On the Dirichlet problem for the nonlinear wave equation in bounded domains with corner points

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Abstract

Using Mawhin's coincidence topological degree arguments and fixed point theory for non-expansive mappings results, we discuss the solvability of the Dirichlet problem for the semilinear equation of the vibrating string $u_{xx} - u_{yy} + f(x,y,u) = 0$ in bounded domain with corner points. When the winding number associated to the domain is rational, we improve and extend some results of Lyashenko [8] and Lyashenko–Smiley [9]. The case where the winding number is irrational is also examined.

1 Introduction

This paper is devoted to the solvability of the Dirichlet problem for the semilinear equation of the vibrating string:

$$\begin{cases} u_{xx} - u_{yy} + f(x,y,u) &= 0, \qquad (x,y) \in \Omega, \\ u|_{\partial\Omega} &= 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, convex relative to the characteristics lines $x \pm y = const$. It is assumed that $\Gamma = \partial\Omega = \cup_{j=1}^{4} \Gamma_j$, where $\Gamma_j \in C^k$ for each j, for some $k \geq 2$, and the endpoints of the curve Γ_j are the so-called vertices of Γ with respect to the lines $x \pm y = const$. A point $(x_0, y_0) \in \Gamma$ is said to be vertex of Γ with

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respect to the lines $x \pm y = const$ if one of the two lines $x \pm y = x_0 \pm y_0$ has an empty intersection with Ω. The domain Ω can be regarded as a "curved rectangle".

Recently, Lyashenko [8] and Lyashenko and Smiley [9] have considered this problem when the winding (or the rotation) number "associated" to Ω is rational and f is monotone.

In this note, first we consider this case and we show that using Mawhin's coincidence topological degree arguments and in particular case, fixed point theory for non-expansive mappings results obtained by the author and Mawhin in [2], we improve and extend the existence and uniqueness results of [8] and [9]. More precisely, from a result of Lyashenko [8], it is possible to map Ω homeomorphically onto a rectangle and then extend to the problem (1.1) some results obtained for the time periodic-Dirichlet nonlinear wave equation [2]:

$$\left\{ \begin{array}{rcl} u_{tt} - u_{xx} + f(t,x,u) &=& 0, \quad x \in]0,\pi[, t \in \mathbb{R} \\ u(t,0) = u(t,\pi) &=& 0, \quad t \in \mathbb{R} \\ u(t+T,x) - u(t,x) &=& 0, \quad x \in]0,\pi[, t \in \mathbb{R}. \end{array} \right. \tag{1.2}$$

Indeed, let us rewrite the problem (1.1) in the characteristic form

$$\left\{ \begin{array}{rcll} u_{xy} + f(x,y,u) &=& 0, & (x,y) \in \Omega, \\ u|_{\partial\Omega} &=& 0. & \end{array} \right. \tag{1.3}$$

Then when the winding number is rational, existence and uniqueness results are obtained by requiring f monotone and to satisfy some jumping nonlinearity conditions.

Secondly we discuss the solvability of the problem (1.3) without the monotonicity assumption on the nonlinear term f. We replace it by a symmetry condition which imply that the linear operator associated to problem (1.3) is a Fredholm operator of index zero and then the coincidence degree arguments are also applied to improve a result of Lyashenko and Smiley [9].

Finally, we examine the case when the winding number is irrational. More exactly when the winding number belongs to some class of irrational numbers specified below, we obtain an existence and uniqueness results of the nonlinear problem (1.3), using a consequence of an existence theorem of Fokin [5], for the linear problem, degree arguments and fixed point theory.

2 Domains with Corner Points

First of all, by a weak solution of problem (1.3), we mean a function $u(x,y) \in L^2(\Omega)$ such that

$$\int_\Omega u(x,y)\phi_{xy}(x,y) + f(x,y,u)\phi(x,y)dxdy = 0$$

for all $\phi \in W_0^{1,2}(\Omega), \phi_{xy} \in L^2(\Omega)$. Following Lyashenko [8] or Lyashenko and Smiley [9], we describe now, here, the domains considered.

The domain $\Omega \subset \mathbb{R}^2$ is assumed to be bounded, with a boundary $\Gamma = \partial\Omega$ satisfying:

A1) $\Gamma = \partial\Omega = \cup_{j=1}^{4}\Gamma_j$, $\Gamma_j = \{(x, y_j(x)) | x_j^0 \leq x \leq x_j^1\}$, $y_j(x) \in C^k([x_j^0, x_j^1])$ for any $j = 1, 2, 3, 4$ and for some $k \geq 2$.

A2) $|y_j'(x)| > 0, x \in [x_j^0, x_j^1]$, $j = 1, 2, 3, 4$.

A3) The endpoints $P_j = (x_j^0, y_j(x_j^0))$ of the curves $\Gamma_1, ..., \Gamma_4$ are the vertices of Γ with respect to the lines $x = const.$, $y = const$. By this we mean that for any $j = 1, ..., 4$ one of the two lines $x = x_j^0$, $y = y_j(x_j^0)$ has empty intersection with Ω and there are no other points on Γ with this property.

These conditions imply that the domain Ω is stricly convex relative to the lines $x = const.$, $y = const$. Therefore, following [7], we can define homeomorphisms T^+, T^- on the boundary Γ as follows:

T^+ assigns to a point on the boundary the other boundary point with the same y coordinate. T^- assigns to a point on the boundary the other boundary point with the same x coordinate. Notice that each vertex P_j is fixed point of either T^+ or T^-. We define $F := T^+ \circ T^-$. It is easy to see that F preserves the orientation of the boundary.

