Measurements of curvilineal angles

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Abstract

The angle between curves is in most cases defined as the angle between their tangents at the point of intersection. From an heuristic point of view this is unsatisfactory. For instance the angle between a circle and a tangent gets a measure 0 in this way. Yet we see a space between the two quite different from the space between two coinciding half-lines. The essential part of the paper concerns the definition of a non-trivial measure for the angle between tangent curves.

0 Introduction

In geometry plane angles are usually introduced as figures associated with two intersecting straight lines, or as the region of the plane bounded by two such lines. And angle measurement is basically restricted to rectilineal angles, too.
Hilbert (see [5]) considers pairs of rays h, k, lying in distinct lines in some plane, and emanating from the same point. Such a pair is called an angle, and denoted by $\sphericalangle(h, k)$ or $\sphericalangle(k, h)$. An angle divides the plane in two parts. The interior of an angle $\sphericalangle(h, k)$ $(< \pi)$ is defined as the set of points in the plane such that for any two points A, A' the segment AA' does not intersect the rays h and k. The other part of the plane is called the exterior.
In Bourbaki (see [2]) the starting point is a 2-dimensional euclidean space E. For $x \in E,\ x \neq 0$, we define the half-line $[x] = \{\lambda x | \lambda \in \mathbb{R},\ \lambda \geq 0\}$. Two pairs of half-lines $([x], [y])$ and $([x'], [y'])$ are called equivalent if there is an orthogonal transformation R of E, with determinant equal to $+1$, such that $R[x] = [x']$ and

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$R[y] = [y']$. Each equivalence class is called an angle.

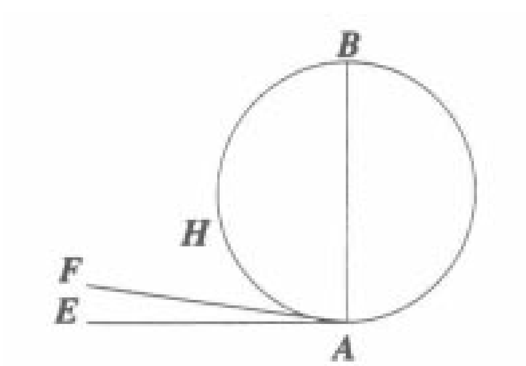
The angle between curves is in most cases defined as the angle between their tangents at the point of intersection. Intuitively this is unsatisfactory. The angle between a circle and a tangent gets a measure 0, and we can clearly distinguish a space between the two, different from the space between two coinciding half-lines. Euclid mentioned this already. From [4] we quote the following definitions:

– A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

– And when the lines containing the angle are straight, the angle is called rectilineal.

So Euclid admits non-rectilineal angles. In one of his propositions in Book II these angles are mentioned explicitly.

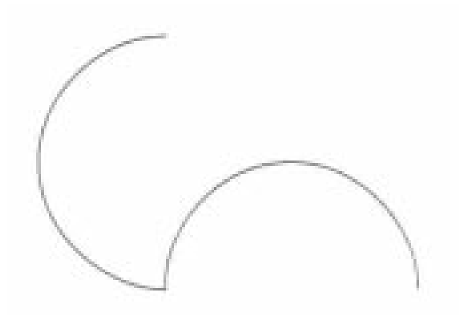
> **Proposition XVI**.
> The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less than any acute rectilineal angle.



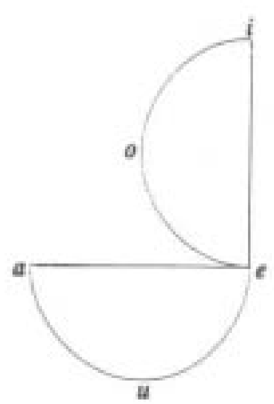
This part of the Elements gave rise to many qualitative descriptions of such "horn-like" angles.

