_{(1)}) \otimes N^2 x_{(2)} &= \sum N^1\sigma(n^2, x_{(1)})n^1 \otimes N^2 x_{(2)} \\ &= \sum N^1\sigma(N^2_{(1)}, x_{(1)}) \otimes N^2_{(2)}x_{(2)} && \text{(by (CP2))} \\ &= \sum N^1 \otimes \sigma(N^2_{(1)}, x_{(1)})N^2_{(2)}x_{(2)} \\ &= \sum N^1 \otimes x_{(1)}N^2_{(1)}\sigma(N^2_{(2)}, x_{(2)}) && \text{(by (BB1))} \\ &= \sum N^1 n^1 \otimes x_{(1)}n^2\sigma(N^2, x_{(2)}) \\ &= \sum \alpha(x_{(2)})n^1 \otimes x_{(1)}n^2 \end{aligned}$$

Suppose finally $j(h) = \sum \sigma(N^2, h_{(1)})N^1 \otimes h_{(2)} = 0$. Applying $\varepsilon \otimes I$ to this equality, and using (CP3) we obtain that $\sum \sigma(1_H, h_{(1)})h_{(2)} = 0$. It then follows from (BB4) that $h = 0$ and this shows that j is injective. ∎

Proposition 3.7. *Let H be a bialgebra and suppose that $R \in H \otimes H$ is invertible and satisfies (QT1) and (QT3). Then H is a quasitriangular bialgebra if and only if $\Delta = \Delta_H : H \to H \bowtie^R H$ is a coalgebra morphism.*

Proof. We denote the inverse of R by U. Of course (H, R) is a quasitriangular bialgebra if and only if (QT5) holds. First suppose that (QT5) holds; then

$$\begin{aligned} \Delta_{(R)}(\Delta(a)) &= \sum a_{(1)} \otimes R^2 a_{(3)}U^2 \otimes R^1 a_{(2)}U^1 \otimes a_{(4)} \\ &= \sum a_{(1)} \otimes a_{(2)}R^2U^2 \otimes a_{(3)}R^1U^1 \otimes a_{(4)} && \text{(by (QT5))} \\ &= \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \otimes a_{(4)} \\ &= (\Delta \otimes \Delta)(\Delta(a)) \end{aligned}$$

so Δ is a coalgebra morphism (the counit property is clear).
Conversely, suppose that Δ is a coalgebra morphism. Then writing $\Delta_{(R)}(\Delta(a)) = (\Delta \otimes \Delta)(\Delta(a))$ and applying $\varepsilon \otimes I \otimes I \otimes \varepsilon$ we obtain

$$\sum R^2 a_{(2)} U^2 \otimes R^1 a_{(1)} U^1 = \sum a_1 \otimes a_2$$

Multiplying both sides on the right by $\sum R^2 \otimes R^1$, we obtain (QT5). ∎

The following lemma shows that the inverse of N in a bialgebra copairing satisfies relations similar to (CP1)-(CP4).

Lemma 3.8. *Let (B, H, N) be a bialgebra copairing and suppose that N is invertible. Denote $N^{-1} = M = m = \mu$. Then:*

(CP1') $\sum M^1_{(1)} \otimes M^1_{(2)} \otimes M^2 = \sum M^1 \otimes m^1 \otimes m^2 M^2$

(CP2') $\sum M^1 \otimes M^2_{(1)} \otimes M^2_{(2)} = \sum M^1 m^1 \otimes M^2 \otimes m^2$

(CP3') $\sum \varepsilon(M^1) M^2 = 1_H$

(CP4') $\sum M^1 \varepsilon(M^2) = 1_B$

Proof. For $i = 1, 2, 3, 4$, (CPi') is nothing else then the inverse of (CPi). ∎

Lemma 3.9. *Let (B, H, N) be a bialgebra copairing. If B has an antipode S (respectively H has a pode $\overline{S}$), then N is invertible with inverse $N^{-1} = \sum S_B(N^1) \otimes N^2$ (respectively $N^{-1} = \sum N^1 \otimes \overline{S}_H(N^2)$).*

Proof. Straightforward. ∎

Proposition 3.10. *Let (B, H, N) be a bialgebra copairing. Suppose that both B and H are Hopf algebras with antipodes S_B and S_H (respectively both B and H are anti-Hopf algebras with podes $\overline{S}_B$ and $\overline{S}_H$). Let us denote by M the inverse of N (which exits by the previous lemma). Then*

1. $\sum M^1 \otimes S_H(M^2) = N$ *(resp.* $\sum \overline{S}_B(M^1) \otimes M^2 = N$*):*
2. $\sum S_B(N^1) \otimes S_H(N^2) = N$ *(resp.* $\sum \overline{S}_B(N^1) \otimes \overline{S}_H(N^2) = N$*).*