Let $\Gamma = \{(x(s), y(s)) |\ 0 \leq s < l\}$ be the parametrization of Γ by the arc length parameter, so that l is the total length of Γ. For each point $P \in \Gamma$ we denote its coordinate by $S(P) \in [0, l[$. Then the homeomorphism F can be lifted [16] to a continuous map $f_1 : \mathbb{R} \to \mathbb{R}$, which is an increasing function onto $\mathbb{R}$ such that $0 \leq f_1(0) < l$ and

$$f_1(s + l) = f_1(s) + l, \quad s \in \mathbb{R}, \text{ and } \quad S(F(P)) = f_1(S(P)) \pmod{l}, \quad P \in \Gamma.$$

The function f_1 is called *the lift* of F [16]. If we inductively set $f_k(s) := f_1(f_{k-1}(s))$ for integer $k \geq 2$, then it is known that the limit

$$\lim_{k\to\infty} \frac{f_k(s)}{kl} =: \alpha(F) \in [0, 1]$$

exists and is independent of $s \in \mathbb{R}$. The number $\alpha(F)$ is called *the winding number* or *rotation number* of F. The following cases are possible:

(A) $\alpha(F) = \frac{m}{n}$ is a rational number, and $F^n = I$ where I is the identity mapping of Γ onto itself.

(B) $\alpha(F) = \frac{m}{n}$ is a rational number, F^n has a fixed point on Γ, but $F^n \neq I$.

(C) $\alpha(F)$ is an irrational number, and F^k has no fixed point on Γ for any $k \in \mathbb{N}$.

The solvability of problem (1.3) is quite different in the three cases (A), (B), (C) (see [5] and [2]). The condition of rationality of T/π in problem (1.2) actually means that the winding number $\alpha(F)$ of the corresponding diffeomorphism F is rational and the condition (A) holds as we see from the following example: Consider the problem (1.2) with $T = 2\pi/\beta$, $\beta \in \mathbb{R}$. We shall see below, that in this case $\alpha(F) = \beta/2$ and then the solvability of (1.2) depends on the rationality of β. In [2], the case where β is irrational was considered for the problem (1.2).

Here we shall consider the cases (A) and (C).

3 Reduction to Rectangular Domains

This section is devoted to the reduction of our problem to equivalent one in rectangle. For this, define an equivalence relation "$\sim$" between open sets of $\mathbb{R}^2$:

$\Omega_1 \sim \Omega_2 \Longleftrightarrow$ there are continuous increasing functions $h, g : \mathbb{R} \to \mathbb{R}$ such that

$$\Omega_2 = \{(h(x), g(y)) \mid (x, y) \in \Omega_1\}.$$

Denote the equivalence class of Ω_1 by

$$E(\Omega_1) := \{\Omega \subset \mathbb{R}^2 | \, \Omega \sim \Omega_1\}.$$

The following theorem due to Lyashenko [8] shows that in the case where the open bounded domains have the same rational winding number, then there are equivalent. More precisely, Lyashenko [8] has proved:

Theorem 3.1 *Let Ω_1, Ω_2 be bounded domains, convex relative to the lines $x =$ const., $y =$ const., and such that* $\alpha(F_{\Gamma_1}) = \alpha(F_{\Gamma_2}) = \frac{m}{n}$, $(m, n) = 1, (F_{\Gamma_1})^n = I_{\Gamma_1}, (F_{\Gamma_2})^n = I_{\Gamma_2}$. *Then* $\Omega_1 \sim \Omega_2$.

It is clear that the above theorem states that the set Σ of domains considered is composed of topological equivalence classes. For a given triple of natural numbers m, n, k, let $E(m, n, k)$ denote the set of domains Ω with smoothness k, winding number $\alpha(F) = \frac{m}{n}$, and $F^n = I$. It follows from the definitions that

1. $E(mj, nj, k) = E(m, n, k)$ for any $m, n, k, j \in \mathbb{N}$
2. $E(m, n, k) = \emptyset$ for any $m, n, k, \in \mathbb{N}$, $m > n$.

More generaly, denoting by:

$$\Omega(a, b) := \{(x, y) \in \mathbb{R}^2 \mid 0 < x + y < a, \; 0 < x - y < b\},$$

we prove the following general result, giving the calculation of the winding number $\alpha(F)$ of the corresponding diffeomorphism F for $\Omega(a, b)$.

Lemma 3.2 *i)* $\Omega(ra, rb) \in E(\Omega(a, b))$ *for all* $r > 0$,

ii) $\alpha(F) = \frac{a}{a+b}$ *for all* $\Omega(a, b)$ *where F is the corresponding diffeomorphism.*

Proof : It is easy to see that $S(F(P)) - S(P) = \sqrt{2}\, a$, for all $P \in \Gamma$. Moreover the function

$$g(s) := f_1(s) - s - \sqrt{2}\, a$$

where f_1 is the lift of F, is such that $g(s+l) = g(s)$ and $g(0) = f_1(0) - \sqrt{2}\, a \in]-l, l[$ where $l = \sqrt{2}(a + b)$. Since $S(F(P)) = f_1(S(P))$ (mod l), we can write:

$$g(S(P)) = f_1(S(P)) - S(P) - \sqrt{2}\, a = S(F(P)) - S(P) - \sqrt{2}\, a + n_P$$

where $n_P \in \mathbb{Z}$, and from above $g(S(P)) = n_P l$. If $P = 0$, then $n_0 l = g(0) \in]-l, l[$ and $n_0 = 0$. Since g is continuous, n_P is constant and then $n_P = n_0 = 0$ and hence $g(s) = 0$ i.e. $f_1(s) = s + \sqrt{2}\, a$. Therefore