Nicole Oresme (14th century, see [3]) calls an angle a genus, which is subdivided as follows. There is an angulus corporales, which is formed by surfaces, and an angulus superficiales, which is formed by lines. The angulus superficiales in turn can be divided into an angulus planus, which is formed in a plane figure, and an angulus curvus, which is formed on a curved surface, such as a sphere. The angulus planus in turn can be divided into an angle which is formed by two straight lines, one which is formed by a straight and a curved line, or one which is formed by two curved lines. From two straight lines an angle is formed in only one way; from a straight and a curved line, in two ways, for the curved line may be concave as well as

convex; from two curved lines, in three ways, for they are both concave, or convex, or one is concave and the other convex.
Oresme also compares these different types of angles. For instance it is mentioned that an angle formed by two concave lines can be either greater than or smaller than a right angle, but can never be equal to a right angle. And an angle formed by a concave and a convex line may be equal to a right angle, but only in the case that the curved lines have both the same curvature:



Petrus Ramus (16th century, see [7]) was a critic of Euclid's Elements. He wrote his own textbook on geometry, *Geometric septem et viginti*. According to Ramus an angle is a quantity. If the sides of two angles fit the angles are equal. The opposite need not be true, illustrated by the following example:



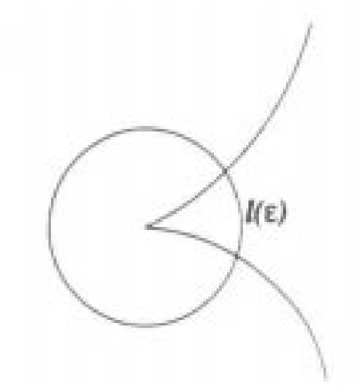
$\angle\, ieo = \angle\, aeu$, so
$\angle\, ieo + \angle\, aeo = \angle\, aeu + \angle\, aeo$,
and therefore $\angle\, aei = \angle\, ueo$

John Wallis (17th century, see [6]) published a treatise on the *Angle of Contact* (i.e. the angle between the curved line and its tangent). He asserts that the angle of contact is no magnitude and not any part of a rightlined angle, but is, to a real angle, whether rectilineal or curvilineal as 0 to a number. He introduced the idea of inceptive quantities. To some kind of magnitude they are nothing, yet they are in the next possibility of being somewhat and the beginning of it. For instance, a point is in the next possibility to length, and inceptive of it. And the same applies to an angle of contact.

We will consider angles formed by curved or straight lines. Definitions of the concept of angle, and congruence of angles will be given. The most important thing will be the introduction of a measure (in fact a series of measures) of such angles. In case of a rectilineal angle we get nothing new, in case of a curvilineal angle we do. For instance we get a non-archimedian ordering on the set of angles. In a sense the angle-measure describes the rate of contact of the two curves.

1 Definitions and examples

We consider angles formed by curved and/or straight lines, and we restrict ourselves to plane curves. In most cases the measure of these angles can be determined only locally. The concept of angle is always associated with the space between the two sides. One way of assigning some kind of measure to this space is the following. Draw a (small) circle of radius ε with center the vertex of the angle.



Call $l(\varepsilon)$ the length of the arc cut out of the circle by the two legs.

If the two legs are straight lines then the quotient $\dfrac{l(\varepsilon)}{\varepsilon}$ = constant = φ, the angle between the two lines.

We generalize this notion by putting $\varphi(\varepsilon) = \dfrac{l(\varepsilon)}{\varepsilon}$, and we get a measure which in general is not a constant, but a function of ε.

Before we go any further into this matter we will define the concept of angle a bit more precise.

Definition.
A directed angle is an ordered pair (F_1, F_2) of curves emanating from the same vertex.

For curvilinear angles most concepts are meaningful only locally, by which we mean within a circular neighbourhood of the vertex.

Definition.
Two angles (F_1, F_2) and (G_1, G_2) are called congruent if there exists some rotation and/or translation of the plane which maps the first vertex onto the second one, and which transforms F_1 and F_2 locally into G_1 and G_2.