Proof. Suppose that B and H are Hopf algebras (the proof in the case where B and H are anit-Hopf algebras is similar).
1. Using (CP2') and (CP4'), we obtain that

$$\begin{aligned} M(\sum m^1 \otimes S_H(m^2)) &= \sum M^1 m^1 \otimes M^2 S_H(m^2) \\ &= \sum M^1 \otimes M^2_{(1)} S_H(M^2_{(2)}) = \sum M^1 \otimes \varepsilon(M^2) = 1 \otimes 1 \end{aligned}$$

This shows that $N = M^{-1} = \sum M^1 \otimes S_H(M^2)$.
2. We know from Lemma 3.9 that $M = N^{-1} = \sum S_B(n^1) \otimes n^2$. It then follows from 1. that

$$N = \sum M^1 \otimes S_H(M^2) = \sum S_B(N^1) \otimes S_H(N^2)$$

∎

Remark 3.11. If (B, H, N) is a bialgebra copairing , then N also defines a bialgebra copairing (B^{cop}, H^{op}, N). Moreover, if both B and H are Hopf algebras (respectively anti-Hopf algebras), then N is invertible in $B^{cop} \otimes H^{op}$. This follows easily from (CP1) and (CP2).

Proposition 3.12. *Let (A, N) be a quasitriangular bialgebra. Then there exists a map $f : A^0 \to A \bowtie^N A$ which is a coalgebra morphism and an antimorphism of algebras (here A^0 is the finite dual of the bialgebra A).*

Proof. We define the maps

$$\mu_l : \; A^0 \to A \;\; \text{and} \;\; \mu_r : \; A^0 \to A$$

by

$$\mu_l(a^*) = \sum a^*(N^1)N^2 \;\; \text{and} \;\; \mu_r(a^*) = \sum a^*(N^2)N^1$$

for any $a^* \in A^0$. It may be checked easily that μ_r is a coalgebra morphism and an antimorphism of algebras, and μ_l is an algebra morphism and an antimorphism of coalgebras. Moreover, μ_l is convolution invertible and $\mu_l^{-1}(a^*) = \sum a^*(M^1)M^2$, and μ_l^{-1} is a coalgebra morphism and an antimorphism of algebras. Regarding μ_r and μ_l^{-1} as maps from $(A^0)^{\mathrm{op}}$ to A, we obtain that the map $f : A^0 \to A \bowtie^N A$ defined by

$$f(a) = \sum \mu_r(a_{(1)}) \otimes \mu_l^{-1}(a_{(2)})$$

is an algebra antimorphism and a coalgebra morphism if we can prove that μ_r and μ_l^{-1} satisfy the relation:

$$\sum N^1\mu_r(a^*_{(1)}) \otimes N^2\mu_l^{-1}(a^*_{(2)}) = \sum \mu_r(a^*_{(2)})n^1 \otimes \mu_l^{-1}(a^*_{(1)})n^2$$

or, equivalently

$$\sum a^*(n^2M^1)N^1n^1 \otimes N^2M^2 = \sum a^*(m^1N^2)N^1n^1 \otimes m^2n^2$$

So we are done if we prove that

$$\sum N^1n^1 \otimes N^2M^2 \otimes n^2M^1 = \sum N^1n^1 \otimes m^2n^2 \otimes m^1N^2$$

But this follows directly from (QT5) and the proof is finished. ■

4 Double Crosscoproducts

The purpose of this Section is to carry out the construction of the new "product" of Hopf algebras that was suggested by Radford in [16]. This will shed more light on the constructions from the previous Section.

Troughout this Section, B and H will be bialgebras, B will be a left H-comodule algebra, with the comodule structure given by $\rho : \; B \to H \otimes B$ (we will denote $\rho(b) = \sum b_{[-1]} \otimes b_{[0]}$), and H will be a right B-comodule algebra, with the comodule

structure given by $\psi: \; H \to H \otimes B$ (we will denote $\psi(h) = \sum h_{<0>} \otimes h_{<1>}$).
We define a comultiplication map Δ and a counit map ε on $B \otimes H$ as follows

$$\Delta(b \otimes h) = \sum b_{(1)} \otimes b_{(2)[-1]} h_{(1)<0>} \otimes b_{(2)[0]} h_{(1)<1>} \otimes h_{(2)}$$

$$\varepsilon(b \otimes h) = \varepsilon_B(b) \varepsilon_H(h)$$

for all $b \in B$ and $h \in H$, and we also consider the tensor product algebra structure on $K = B \otimes H$. In the next Theorem, we introduce the construction of the double crosscoproduct.