$$\alpha(F) = \lim_{k\to\infty} \frac{f_k(0)}{kl} = \frac{k\sqrt{2}\, a}{k\sqrt{2}(a + b)} = \frac{a}{a + b}$$

which finishes the proof. ∎

From Theorem 3.1 and Lemma 3.2 the simplest representative of the equivalence class $E(m,n,k)$ is the rectangle

$$\prod_n^m := \{(x,y) \mid 0 < x+y < m,\ 0 < x-y < n-m\}$$

with $\Gamma_n^m := \partial \prod_n^m$, and therefore $E(m,n,k) \neq \emptyset$ if $m < n$. From Theorem 3.1 it follows that for any domain $\Omega \in E(m,n,k)$, there are continuous increasing functions $h,\ g : \mathbb{R} \to \mathbb{R}$ such that

$$\Omega = \{(h(x), g(y)) \mid (x,y) \in \prod_n^m\}. \tag{3.4}$$

It is shown in [8] that h and g can be chosen such that there are finite sets of points $S_x := \{x_1, \ldots, x_{n-1}\}$ and $S_y := \{y_1, \ldots, y_{n-1}\}$ such that:

$$\begin{cases} h \in C^k(\mathbb{R}\backslash S_x),\ g \in C^k(\mathbb{R}\backslash S_y) \\ 0 < \delta \leq h'(x),\ g'(y) \leq C,\ \ x \in \mathbb{R}\backslash S_x,\ y \in \mathbb{R}\backslash S_y. \end{cases} \tag{3.5}$$

We now return to the problem (1.3).

Lemma 3.3 [8] *Let h, g satisfy (3.4) (3.5). Then $u(x,y)$ is a weak solution of (1.3) if and only if $v(x,y) := u(h(x), g(y))$ is a weak solution of the following problem*

$$\begin{cases} v_{xy} + f(h(x), g(y), v(x,y))\, h'(x)\, g'(y) &= 0, \qquad (x,y) \in \prod_n^m, \\ v|_{\partial \prod_n^m} &= 0. \end{cases} \tag{3.6}$$

Consider problem (3.6). We make the change of variables:

$$z := \frac{\pi}{n-m}(x-y),\ \ t := \frac{\pi}{n-m}(x+y)$$

$$w(z,t) := v\left(\frac{n-m}{2\pi}(z+t), \frac{n-m}{2\pi}(z-t)\right)$$

and set

$$\begin{aligned} \tilde{f}(z,t,w) \ \ := \ \ & \frac{(n-m)^2}{2\pi^2}.\ f\left(h\left(\frac{n-m}{2\pi}(z+t)\right), g\left(\frac{n-m}{2\pi}(z-t)\right), w\right) \\ & .\ h'\left(\frac{n-m}{2\pi}(z+t)\right).\ g'\left(\frac{n-m}{2\pi}(z-t)\right), \end{aligned}$$

$$T := \frac{m\pi}{n-m}.$$

Then $z \in\,]0,\pi[$, $t \in\,]0, \frac{m\pi}{n-m}[$, and $v(x,y)$ is a solution of (3.6) if and only if $w(z,t)$ is a solution of

$$\begin{cases} w_{tt} - w_{zz} + \tilde{f}(z,t,w) = 0, \qquad z \in\,]0,\pi[,\ t \in\,]0,T[\\ w(0,t) = w(\pi,t) = w(z,0) = w(z,T) = 0, \quad z \in\,]0,\pi[,\ t \in\,]0,T[. \end{cases} \tag{3.7}$$

Thus $u(x,y)$ is a weak solution of (1.3) if and only if the function

$$w(z,t) := u\left(h\left(\frac{n-m}{2\pi}(z+t)\right), g\left(\frac{n-m}{2\pi}(z-t)\right)\right) \tag{3.8}$$

is a solution of (3.7).

4 Existence and Uniqueness in the Case (A)

We suppose that Ω satisfies the condition (A) i.e. the case where $\alpha(F) = \frac{m}{n}$ is rational number, and $F^n = I$, where I is the identity mapping of Γ onto itself. Then, consider the problem (3.7). Let L be the abstract realisation in $H := L^2(J)$, where $J :=]0, \pi[\times]0, T[$, of the wave operator with Dirichlet boundary conditions on J. It is standard to show that L is a selfadjoint operator in H, with spectrum

$$\sigma(L) := \left\{ \lambda_{jl} := j^2 - l^2 \left(\frac{n-m}{m}\right)^2 \mid j,\ l \in \mathbb{N} \right\}.$$

Observe that the zero eigenvalue $\lambda = 0$ has infinite multiplicity while each nonzero eigenvalue has finite multiplicity. As consequence the right inverse operator K of L is compact in H.