For the time being we fix the vertex and choose it as origin for a rectangular system of coordinates. We consider only angles in the right half plane, which is not a serious restriction in view of the given definition of congruence. We put some contraints on the curves we will consider, in order to get an angle-measure-function which behaves properly. The legs of the angles will be (locally) the graphs of certain

functions.
Let $F, F_1, F_2, \ldots$ denote functions of x. At first we only suppose them to be differentiable and monotonically increasing or decreasing in a neighbourhood of the origin from the right. Furthermore we suppose $F(0) = F_1(0) = \ldots = 0$. We use the same symbol to denote both the function and its graph (in a neighbourhood of the origin).

A partial addition is defined by $(F_1, F_2) + (F_2, F_3) = (F_1, F_3)$

So two angles can be added if and only if the second leg of the first angle coincides (locally) with the first leg of the second angle.
Let us call Φ the function defined by

$$\Phi(x) = 0 \text{ if } x \geq 0 \ .$$

The following identities are obvious:

$$\begin{aligned} &(F, \Phi) + (\Phi, \Phi) = (F, \Phi) \\ &(F_1, F_2) = (F_1, \Phi) + (\Phi, F_2) \ . \end{aligned}$$

The next thing is to introduce a measure μ for these angles. Before giving an explicit definition we note some requirements we want this measure to satisfy:

- if (F_1, F_2) and (G_1, G_2) are congruent angles then $\mu(F_1, F_2) = \mu(G_1, G_2)$,
- μ must be additive, i.e. $\mu(F_1, F_2) + \mu(F_2, F_3) = \mu(F_1, F_3)$,
- $\mu(F, F) = 0$,
- $\mu(F_1, F_2) = -\mu(F_2, F_1)$.

Now we could proceed in the geometric way mentioned in the introduction of this section, and define $\mu(F_1, F_2) = \dfrac{l(\varepsilon)}{\varepsilon}$. It is obvious that this function μ of ε satisfies the required conditions. However, in order to do some explicit arithmetic we will proceed in a different way. Besides this will enable us to define not only one, but in most cases a series of measures. Because of $(F_1, F_2) = (F_1, \Phi) + (\Phi, F_2)$ we have to define the values of $\mu(F, \Phi)$ only. The curve F is given by $y = F(x)$.
Using polar coordinates this can be rewritten as

$$F(r \cos \varphi) - r \sin \varphi = 0 \ .$$

By the implicit function theorem we can determine φ as a function of the variable r:

$$\varphi = A(r) \ .$$

Now we suppose that A is differentiable at least once in a neighbourhood of 0.
If A is differentiable n times we denote by A_k the Taylor polynomial of degree $k (0 \leq k \leq n)$. In our case, since we are dealing with a field of characteristic 0, there is no need to distinguish between A_k as an ordered set of coefficients or A_k as

a polynomial function.

Definition.
If A is differentiable n times we define

$$\mu_k(F,\Phi) = A_k(\varepsilon) \text{ for } 0 \le k \le n \ .$$

("par abus de language" we write $\mu_k(F,\Phi)$ instead of $\mu_k(F,\Phi)(\varepsilon)$).
If A is a real analytic function we define

$$\mu_\infty(F,\Phi) = A(\varepsilon) \ ,$$

where we identify $A(\varepsilon)$ with its power series in ε.

The functions $A_k(\varepsilon)$ or $A(\varepsilon)$ are determined by the sets of coefficients. We could have defined μ_k or μ_∞ as the ordered $(k+1)$-tuple or the ordered set of all coefficients, but in order to get a simpler notation in most cases we prefer to take the function.
However, in some cases (see for instance §3) we will switch to the coefficients.