Theorem 4.1. *Let B and H be bialgebras such that B is a left H-comodule algebra and H is a right B-comodule algebra as above. Then $K = B \otimes H$ is a bialgebra with the tensor product algebra structure and coalgebra structure defined by Δ and ε as above if and only if the following relations are satisfied for all $h \in H$ and $b \in B$:*

(i1) $\sum b_{[-1]} \varepsilon_B(b_{[0]}) = \varepsilon_B(b) 1_H$

(i2) $\sum b_{[-1]} \otimes b_{[0](1)} \otimes b_{[0](2)} = \sum b_{(1)[-1]} b_{(2)[-1]<0>} \otimes b_{(1)[0]} b_{(2)[-1]<1>} \otimes b_{(2)[0]}$

(ii1) $\sum \varepsilon_H(h_{<0>}) h_{<1>} = \varepsilon_H(h) 1_B$

(ii2) $\sum h_{<0>(1)} \otimes h_{<0>(2)} \otimes h_{<1>} = \sum h_{(1)<0>} \otimes h_{(1)<1>[-1]} h_{(2)<0>} \otimes h_{(1)<1>[0]} h_{(2)<1>}$

(iii) $\sum h_{<0>} b_{[-1]} \otimes h_{<1>} b_{[0]} = \sum b_{[-1]} h_{<0>} \otimes b_{[0]} h_{<1>}$

In this situation, we call K the double crosscoproduct of the bialgebras B and H. Moreover, if B and H are Hopf algebras with antipodes S_B and S_H, then K is also a Hopf algebra with antipode given by :

$$S(b \otimes h) = \sum S_B(b_{[0]} h_{<1>}) \otimes S_H(b_{[-1]} h_{<0>}) \tag{10}$$

If S_B and S_H are bijective, then S is bijective with inverse:

$$S^{-1}(b \otimes h) = \sum S_B^{-1}(b_{[0]} h_{<1>}) \otimes S_H^{-1}(b_{[-1]} h_{<0>})$$

Proof. Since B is a left H-comodule algebra we have that:

(B1) $\sum b_{[-1](1)} \otimes b_{[-1](2)} \otimes b_{[0]} = \sum b_{[-1]} \otimes b_{[0][-1]} \otimes b_{[0][0]}$
(B2) $\sum \varepsilon_H(b_{[-1]}) b_{[0]} = b$
(B3) $\sum (1_B)_{[-1]} \otimes (1_B)_{[0]} = 1_H \otimes 1_B$
(B4) $\sum (bc)_{[-1]} \otimes (bc)_{[0]} = \sum b_{[-1]} c_{[-1]} \otimes b_{[0]} c_{[0]}$

Since H is a right B-comodule algebra we have that:

(H1) $\sum h_{<0>} \otimes h_{<1>(1)} \otimes h_{<1>(2)} = \sum h_{<0><0>} \otimes h_{<0><1>} \otimes h_{<1>}$
(H2) $\sum h_{<0>} \varepsilon_B(h_{<1>}) = h$
(H3) $\sum (1_H)_{<0>} \otimes (1_H)_{<1>} = 1_H \otimes 1_B$
(H4) $\sum (hg)_{<0>} \otimes (hg)_{<1>} = \sum h_{<0>} g_{<0>} \otimes h_{<1>} g_{<1>}$

If we apply $I \otimes I \otimes \rho$ to (i2) we obtain that:

$$\begin{aligned} &\sum b_{[-1]} \otimes b_{[0](1)} \otimes b_{[0](2)[-1]} \otimes b_{[0](2)[0]} \\ &= \sum b_{(1)[-1]} b_{(2)[-1]<0>} \otimes b_{(1)[0]} b_{(2)[-1]<1>} \otimes b_{(2)[0][-1]} \otimes b_{(2)[0][0]} \end{aligned} \tag{11}$$

From (H1), it follows that:

$$\begin{aligned}&\sum h_{<0>} \otimes h_{<1>(1)} \otimes h_{<1>(2)[-1]} \otimes h_{<1>(2)[0]} \\ &\quad = \sum h_{<0><0>} \otimes h_{<0><1>} \otimes h_{<1>[-1]} \otimes h_{<1>[0]} \end{aligned} \tag{12}$$

We obtain from (B1) that:

$$\begin{aligned}&\sum b_{[-1](1)<0>} \otimes b_{[-1](1)<1>} \otimes b_{[-1](2)} \otimes b_{[0]} \\ &\quad = \sum b_{[-1]<0>} \otimes b_{[-1]<1>} \otimes b_{[0][-1]} \otimes b_{[0][0]} \end{aligned} \tag{13}$$

and we obtain from (ii2) that:

$$\begin{aligned}&\sum h_{<0>(1)<0>} \otimes h_{<0>(1)<1>} \otimes h_{<0>(2)} \otimes h_{<1>} \\ &\quad = \sum h_{(1)<0><0>} \otimes h_{(1)<0><1>} \otimes h_{(1)<1>[-1]}h_{(2)<0>} \otimes h_{(1)<1>[0]}h_{(2)<1>} \end{aligned} \tag{14}$$

Suppose now that the relations (i)-(iii) hold. We prove that K is a bialgebra. The computations are tedious, but straightforward, so we only give a brief sketch of the proof.