Let λ, μ two consecutive eigenvalues of L and H_1 (resp H_2) be the space spanned by the eigenfunctions of L associated with the eigenvalues smaller or equal to λ (resp. larger or equal to μ). For each $w \in H$, we set

$$w^+ := \frac{1}{2}(|w| + w), \text{ and } w^- := \frac{1}{2}(|w| - w).$$

Define the subsets $J_\pm$ of J by

$$J_+(w) := \{(z,t) \in J;\ w(z,t) > 0\}$$

$$J_-(w) := \{(z,t) \in J;\ w(z,t) < 0\}$$

and denote by $\chi_{J_\pm}$ the corresponding characteristic functions. If $p_+ \in L^\infty(J)$ and $p_- \in L^\infty(J)$, define the operator $A_p : H \to L^\infty(J)$ by

$$A_p(w) := p_+ \chi_{J_+(w)} + p_- \chi_{J_-(w)}$$

and the operator $B_p : H \to H$ by

$$B_p(w)(z,t) := (A_p(w))(z,t) w(z,t) = [A_p(w) w](z,t).$$

If $w_1 \in H_1,\ w_2 \in H_2$, we have for all $w \in \operatorname{dom} L$

$$\begin{aligned} & (Lw - B_p(w), u_2 - w_1) \\ = \ & (L(w_2 + w_1) - A_p(w)(w_2 + w_1), w_2 - w_1) \\ = \ & (Lw_2 - A_p(w) w_2, w_2) - (Lw_1 - A_p(w) w_1, w_1) \end{aligned}$$

and for every $\theta \geq 0$, $B_p(\theta w) = \theta B_p(w)$, so that B_p is positive homogeneous. Moreover, if S is a vector space of $\ker L$, and P_S the corresponding orthogonal projector, we shall denote by N_S the mapping defined on $H_S := S \bigoplus \operatorname{Im} L$ by $N_S := Q_S N$, with $Q_S := P_S + Q$. The following lemmas for linear problem are proved in [13] or [3].

Lemma 4.1 *Let $\alpha_\pm$ and $\beta_\pm$ be elements of $L^\infty(J)$ such that*

$$\lambda \le \alpha_+(z,t) \le \beta_+(z,t) \le \mu$$

$$\lambda \le \alpha_-(z,t) \le \beta_-(z,t) \le \mu,$$

a.e. on J and such that

$$\int_J [(\alpha_+ - \lambda)(w^+)^2 + (\alpha_- - \lambda)(w^-)^2] > 0$$

for all $w \in \ker(L - \lambda I)\backslash\{0\}$ and

$$\int_J [(\mu - \beta_+)(v^+)^2 + (\mu - \beta_-)(v^-)^2] > 0$$

for all $v \in \ker(L - \mu I)\backslash\{0\}$. Then, for each subspace $S \subset \ker L$, there exists $\epsilon > 0$, and $\delta > 0$ such that for all real measurable functions p_+ and p_- on J with

$$\alpha_+(z,t) - \epsilon \le p_+(z,t) \le \beta_+(z,t) + \epsilon,$$

$$\alpha_-(z,t) - \epsilon \le p_-(z,t) \le \beta_-(z,t) + \epsilon$$

a.e. on J and for all $w \in \mathrm{dom}\ L \cap H_S$, one has $|Lw - B_{p,S}(w)| \ge \delta |w|$.

Lemma 4.2 *Under the assumptions of Lemma 4.1, one has*

$$|D_{L_S}(L_S - B_{p,S}, B(\gamma))| = 1$$

for every finite dimensional vector subspace $S \subset \ker L$, every open ball $B(\gamma)$ in H_S and every p_+ and p_- satisfying the conditions of Lemma 4.1.

We assume that $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(.,.,w)$ is measurable on J for each $w \in \mathbb{R}$, $\tilde{f}(z,t,.)$ is continuous on $\mathbb{R}$ for a.e. $(z,t) \in J$. Assume moreover that, for each $\rho > 0$, there exists $h_\rho \in H$ such that

$$|\tilde{f}(z,t,w)| \le h_\rho(z,t) \tag{4.9}$$

when $(z,t) \in J$ and $|w| \le \rho$. We shall say that $\tilde{f}$ satisfies the Caratheodory conditions for H. Then we have the following existence result.

Theorem 4.3 *Let $\lambda < \mu$ be two consecutive nonzero eigenvalues of L. Assume that $\tilde{f}$ satisfies (4.9),* sign $\lambda.\tilde{f}(z,t,.)$ *is nondecreasing and that the inequalities*

$$\alpha_+(z,t) \le \liminf_{w \to +\infty} w^{-1}\tilde{f}(z,t,w) \le \limsup_{w \to +\infty} w^{-1}\tilde{f}(z,t,w) \le \beta_+(z,t) \tag{4.10}$$

$$\alpha_-(z,t) \le \liminf_{w \to -\infty} w^{-1}\tilde{f}(z,t,w) \le \limsup_{w \to -\infty} w^{-1}\tilde{f}(z,t,w) \le \beta_-(z,t) \tag{4.11}$$

hold uniformly a.e. in $(z,t) \in J$, where $\alpha_\pm$ and $\beta_\pm$ are functions in $L^\infty(J)$ such that

$$\lambda \le \alpha_+(z,t) \le \beta_+(z,t) \le \mu \tag{4.12}$$

$$\lambda \leq \alpha_-(z,t) \leq \beta_-(z,t) \leq \mu. \tag{4.13}$$

Assume moreover that

$$\int_J [(\alpha_+ - \lambda)(w^+)^2 + (\alpha_- - \lambda)(w^-)^2] > 0 \tag{4.14}$$

for all $w \in \ker(L - \lambda I)\backslash\{0\}$ *and*

$$\int_J [(\mu - \beta_+)(v^+)^2 + (\mu - \beta_-)(v^-)^2] > 0 \tag{4.15}$$

for all $v \in \ker(L - \mu I)\backslash\{0\}$. *Then the problem*

$$\begin{cases} w_{tt} - w_{zz} - \tilde{f}(z,t,w) = 0, & z \in]0,\pi[,\ t \in]0,T[\\ w(0,t) = w(\pi,t) = w(z,0) = w(z,T) = 0, & z \in]0,\pi[,\ t \in]0,T[\end{cases}$$

has at least one weak solution.