In the last case (i.e. A a real analytic function) there is an infinite series of measures for the angle (F,Φ):
$\mu_n(F,\Phi)$ is simply the first part of the power series of $A(\varepsilon)$ for any $n \in \mathbb{N}$. The measure $\mu_0(F_1,F_2)$ is the constant function, corresponding to the ordinary measure of the angle between the two curves, i.e. the measure of the angle between the two tangent lines. The other measures can be considered as successive refinements of this measure. If A is a real analytic function then $\mu_\infty(F,\Phi)$ is equal to the function we considered in the introduction:

$$\mu_\infty(F,\Phi) = \frac{l(\varepsilon)}{\varepsilon} \ .$$

Examples (see also [1]).

1. $F(x) = a\,x$.
   From $r\sin\varphi = a\,r\cos\varphi$ it is clear that $\varphi = \arctan(a)$, and therefore $\mu_n(F,\Phi) = \arctan(a)$, a constant, for any $n \in \mathbb{N} \cup \{\infty\}$, as was to be expected.
   Furthermore, if $F_1(x) = a\,x, F_2(x) = b\,x$ then $\mu_n(F_1,F_2) = \mu_n(G,\Phi)$, where $G(x) = c\,x$, and $\arctan(c) = \arctan(a) - \arctan(b)$.

2. $F(x) = a\,x^2$.
   Now $\mu_\infty(F,\Phi) = \arcsin\left(\dfrac{2a\varepsilon}{1+\sqrt{1+4a^2\varepsilon^2}}\right)$, and
   $\mu_1(F,\Phi) = a\varepsilon$,
   $\mu_2(F,\Phi) = a\varepsilon - \frac{5}{6}a^3\varepsilon^3$.
   The 0-th order term has disappeared: the curve is tangent to Φ at the origin.

3. $F(x) = R - \sqrt{R^2 - x^2}$.

   In this case $\mu_\infty(F, \Phi) = \arcsin\left(\frac{\varepsilon}{2R}\right)$, and

$$\mu_1(F, \Phi) = \frac{1}{2R}\varepsilon \ ,$$

$$\mu_2(F, \Phi) = \frac{1}{2R}\varepsilon + \frac{1}{48R^3}\varepsilon^3 \ .$$

   We can combine this example with the previous one: if we take $R = \frac{1}{2a}$, then the μ_1-measures of both angles are equal. And this fact has a nice geometrical interpretation: $\frac{1}{2a}$ is the radius of curvature of the parabola at the origin.

Now we are able to compare angles. We fix a measure μ_n for some $n \in \mathbb{N} \cup \{\infty\}$.

Definition

Two angles (F_1, F_2) and (G_1, G_2) are called equal in the μ_n sense if $\mu_n(F_1, F_2) = \mu_n(G_1, G_2)$.

Equality of angles depends therefore on the chosen measure.

Theorem

If $\mu_n(F_1, F_2)$ exists then there is at least one curve G such that $\mu_n(F_1, F_2) = \mu_n(G, \Phi)$.

Proof

$\mu_n(F_1, F_2) = \mu_n(F_1, \Phi) - \mu_n(F_2, \Phi)$, and $\mu_n(F_i, \Phi) = A_{n,i}(\varepsilon)$ (if $n = \infty$ then $A_{\infty,i}(\varepsilon) = A_i(\varepsilon)$).

Put $A_n(\varepsilon) = A_{n,1}(\varepsilon) - A_{n,2}(\varepsilon)$. By using the implicit function theorem we can solve y as a function $G(x)$ from the equation

$$y = x \tan\left(A_n\left(\sqrt{x^2 + y^2}\right)\right) \ .$$

Then the curve $y = G(x)$ satisfies the required property.