When computing $((I \otimes \Delta)\Delta)(b \otimes h)$, we apply (B4) for $b_{(2)[0](2)}$ and $h_{(1)<1>(2)}$, (11) for $b_{(2)}$, and (12) for $h_{(1)}$. When computing $((\Delta \otimes I)\Delta)(b \otimes h)$ we apply (H4) for $b_{(2)[-1](1)}$ and $h_{(1)<0>(1)}$, (13) for $b_{(3)}$, and (14) for $h_{(1)}$. After a long computation, we obtain that

$$\begin{aligned}&((I \otimes \Delta)\Delta)(b \otimes h) \\ &\quad = ((\Delta \otimes I)\Delta)(b \otimes h) \\ &\quad = \sum b_{(1)} \otimes b_{(2)[-1]}b_{(3)[-1]<0>}h_{(1)<0><0>} \otimes b_{(2)[0]}b_{(3)[-1]<1>}h_{(1)<0><1>} \\ &\qquad \otimes b_{(3)[0][-1]}h_{(1)<1>[-1]}h_{(2)<0>} \otimes b_{(3)[0][0]}h_{(1)<1>[0]}h_{(2)<1>} \otimes h_{(3)}\end{aligned}$$

We check now that ε is a counit map.

$$\begin{aligned}&\sum \varepsilon((b \otimes h)_{(1)})(b \otimes h)_{(2)} \\ &\quad = \sum \varepsilon(b_{(1)} \otimes b_{(2)[-1]}h_{(1)<0>})b_{(2)[0]}h_{(1)<1>} \otimes h_{(2)} \\ &\quad = \sum \varepsilon_B(b_{(1)})\varepsilon_H(b_{(2)[-1]})\varepsilon_H(h_{(1)<0>})b_{(2)[0]}h_{(1)<1>} \otimes h_{(2)} \\ &\quad = b \otimes h \qquad \text{(by (B2) and (ii1))}\end{aligned}$$

and

$$\begin{aligned}&\sum (b \otimes h)_{(1)}\varepsilon((b \otimes h)_{(2)}) \\ &\quad = \sum b_{(1)} \otimes b_{(2)[-1]}h_{(1)<0>}\varepsilon_B(b_{(2)[0]})\varepsilon_B(h_{(1)<1>})\varepsilon_H(h_{(2)}) \\ &\quad = \sum b_{(1)} \otimes b_{(2)[-1]}\varepsilon_B(b_{(2)[0]})h_{<0>}\varepsilon_B(h_{<1>}) \\ &\quad = b \otimes h \qquad \text{(by (H2) and (i1))}\end{aligned}$$

Therefore (K, Δ, ε) is a coalgebra.

It is obvious that ε is an algebra morphism. Let us show that Δ is also an algebra morphism.

$$\begin{aligned}
&\Delta((b\otimes g)(c\otimes h))\\
&= \sum b_{(1)}c_{(1)}\otimes (b_{(2)}c_{(2)})_{[-1]}(g_{(1)}h_{(1)})_{<0>}\otimes (b_{(2)}c_{(2)})_{[0]}(g_{(1)}h_{(1)})_{<1>}\otimes g_{(2)}h_{(2)}\\
&= \sum b_{(1)}c_{(1)}\otimes b_{(2)[-1]}c_{(2)[-1]}g_{(1)<0>}h_{(1)<0>}\otimes b_{(2)[0]}c_{(2)[0]}g_{(1)<1>}h_{(1)<1>}g_{(2)}h_{(2)}\\
&\qquad\qquad\qquad\qquad\text{(by (B4) and (H4))}\\
&= \sum b_{(1)}c_{(1)}\otimes b_{(2)[-1]}g_{(1)<0>}c_{(2)[-1]}h_{(1)<0>}\otimes b_{(2)[0]}g_{(1)<1>}c_{(2)[0]}h_{(1)<1>}\otimes g_{(2)}h_{(2)}\\
&\qquad\qquad\qquad\qquad\text{(by (iii))}\\
&= \Delta(b\otimes g)\Delta(c\otimes h)
\end{aligned}$$

and

$$\Delta(1_B\otimes 1_H)=\sum 1_B\otimes (1_B)_{[-1]}(1_H)_{<0>}\otimes (1_B)_{[0]}(1_H)_{<1>}\otimes 1_H = 1_B\otimes 1_H\otimes 1_B\otimes 1_H$$

where we used (B3) and (H3) for the last equality. Thus we have proved that K is a bialgebra.
Conversely, suppose that K is a bialgebra, and write