Remark 4.1 If $\lambda < \mu$ denote now the eigenvalues of $-L$, then Theorem 4.3 hold true for the problem (3.7).

Remark 4.2 If $\alpha_+ = \alpha_- = \alpha$ and $\beta_+ = \beta_- = \beta$, then (4.14) and (4.15) respectively become

$$\int_J (\alpha - \lambda) w^2 > 0 \qquad \text{for all } w \in \ker(L - \lambda I)\backslash\{0\}$$

and

$$\int_J (\mu - \beta) v^2 > 0 \qquad \text{for all } v \in \ker(L - \mu I)\backslash\{0\}$$

which is equivalent to $\alpha(z,t) > \lambda$ (resp. $\beta(z,t) < \mu$) on a subset of J of positive measure.

sketch of the proof of Theorem 4.3:

The proof is based on the two Lemmas 4.1 and 4.2. Let $\delta > 0$ and $\epsilon > 0$ be given by Lemma 4.2. We can find $\rho > 0$ such that for a.e. $(z,t) \in J$,

$$\alpha_+(z,t) - \epsilon \leq w^{-1}\tilde{f}(z,t,w) \leq \beta_+(z,t) + \epsilon \qquad \text{if } w \geq \rho,$$

$$\alpha_-(z,t) - \epsilon \leq w^{-1}\tilde{f}(z,t,w) \leq \beta_-(z,t) + \epsilon \qquad \text{if } w \leq -\rho.$$

This implies by (4.9) that

$$|\tilde{f}(z,t,w)| \leq (C + \epsilon)|w| + h_\rho(z,t)$$

for a.e. $(z,t) \in J$ and all $w \in \mathbb{R}$, with $C = \mu$. Consequently, the mapping N defined on H by

$$(Nw)(z,t) := \tilde{f}(z,t,w)$$

will map H continuously into itself and takes bounded sets into bounded sets. Moreover, the weak solutions of the problem

$$\begin{cases} w_{tt} - w_{zz} - \tilde{f}(z,t,w) = 0, & z \in]0,\pi[,\ t \in]0,T[\\ w(0,t) = w(\pi,t) = w(z,0) = w(z,T) = 0, & z \in]0,\pi[,\ t \in]0,T[\end{cases}$$

will be the solutions in dom L of the abstract equation in H

$$Lw - Nw = 0. \tag{4.16}$$

Our assumption on $\tilde{f}$ implies that N is monotone in H. As the right inverse K of L is compact, we see that KQN is compact on bounded sets on H. The (nonlinear) operator B_α defined by

$$B_\alpha(w) := \alpha_+ w^+ - \alpha_- w^-$$

is continuous, takes bounded sets into bounded sets and is such that

$$(B_\alpha(w) - B_\alpha(v), w - v) \geq \lambda |w - v|^2$$

for all $w, v \in H$ as is easily checked. Thus KQB_α is compact on bounded sets and B_α is strongly monotone. It follows from Lemma 4.2, that

$$|D_{L_S}(L_S - B_{\alpha,S}, B(\gamma))| = 1$$

for every $\gamma > 0$ and every finite dimensional vector subspace S of $\ker L$.
According to the Theorem I.1 of [12], equation (4.16) will have a solution if the set of possible solutions of the family of equations

$$Lw - (1-s)B_\alpha(w) - sN(w) = 0 \quad s \in [0,1]$$

is a priori bounded independently of s, which can be obtained as in the proof of Theorem 2.1. in [3]. ■

The following result extends those of Lyashenko and Smiley [9]

Corollary 4.4 *Let $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ satisfy the Caratheodory conditions for H and be such that*

$$\alpha(z,t) \leq \frac{\tilde{f}(z,t,w) - \tilde{f}(z,t,v)}{w - v} \leq \beta(z,t) \tag{4.17}$$

for a.e. $(z,t) \in J$ and all $w \neq v \in J$, $\alpha_- = \alpha_+ = \alpha$ and $\beta_- = \beta_+ = \beta$ like in Theorem 4.3 with $\lambda, \mu \in \sigma(-L)$. Then the problem (3.7) has a unique solution.

Proof : It follows from (4.17) that conditions (4.10) (4.11) hold. Thus the existence follows from Theorem 4.3. If, now u and v are solutions, then letting $w = u - v$, w will be a weak solution of Dirichlet problem in J for equation

$$w_{tt} - w_{zz} + [\tilde{f}(z,t,v+w) - \tilde{f}(z,t,v)] = 0. \tag{4.18}$$

Setting

$$g(z,t,w) := \begin{cases} w^{-1}[\tilde{f}(z,t,v+w) - \tilde{f}(z,t,v)], & \text{if } w \neq 0, \\ \alpha(z,t), & \text{if } w = 0, \end{cases}$$

we see that (4.18) can be written as

$$w_{tt} - w_{zz} + g(z,t,w)w = 0 \tag{4.19}$$

with

$$\alpha(z,t) \leq g(z,t,w) \leq \beta(z,t)$$

for a.e. $(z,t) \in J$ and all $w \in \mathbb{R}$. Consequently, by Lemma 4.1, we easily see from (4.19) that $w = 0$, i.e. $u = v$. ■

Now, when λ may be zero, we assume that $\alpha_- = \alpha_+ = const.$ and $\beta_- = \beta_+ = const.$ Then we obtain, as consequence of the following theorem, the results of Lyashenko and Smiley [9].