Example

$$F_1(x) = R_1 - \sqrt{R_1^2 - x^2} \ ,$$

$$F_2(x) = R_2 - \sqrt{R_2^2 - x^2} \ , \quad (R_1 < R_2) \ .$$

The angle (F_1, F_2) is in the μ_∞ sense equal to the angle (G, Φ), where G is part of the curve given in polar coordinates by

$$r^2 = \frac{4R_1^2 R_2^2 \sin^2\varphi}{R_1^2 + R_2^2 - 2R_1R_2\cos\varphi} \ .$$

In rectangular coordinates this curve is given by

$$\left[(x^2 + y^2)^2(R_1^2 + R_2^2) - 4R_1^2R_2^2y^2\right]^2 = 4R_1^2R_2^2x^2(x^2 + y^2)^3 \ .$$

If we choose the measure μ_1 we can find a curve G of a much simpler form. This situation will be looked at in detail in section 3.

Next we will define a bisectrix of an angle. Again this definition depends on the chosen measure μ_n, $n \in \mathbb{N} \cup \{\infty\}$.

Definition

A curve $y = G(x)$ is called a bisectrix of an angle (F_1, F_2) if $\mu_n(F_1, G) = \mu_n(G, F_2) = \frac{1}{2}\mu_n(F_1, F_2)$.

The definition depends on the chosen measure μ_n. Moreover, the bisectrix is uniquely defined within a class of functions determined by the chosen measure.

Given F_1 and F_2 we can find a G in the following way. Suppose $\mu_n(F_i, \Phi) = A_{n,i}(\varepsilon)$, then we must have $\mu_n(G, \Phi) = \frac{1}{2}A_{n,1}(\varepsilon) + \frac{1}{2}A_{n,2}(\varepsilon)$ and therefore $y = G(x)$ satisfies

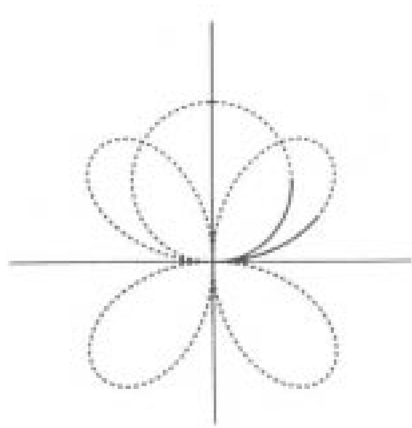
$$y = x \tan\left(\frac{1}{2}A_{n,1}\left(\sqrt{x^2+y^2}\right) + \frac{1}{2}A_{n,2}\left(\sqrt{x^2+y^2}\right)\right) .$$

Examples

1. $F_1(x) = ax$, $F_2(x) = bx$.
   Then $G(x) = cx$, where $c = \tan(\frac{1}{2}\arctan(a) + \frac{1}{2}\arctan(b))$. Of course this holds for any measure μ_n.

2. $F_1(x) = R - \sqrt{R^2 - x^2}$, $F_2(x) = \Phi(x)$.
   In the μ_∞ sense we find a bisectrix G which is part of the curve, given in polar coordinates by $r = 2R\sin 2\varphi$, or in rectangular coordinates by

$$(x^2 + y^2)^3 = 16R^2x^2y^2 .$$



Finally we remark that using the coefficients of the polynomial $\mu_n(F_1, F_2)$, or the coefficients of the power series $\mu_\infty(F_1, F_2)$ we can introduce a lexicographic ordering in the set of angles. The angle between a curve and a tangent is smaller then any non-trivial angle between two straight lines. This is a different formulation of Euclid's proposition in section 0.

2 Intrinsic equations and angle measurement

By using the curvature κ as a function of the length s of the arc we get the so-called intrinsic equation of a curve, i.e., $\kappa = \kappa(s)$. One might expect that the given measure of curvilineal angles turns out to be a simple expresssion if we use these intrinsic coordinates. We start with a curve F, given in polar coordinates, both of them depending at first from a parameter t:

$$\begin{aligned} &r = r(t) \\ &\varphi = \varphi(t), \text{ and } r(0) = 0 \ . \end{aligned}$$