$$\Delta((b\otimes g)(c\otimes h))=\Delta(b\otimes g)\Delta(c\otimes h)$$

with $b=1_B, h=1_H$. Applying $\varepsilon_B\otimes I\otimes I\otimes\varepsilon_H$ to both sides, we obtain (iii).
Now write

$$\sum \varepsilon((b\otimes h)_{(1)})(b\otimes h)_{(2)}=b\otimes h$$

with $b=1$ and apply $I\otimes\varepsilon_H$to both sides. Then (ii1) follows. Similarly, if we apply $\varepsilon_B\otimes I$ to both sides of the other counit property, with $h=1$, then we obtain (i1).
Now we write

$$((I\otimes\Delta)\Delta)(b\otimes h)=((\Delta\otimes I)\Delta)(b\otimes h) \tag{15}$$

with $h=1$, and apply ε on the first and the fourth factor of both sides. Using (B2) we obtain (i2).
Take $b=1$ in (15), and apply ε to the second factor. Then use (H2) and apply ε on the fourth position. This yields (ii2).
Let us suppose now that B and H are Hopf algebras. To prove that K is a Hopf

algebra it is enough to show that (10) provides an antipode S for K. We have that

$$
\begin{aligned}
&\sum S((b\otimes h)_{(1)})(b\otimes h)_{(2)}\\
&= \sum S(b_{(1)}\otimes b_{(2)[-1]}h_{(1)<0>})(b_{(2)[0]}h_{(1)<1>}\otimes h_{(2)})\\
&= \sum (S_B(b_{(1)[0]}(b_{(2)[-1]}h_{(1)<0>})_{<1>})\\
&\qquad \otimes S_H(b_{(1)[-1]}(b_{(2)[-1]}h_{(1)<0>})_{<0>}))(b_{(2)[0]}h_{(1)<1>}\otimes h_{(2)})\\
&= \sum S_B(b_{(1)[0]}b_{(2)[-1]<1>}h_{(1)<0><1>})b_{(2)[0]}h_{(1)<1>}\\
&\quad \otimes S_H(b_{(1)[-1]}b_{(2)[-1]<0>}h_{(1)<0><0>})h_{(2)}\\
&\qquad\qquad\qquad\qquad \text{(by (H4))}\\
&= \sum S_B(b_{[0](1)}h_{(1)<0><1>})b_{[0](2)}h_{(1)<1>}\otimes S_H(b_{[-1]}h_{(1)<0><0>})h_{(2)}\\
&\qquad\qquad\qquad\qquad \text{(by (i2))}\\
&= \sum S_B(b_{[0](1)}h_{(1)<1>(1)})b_{[0](2)}h_{(1)<1>(2)}\otimes S_H(b_{[-1]}h_{(1)<0>})h_{(2)}\\
&\qquad\qquad\qquad\qquad \text{(by (H1) for h}_{(1)}\text{)}\\
&= \sum \varepsilon(b_{[0]})\varepsilon(h_{(1)<1>})\otimes S_H(b_{[-1]}h_{(1)<0>})h_{(2)}\\
&= \sum 1\otimes\varepsilon(b)S_H(h_{(1)})h_{(2)}\\
&\qquad\qquad\qquad\qquad \text{(by (i1) and (H2))}\\
&= \varepsilon(b\otimes h)1\otimes 1
\end{aligned}
$$

and it follows that S is a left convolution inverse of I. A similar computation shows that S is also a right convolution inverse of I.

Finally, suppose that S_B and S_H are bijective. Let us recall the well-known fact that for a Hopf algebra A with antipode S, if there exists a k-linear map S' satisfying $\sum S'(h_{(2)})h_{(1)} = \sum h_{(2)}S'(h_{(1)}) = \varepsilon(h)1$ for any $h\in A$, then S is bijective and $S^{-1}=S'$.

Define S' by the formula

$$S'(b\otimes h)=\sum S_B^{-1}(b_{[0]}h_{<1>})\otimes S_H^{-1}(b_{[-1]}h_{<0>})$$

The proof is finished if we can show that

$$\sum S'((b\otimes h)_{(2)})(b\otimes h)_{(1)}=\sum(b\otimes h)_{(2)}S'((b\otimes h)_{(1)})=\varepsilon(b\otimes h)1\otimes 1$$