Indeed, let λ, μ, two nonegative constants and β_0, β_1, two positive constants such that

$$0 \leq \lambda < \beta_0 \leq \beta_1 < \mu$$

and the assumptions

$$0 \leq \lambda < \beta_0 \leq \frac{\tilde{f}(z,t,w) - \tilde{f}(z,t,v)}{w-v} \leq \beta_1 < \mu \tag{4.20}$$

or

$$0 \leq \lambda < \beta_0 \leq \frac{-\tilde{f}(z,t,w) + \tilde{f}(z,t,v)}{w-v} \leq \beta_1 < \mu \tag{4.21}$$

are satisfied for all $w, v \in \mathbb{R}, \quad w \neq v$.

If $\mathbb{R}_0^-$ (resp. $\mathbb{R}_0^+$) denotes the set of negative (resp. positive) real numbers, we shall set

$$d_0^-(L) := \text{ dist }(0, \sigma(L) \cap \mathbb{R}_0^-),$$

and

$$d_0^+(L) := \text{ dist }(0, \sigma(L) \cap \mathbb{R}_0^+),$$

with the convention $d_0^-(L) = +\infty$ (resp. $d_0^+(L) = +\infty$) if $\sigma(L)\backslash\{0\} \subset \mathbb{R}_0^+$ (resp. $\sigma(L)\backslash\{0\} \subset \mathbb{R}_0^-$). The following result generalizes in several ways Theorem 7 of [9].

Theorem 4.5 *Assume that λ, μ are consecutive and $\lambda, \mu \in \sigma(-L)$ or $\lambda, \mu \in \sigma(L)$ according to whether $\tilde{f}$ satisfies (4.20) or (4.21). Then there is a unique weak solution of problem (3.7).*

Proof : Assume that $\lambda, \mu \in \sigma(-L)$ and the condition (4.20) holds. The case where $\lambda, \mu \in \sigma(L)$ and (4.21) holds is similar. It is clear, that a solution of the abstract equation in H

$$Lw + Nw = 0 \tag{4.22}$$

where N is the mapping defined on H by

$$(Nw)(z,t) := \tilde{f}(z,t,w)$$

will be a solution of problem (3.7).

Let $L_\lambda := L + \lambda I$ and $N_\lambda := N - \lambda I$. Then, equation (4.22) is equivalent to

$$L_\lambda w + N_\lambda w = 0. \tag{4.23}$$

From (4.20) we have

$$\beta_0(L_\lambda)|w-v|^2 \leq (N_\lambda w - N_\lambda v, w-v) \leq \beta_1(L_\lambda)|w-v|^2,$$

with

$$\beta_0(L_\lambda) := \beta_0 - \lambda > 0,$$

$$\beta_1(L_\lambda) := \beta_1 - \lambda < \mu - \lambda.$$

On the other hand, we have

$$\beta_1(L_\lambda) = \beta_1 - \lambda < \mu - \lambda = d_0^-(L_\lambda),$$

and then we can conclude, from Theorem 2 of [2], that there exists a unique solution of equation (4.22) and hence of problem (3.7). ∎

Remark 4.3 Theorem 4.5 improves Theorem 7 of Lyashenko and Smiley in [9] which requires $\lambda = 0$ and condition (4.20) (resp. (4.21)) with condition $\beta_1 < d_0^-(L) = \mu$ (resp. $\beta_1 < d_0^+(L) = \mu$) replaced by the stronger assumptions

$$\beta_1^2 < d_0^-(L)\beta_0, \ \ (\text{resp. } \beta_1^2 < d_0^+(L)\beta_0).$$

5 Existence Without Monotonicity

Briefly and following Lyashenko and Smiley [9] we shall discuss the solvablity of (3.7) without the assumption of monotonicity on the nonlinear term $\tilde{f}$. It is classical now, from an idea of Coron [4], to eliminate this assumption by searching for a solution of the problem (3.7) in a subspace $H_1 \subset L^2(J)$ of functions satisfying some symmetry properties, which is invariant under L and N such that $H_1 \cap \ker L = \{0\}$.

Consider the problem (3.7) with $T = \pi\frac{m}{n-m}$ for some natural numbers m, n such that $m < n$ with $(m, n) = 1$ and n is odd.

Denote

$$H_1 := \begin{cases} \text{cl } \{w(z,t) \in C^\infty(J) \,|w(z,t) = w(z, T-t)\}^{L^2(J)} & \text{if} \quad m \text{ is even} \\ \text{cl } \{w(z,t) \in C^\infty(J) \,|w(z,t) = w(\pi - z, t)\}^{L^2(J)} & \text{if} \quad m \text{ is odd} \end{cases}$$

where for any set $H \subset L^2(J)$ we mean by cl $H^{L^2(J)}$ the closure of H in $L^2(J)$. Then we have

$$H_1 \cap \ker L = \{0\}$$

and the spectrum of $L_1 := L|_{H_1}$ consists of nonzero isolated eigenvalues with finite multiplicity which imply that L_1 is a linear Fredholm operator of index zero. Then instead to use the Schauder fixed point theorem as Lyashenko and Smiley, we apply the continuation theorem of the Leray-Schauder type based on Mawhin's coincidence degree theory which corresponds to Theorem I.1 of [12], to improve the Theorem 12 in [9].