Indeed, we have:

$$
\begin{aligned}
&\sum S'((b\otimes h)_{(2)})(b\otimes h)_{(1)}\\
&= \sum S'(b_{(2)[0]}h_{(1)<1>}\otimes h_{(2)})(b_{(1)}\otimes b_{(2)[-1]}h_{(1)<0>})\\
&= \sum (S_B^{-1}((b_{(2)[0]}h_{(1)<1>})_{[0]}h_{(2)<1>})\otimes S_H^{-1}((b_{(2)[0]}h_{(1)<1>})_{[-1]}h_{(2)<0>}))\\
&\quad (b_{(1)}\otimes b_{(2)[-1]}h_{(1)<0>})\\
&= \sum S_B^{-1}(b_{(2)[0][0]}h_{(1)<1>[0]}h_{(2)<1>})b_{(1)}\\
&\quad \otimes S_H^{-1}(b_{(2)[0][-1]}h_{(1)<1>[-1]}h_{(2)<0>})b_{(2)[-1]}h_{(1)<0>}\\
&\qquad\qquad\qquad\qquad \text{(by (B4))}\\
&= \sum S_B^{-1}(b_{(2)[0]}h_{<1>})b_{(1)}\otimes S_H^{-1}(b_{(2)[-1](2)}h_{<0>(2)})b_{(2)[-1](1)}h_{<0>(1)}\\
&\qquad\qquad\qquad\qquad \text{(by (ii2) and (B1))}\\
&= \sum S_B^{-1}(b_{(2)[0]}h_{<1>})b_{(1)}\otimes \varepsilon(h_{<0>})\varepsilon(b_{(2)[-1]})1\\
&= \sum S_B^{-1}(b_{(2)})b_{(1)}\otimes \varepsilon(h)1\\
&\qquad\qquad\qquad\qquad \text{(by (B2) and (ii1))}\\
&= \varepsilon(b\otimes h)1\otimes 1
\end{aligned}
$$

The proof of the second equality is similar. ■

We will now show that the construction of the bialgebra $B \bowtie^N H$ in Section 3 is a double crosscoproduct. Consider a bialgebra copairing (B, H, N) with invertible $N \in B \otimes H$. Let $M = N^{-1}$. We define:

$$\rho : B \to H \otimes B, \quad \rho(b) = \sum N^2 M^2 \otimes N^1 b M^1$$

for all $b \in B$, and

$$\psi : H \to H \otimes B, \quad \psi(h) = \sum N^2 h M^2 \otimes N^1 M^1$$

for any $h \in H$.

Proposition 4.2. *Let (B, H, N) be a bialgebra copairing with invertible N. Then the above structures make B into a left H-comodule algebra, and H into a right B-comodule algebra. Moreover, the conditions (i)-(iii) hold, and the double crossco-product of B and H is just the bialgebra $B \bowtie^N H$.*

Proof. We check only that B is a left H-comodule algebra. Indeed:

$$
\begin{aligned}
&(\Delta_H\otimes I)\rho(b)\\
&= \sum N^2_{(1)}M^2_{(1)}\otimes N^2_{(2)}M^2_{(2)}\otimes N^1bM^1\\
&= \sum n^2M^2\otimes N^2m^2\otimes N^1n^1bM^1m^1\\
&\qquad\qquad\qquad \text{(by (CP2) and (CP2}') \text{)}
\end{aligned}
$$

and

$$
\begin{aligned}
((I\otimes\rho)\rho)(b) &= \sum N^2M^2\otimes\rho(N^1bM^1)\\
&= \sum N^2M^2\otimes n^2m^2\otimes n^1N^1bM^1m^1 = (\Delta_H\otimes I)\rho(b)
\end{aligned}
$$

The counit property is clear, so B is a left H-comodule. To prove that ρ is an algebra morphism, we observe that:

$$\begin{aligned}\rho(b)\rho(c) &= (\sum N^2M^2 \otimes N^1bM^1)(\sum n^2m^2 \otimes n^1cm^1)\\ &= \sum N^2M^2n^2m^2 \otimes N^1bM^1n^1cm^1\\ &= \sum N^2m^2 \otimes N^1bcm^1 = \rho(bc)\end{aligned}$$

for all $b, c \in B$. It is clear that $\rho(1) = 1 \otimes 1$. We already know from the second section that $B \bowtie^N H$ is a bialgebra with certain coalgebra and algebra structures, and these structures are just the ones we have defined at the beginning of this section for $K = B \otimes H$. Therefore we obtain the required statement directly from the theorem. ∎

Corollary 4.3. *Let (B, H, N) be a bialgebra copairing such that B and H are Hopf algebras with antipodes S_B and S_H. Let $M = N^{-1}$. Then $B \bowtie^N H$ is a Hopf algebra with antipode:*

$$S(b \otimes h) = \sum S_B(N^1bM^1) \otimes S_H(N^2hM^2)$$

Moreover, if S_B and S_H are bijective, then S is bijective with inverse given by:

$$S^{-1}(b \otimes h) = \sum S_B^{-1}(N^1bM^1) \otimes S_H^{-1}(N^2hM^2)$$

Examples 4.4. 1. If B and H are finite dimensional bialgebras and $B \bowtie H$ is a double crossproduct in the sense of Majid ([11]), then its dual is a double crosscoproduct of B^* and H^*.
2. In particular, the dual of the Drinfeld double $D(H)$ is the double crosscoproduct, as Radford suggested in [16, Proposition 11].
3. Let B and H be two bialgebras as in the beginning of this section, such that the coaction of B on H is trivial, that is $\psi(h) = h \otimes 1_B$, for all $h \in H$. Then observe the following:

- (i1) and (i2) mean that B is a left H-comodule coalgebra;
- (ii1) and (ii2) are clearly satisfied;
- (iii) means that $\sum hb_{[-1]} \otimes b_{[0]} = \sum b_{[-1]}h \otimes b_{[0]}$.