We assume that $\tilde{f}(z,t,w) \in H_1$ for all $w \in \mathbb{R}$. Using again the Lemmas 4.1 and 4.2 and following the proof of Theorem 2.1 in [3] we obtain

Theorem 5.1 *Let $\lambda < \mu$ be two consecutive eigenvalues of L_1. Assume that $\tilde{f}$ satisfies (4.9), and that the inequalities*

$$\alpha_+(z,t) \le \liminf_{w\to+\infty} w^{-1}\tilde{f}(z,t,w) \le \limsup_{w\to+\infty} w^{-1}\tilde{f}(z,t,w) \le \beta_+(z,t)$$

$$\alpha_-(z,t) \le \liminf_{w\to-\infty} w^{-1}\tilde{f}(z,t,w) \le \limsup_{w\to-\infty} w^{-1}\tilde{f}(z,t,w) \le \beta_-(z,t)$$

hold uniformly a.e. in $(z,t) \in J$*, where* $\alpha_\pm$ *and* $\beta_\pm$ *are functions in* $L^\infty(J)$ *such that*

$$\lambda \leq \alpha_+(z,t) \leq \beta_+(z,t) \leq \mu$$

$$\lambda \leq \alpha_-(z,t) \leq \beta_-(z,t) \leq \mu.$$

Assume moreover that

$$\int_J [(\alpha_+ - \lambda)(w^+)^2 + (\alpha_- - \lambda)(w^-)^2] > 0$$

for all $w \in \ker(L_1 - \lambda I)\backslash\{0\}$ *and*

$$\int_J [(\mu - \beta_+)(v^+)^2 + (\mu - \beta_-)(v^-)^2] > 0$$

for all $v \in \ker(L_1 - \mu I)\backslash\{0\}$*. Then the problem*

$$\begin{cases} w_{tt} - w_{zz} - \tilde{f}(z,t,w) = 0, & z \in \,]0,\pi[,\ t \in \,]0,T[\\ w(0,t) = w(\pi,t) = w(z,0) = w(z,T) = 0, & z \in \,]0,\pi[,\ t \in \,]0,T[\end{cases}$$

has at least one weak solution in H_1*.*

Remark 5.1 If $\lambda < \mu$ denote now the eigenvalues of $-L_1$, then the Theorem 5.1 hold true for the problem (3.7).

The following uniqueness result is obtained as in Corollary 4.4.

Corollary 5.2 *Let* $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ *satisfy the Caratheodory conditions for* H_1 *and be such that*

$$\alpha(z,t) \leq \frac{\tilde{f}(z,t,w) - \tilde{f}(z,t,v)}{w - v} \leq \beta(z,t)$$

for a.e. $(z,t) \in J$ *and all* $w \neq v \in J$*,* $\alpha_- = \alpha_+ = \alpha$ *and* $\beta_- = \beta_+ = \beta$ *like in Theorem 5.1 with* $\lambda, \mu \in \sigma(-L_1)$*. Then the problem (3.7) has a unique solution.*

6 Existence and Uniqueness in the Case (C)

The results of this section require some results in number theory. Those results can be essentially found in [15] or in [2].

Let us consider the problem (1.3) and suppose that for the domain Ω, condition (C) holds i.e. the winding number $\alpha(F)$ is irrational, and F^n has no fixed points on Γ for any $n \geq 1$, (where F is the corresponding diffeomorphism) and bounded by a analytic contour Γ. For more simplicity we suppose that $f(x,y,u) = f(u)$. We shall consider only the case where $\alpha(F)$ belongs to some class of irrational numbers which we define below.

In order to use an existence theorem of Fokin [5], we want to determine a class of irrational numbers α such that

$$|\alpha - m/n| \geq C(\alpha)/n^2 \tag{6.24}$$

for some $C(\alpha) > 0$ and any rational number m/n.

It has been shown in [15], in [14] or in [2] in a much simpler way, that an irrational number α satisfies the condition

$$\Delta_\alpha \quad := \inf_{(m,n)\neq(0,0)} |(\alpha m)^2 - n^2| > 0 \qquad (6.25)$$

if and only if α satisfies the condition (6.24).

We shall now characterize the set of irrational numbers which satisfies the condition (6.25).

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. We can further assume without loss of generality, that $\alpha > 0$. Let

$$\alpha := [a_0, a_1, \ldots]$$

be the continuous fraction decomposition of α. Recall that it is obtained as follows; put $a_0 := [\alpha]$, where $[.]$ denotes the integer part. Then $\alpha = a_1 + \frac{1}{\alpha_1}$ with $\alpha_1 > 1$, and we set $a_1 := [\alpha_1]$. If $a_0, a_1, \ldots, a_{n-1}$ and $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are known, then $\alpha_{n-1} = a_{n-1} + \frac{1}{\alpha_n}$, with $\alpha_n > 1$ and we set $a_n := [\alpha_n]$. It is well known that this process does not terminate if and only if α is irrational. The integers $a_0, a_1, \ldots$ are the partial quotients of α; the numbers $\alpha_1, \alpha_2, \ldots$ are the complete quotients of α and the rationals

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \ldots \frac{1}{+ a_n},$$

with p_n, q_n relatively prime integers, are the convergents of α and are such that $p_n/q_n \to \alpha$ as $n \to \infty$. It is well known that the p_n, q_n are recursively defined by the relations

$$p_0 := a_0,\ q_0 := 1,\ p_1 := a_0 a_1 + 1,\ q_1 := a_1,$$

$$p_n := a_n p_{n-1} + p_{n-2},\quad q_n := a_n q_{n-1} + q_{n-2}.$$

Then we have the following characterization of irrational numbers satisfying condition (6.25) proved in [15] or in [2].

Proposition 6.1 $\Delta_\alpha > 0$ *if and only if the sequence* $(a_i)_{i\in\mathbb{N}}$ *of partial quotients of* α *is bounded above.*

As an example, the golden number $\alpha := \frac{1+\sqrt{5}}{2}$ is such that

$$\frac{1+\sqrt{5}}{2} = [1, 1, 1, \cdots]$$

and then satisfies the condition (6.25).

We now return to the problem (1.3) and we first recall a special theorem of Fokin [5] for the linear problem.

Let L be the linear differential operator on $H := L^2(\Omega)$ (with corresponding norm $|.|$) associated to problem (1.3) and we shall denote by $\sigma(L)$ and $\sigma_{\mathrm{ess}}(L)$ the spectrum and the essential spectrum of L respectively.

Theorem 6.2 [5] *Suppose that for the domain Ω condition (C) holds and the diffeomorphism F is analytically conjugate to the shift $R_{\alpha(F)} : t \to t+\alpha(F)$. Then L is selfadjoint and the linear problem*

$$\begin{cases} u_{xy} + h(x,y) &= 0, \qquad (x,y) \in \Omega, \\ u|_{\partial\Omega} &= 0, \end{cases}$$

has a unique solution u in H for any $h \in H$ if and only if for some $C(\alpha(F)) > 0$ and any rational number m/n,

$$|\alpha(F) - m/n| \geq C(\alpha(F))/n^2$$

in which case

$$|u| \leq C|h|.$$

Remark 6.1 In the case where condition (C) holds for Ω, it is known from [14] that if $\alpha(F) \in M$, where M is a set of irrational numbers of full Lebesgue measure, then F is analytically conjugate to the shift $R_{\alpha(F)}$. Then if $\alpha(F)$ satisfies the condition (6.25), also by [14], $\alpha(F)$ belongs to some set of full Lebesgue measure and hence all the assumptions of the Theorem 6.2 are satisfied.

Consequently we obtain the following results proved in [2] for the periodic–Dirichlet problem for semilinear wave equation for some irrational ratios between the period and interval length.

Theorem 6.3 *Assume that for the domain Ω condition (C) holds and that $\alpha(F)$ has a bounded sequence of partial quotients. Then there exists $\epsilon > 0$ such that the problem (1.3) has a unique weak solution for each $f \in H$ when the condition*

$$\left| \frac{f(u) - f(v)}{u - v} \right| \leq \epsilon,$$

holds for all $u, v \in \mathbb{R}, u \neq v$.

Proof : It follows from our assumptions, Proposition 6.1 and Theorem 6.2 that there exists $\epsilon_1 > 0$ such that $\sigma(L) \cap]-\epsilon_1, \epsilon_1[= \emptyset$, and if we choose any $0 < \epsilon < \epsilon_1$, then $\epsilon < d_0 := \mathrm{dist}(0, \sigma(L))$, so that the result follows from Theorem 2 of [3]. ■

Theorem 6.4 *Assume again that for the domain Ω condition (C) holds and that $\alpha(F)$ has a bounded sequence of partial quotients. Assume moreover that there exist real numbers a and b with $a \leq b$ such that the following conditions hold.*

(i) $[a,b] \cap \sigma_{\mathrm{ess}}(L) = \emptyset$;

(ii) $a \leq \frac{f(u)-f(v)}{u-v} \leq b$ *for all* $u, v \in \mathbb{R},\ u \neq v$;

(iii) $\left[\liminf_{|u|\to\infty} \frac{f(u)}{u}, \limsup_{|u|\to\infty} \frac{f(u)}{u}\right] \cap \sigma(L) = \emptyset$.

Then the problem

$$\begin{cases} u_{xy} - f(u) &= 0, \qquad (x,y) \in \Omega, \\ u|_{\partial\Omega} &= 0, \end{cases}$$

has at least one weak solution for each $f \in H$.

Proof : We shall show that the conditions of Corollary 1 in [11] are satisfied. The first assumption in this theorem follows from conditions (i) and (ii). Letting

$$f_- := \liminf_{|u|\to\infty} \frac{f(u)}{u} \quad \text{and} \quad f_+ := \limsup_{|u|\to\infty} \frac{f(u)}{u},$$

it follows from condition (iii) that we can find $\lambda, \mu \in \sigma(L)$ such that $]\lambda,\mu[\subset \rho(L)$, with $\rho(L)$ the resolvant of L, and

$$\lambda < f_- \leq f_+ < \mu.$$

Let $\beta > 0$ be such that

$$\beta < \min(\mu - f_+, f_- - \lambda).$$

Then there exists $R > 0$ such that

$$\lambda < f_- - \beta \leq \frac{f(u)}{u} \leq f_+ + \beta < \mu,$$

for all $|u| \geq R$, and hence

$$\begin{aligned} \left| \frac{f(u)}{u} - \frac{\lambda+\mu}{2} \right| &\leq \min\left(f_+ + \beta - \frac{\lambda+\mu}{2}, \frac{\lambda+\mu}{2} - f_- + \beta \right) \\ &= \gamma < \frac{\mu-\lambda}{2} = \operatorname{dist}\left(\frac{\lambda+\mu}{2}, \sigma(L)\right). \end{aligned}$$

The conclusion follows then from Corollary 1 in [11] as clearly one has $\frac{\lambda+\mu}{2} \in [a,b] \setminus \sigma(L)$. ■

Remark 6.2 For problem (1.3) it suffices to consider $f(u) = f(x,y,u)$ and replace L by $-L$ in the above theorem.

Remark 6.3 One question still unanswered is whether the equivalent of Theorem 3.1 holds for the cases (B) and (C). If it is the case, the same method works as the first one.

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