In this case the coalgebra structure of $K = B \otimes H$ is just the smash coproduct of Molnar ([12]).
4. Take $H = B$ in 3., and suppose that H is a commutative Hopf algebra. We define $\rho : B \to H \otimes B$ by $\rho(b) = \sum b_{(1)}S(b_{(3)}) \otimes b_{(2)} = \sum b_{[-1]} \otimes b_{[0]}$, $\psi : H \to H \otimes B$ by $\psi(h) = h \otimes 1_B = \sum h_{<0>} \otimes h_{<1>}$.
If $G = M_k(H, k)$ is the affine algebraic group corresponding to H (see [1, p.175]),then ρ is H-comodule algebra structure map on H corresponding to the left action of G on itself, namely $\sigma : G \times G \to G : (g, h) \mapsto ghg^{-1}$. In fact, ρ is also H-comodule coalgebra structure map on H, named the coadjoint action in [12].

The conditions (i)-(iii) are satisfied according to 3. Thus $H \otimes H$ is a commutative Hopf algebra, with the tensor product algebra multiplication, and the smash coproduct comultiplication given by:

$$\Delta(b \otimes h) = \sum b_{(1)} \otimes b_{(2)}S(b_{(4)})h_{(1)} \otimes b_{(3)} \otimes h_{(2)} = \sum b_{(1)} \otimes b_{(2)[-1]}h_{(1)} \otimes b_{(2)[0]} \otimes h_{(2)}$$

The antipode is :

$$\begin{aligned} S(b \otimes h) &= \sum S(b_{(2)}) \otimes S(b_{(1)}S(b_{(3)})h) \\ &= \sum S(b_{(2)}) \otimes S(b_{(1)})b_{(3)}S(h) \end{aligned}$$

since H is commutative.

5 Inner coactions and crossed coproducts

Let C be a coalgebra and A an algebra over a field k. A k-linear map

$$\nu : \ C \longrightarrow A \otimes C : \ c \mapsto c_{\langle -1 \rangle} \otimes c_{\langle 0 \rangle}$$

is called a *comeasuring* if the following conditions hold for all $c \in C$:
(W1) $\sum c_{\langle -1 \rangle} \otimes c_{\langle 0 \rangle_{(1)}} \otimes c_{\langle 0 \rangle_{(2)}} = \sum c_{(1)_{\langle -1 \rangle}} c_{(2)_{\langle -1 \rangle}} \otimes c_{(1)_{\langle 0 \rangle}} \otimes c_{(2)_{\langle 0 \rangle}}$;
(W2) $\sum \varepsilon(c_{\langle 0 \rangle})c_{\langle -1 \rangle} = \varepsilon(c)1_A$.
A comeasuring $\nu : \ C \to A \otimes C$ is called *inner* if there exists a convolution invertible map $v : \ C \to A$ such that

$$\nu(c) = \sum v(c_{(1)})v^{-1}(c_{(3)}) \otimes c_{(2)}$$

for all $c \in C$. The standard example of an inner comeasuring is the adjoint coaction of a Hopf algebra H on itself:

$$H \longrightarrow H \otimes H : \ h \mapsto \sum h_{(1)}S(h_{(3)}) \otimes h_{(2)}$$

In this Section, we will show that the crossed coproduct over an inner coaction is isomorphic to a twisted coproduct. The advantage of this twisted coproduct is that the formula for the comultiplication is easier.

Theorem 5.1. *Let $C \rtimes_\alpha H$ be a crossed coproduct, and assume that the weak coaction $\omega : \ C \to H \otimes C$ is inner. Then $C \rtimes_\alpha H$ is isomorphic as a right H-module coalgebra to a twisted coproduct $C_\tau[H]$, for some cocycle τ.*

Proof. Let $v : \ C \to H$ be convolution invertible, and assume that

$$\omega(c) = \sum v(c_{(1)})v^{-1}(c_{(3)}) \otimes c_{(2)}$$

for all $c \in C$. Define $\tau : \ C \to H \otimes H$ as follows:

$$\tau(c) = \sum v^{-1}(c_{(2)})\alpha_1(c_{(3)})v(c_{(4)})_{(1)} \otimes v^{-1}(c_{(1)})\alpha_2(c_{(3)})v(c_{(4)})_{(2)}$$

for all $c \in C$. We will show that the map

$$g:\ C \bowtie_\alpha H \longrightarrow C_\tau[H]:\ c \bowtie_\alpha h \mapsto \sum c_{(1)} \bowtie_\tau v^{-1}(c_{(2)})h$$

is an isomorphism of right H-module coalgebras. It is clear that g is H-linear. We also have that

$$\begin{aligned}
&\Big[(g\otimes g)\circ\Delta_\alpha\Big](c \bowtie_\alpha h)\\
&= (g\otimes g)\Big(c_{(1)} \bowtie_\alpha v(c_{(2)})v^{-1}(c_{(4)})\alpha_1(c_{(5)})h_{(1)} \otimes c_{(3)} \bowtie_\alpha \alpha_2(c_{(5)})h_{(2)}\Big)\\
&\qquad\qquad\qquad\qquad\text{(using the definition of } \Delta_\alpha)\\
&= \sum c_{(1)} \bowtie_\tau v^{-1}(c_{(2)})v(c_{(3)})v^{-1}(c_{(6)})\alpha_1(c_{(7)})h_{(1)} \otimes c_{(4)} \bowtie_\tau v^{-1}(c_{(5)})\alpha_2(c_{(7)})h_{(2)}\\
&= \sum c_{(1)} \bowtie_\tau v^{-1}(c_{(4)})\alpha_1(c_{(5)})h_{(1)} \otimes c_{(2)} \bowtie_\tau v^{-1}(c_{(3)})\alpha_2(c_{(5)})h_{(2)}
\end{aligned}$$

and

$$\begin{aligned}
&(\Delta_\tau\circ g)(c \bowtie_\alpha h)\\
&= \Delta_\tau\big(\sum c_{(1)} \bowtie_\tau v^{-1}(c_{(2)})h\big)\\
&= \sum c_{(1)} \bowtie_\tau \tau_1(c_{(3)})v^{-1}(c_{(4)})_{(1)}h_{(1)} \otimes c_{(2)} \bowtie_\tau \tau_2(c_{(3)})v^{-1}(c_{(4)})_{(2)}h_{(2)}\\
&= \sum c_{(1)} \bowtie_\tau v^{-1}(c_{(4)})\alpha_1(c_{(5)})v(c_{(6)})_{(1)}v^{-1}(c_{(7)})_{(1)}h_{(1)}\\
&\qquad \otimes c_{(2)} \bowtie_\tau v^{-1}(c_{(3)})\alpha_2(c_{(5)})v(c_{(6)})_{(2)}v^{-1}(c_{(7)})_{(2)}h_{(2)}\\
&= \sum c_{(1)} \bowtie_\tau v^{-1}(c_{(4)})\alpha_1(c_{(5)})h_{(1)} \otimes c_{(2)} \bowtie_\tau v^{-1}(c_{(3)})\alpha_2(c_{(5)})h_{(2)}
\end{aligned}$$

and it follows that g is a coalgebra map. Now define $f:\ C_\tau[H]\longrightarrow C \bowtie_\alpha H$ by

$$f(c \bowtie_\tau h) = \sum c_{(1)} \bowtie_\alpha v(c_{(2)})h$$

for all $c \in C$ and $h \in H$. It is easy to show that f and g are each others inverses, and it follows that g is an isomorphism of right H-module coalgebras. From the fact that $C \bowtie_\alpha H$ is coassociative and counitary, it follows that $C_\tau[H]$ is also coassociative and counitary, and that τ satisfies the conditions (CU), (C) and (CC) (see [6, Lemma 2.2 and 2.3]) ∎

We will give an application of the above theorem. We say that a comeasuring $\omega : C \to H\otimes C$ is *strongly inner* if there exists $v : C \to H$ a coalgebra map such that $\omega(c) = \sum v(c_{(1)})S(v(c_{(3)}))\otimes c_{(2)}$, for all $c \in C$. It is easy to see that if $\omega : C \to H\otimes C$ is a weak coaction which is also strongly inner, then ω is just a structure of H-comodule algebra and we can construct the usual smash coproduct $C \bowtie H$.

Corollary 5.2. *Let C be a left H-comodule algebra such that the left coaction of H on C, $C \to H\otimes C$ is strongly inner. Then*

$$C \bowtie H \simeq C\otimes H$$

as right H-module coalgebras.

Proof. We apply theorem 3.1 for the trivial cocycle $\alpha : C \to H\otimes H$, $\alpha(c) = \varepsilon(c)1_H\otimes 1_H$.In this case the cocycle τ given in the proof of theorem 3.1 is also trivial. Hence the twisted coproduct $C_\tau[H]$ reduced to the usual tensor product of coalgebras. ∎

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