Eliminating t we get

$$\varphi = A(r) \ ,$$

and we defined the measure μ_∞ of the angle between the curve and its tangent at the origin as

$$\mu_\infty(F, \Phi) = A(\varepsilon) \ .$$

We denote differentiation with respect to the parameter by a dot. From differential geometry we know that

$$\begin{aligned} \dot{s}^2 \ &= \dot{r}^2 + r^2\dot{\varphi}^2 \ , \\ \kappa\dot{s}^3 \ &= r^2\dot{\varphi}^3 + r\dot{r}\ddot{\varphi} + 2\dot{r}^2\dot{\varphi} - r\ddot{r}\dot{\varphi} \ . \end{aligned}$$

In the following we use r as a parameter, thus replacing t by r, and then we have

$$\dot{r} = 1 \text{ and } \ddot{r} = 0 \ .$$

The above formulas are reduced to

$$\begin{aligned} \dot{s}^2 \ &= 1 + r^2\dot{\varphi}^2 \ , \\ \kappa\dot{s}^3 \ &= r^2\dot{\varphi}^3 + r\ddot{\varphi} + 2\dot{\varphi} \ . \end{aligned}$$

If the curve F is given by the intrinsic equation $\kappa = \kappa(s)$, we will use the derivatives $\left.\dfrac{d^k\kappa}{ds^k}\right|_{s=0}$ as known constants, and we will try to express $\mu_n(F, \Phi)$ as a function of ε, using these constants.
From the two equations we get (since $r(0) = 0$) that $\dot{s}(0) = 1$ and $\kappa(0) = 2\dot{\varphi}(0)$.
So for $n = 1$ we get

$$\mu_1(F, \Phi) = \frac{1}{2}\kappa(0)\varepsilon \ .$$

As we already remarked in section 1 (in a special case) the coefficient of the first order term is equal to half of the curvature at the origin.
By differentiation of both equations (with respect to the parameter r) we find that $\ddot{s}(0) = 0$ and $\dot{\kappa}(0) = 3\ddot{\varphi}(0)$. Now

$$\begin{aligned} &\dot{\kappa} = \frac{d\kappa}{dr} = \frac{d\kappa}{ds} \cdot \frac{ds}{dr} \ , \text{ and therefore} \\ &\dot{\kappa}(0) = \left.\frac{d\kappa}{ds}\right|_{s=0} \ . \end{aligned}$$

Thus

$$\mu_2(F,\Phi) = \frac{1}{2}\kappa(0)\varepsilon + \frac{1}{6}\left.\frac{d\kappa}{ds}\right|_{s=0}\varepsilon^2 \ .$$

So the coefficients which turn up in μ_1 and μ_2 are the ones which one might expect. However, starting with μ_3 things go "wrong".
Differentiation of both equations once more, and substitution of previous results yields

$$\begin{aligned} &\dddot{\varphi}(0) = \tfrac{1}{8}\kappa(0)^3 + \tfrac{1}{4}\ddot{\kappa}(0)\,, \text{ and since} \\ &\ddot{\kappa}(0) = \left.\frac{d^2\kappa}{ds^2}\right|_{s=0}\,, \text{ we get} \\ &\mu_3(F,\Phi) = \tfrac{1}{2}\kappa(0)\varepsilon + \tfrac{1}{6}\left.\frac{d\kappa}{ds}\right|_{s=0}\varepsilon^2 + \left[\tfrac{1}{48}\kappa(0)^3 + \tfrac{1}{24}\left.\frac{d^2\kappa}{ds^2}\right|_{s=0}\right]\varepsilon^3 \ . \end{aligned}$$

Besides the expected term with $\left.\frac{d^2\kappa}{ds^2}\right|_{s=0}$ an extra term with $\kappa(0)^3$ turns up.

So the use of the intrinsic equation doesn't produces "nice" formulas for the measures $\mu_n(F,\Phi)$.

3 Circles in the plane

In this section we study more closely the angle of tangent circles, or the angle of a circle and a tangent. To the angle of a circle and a tangent we assigned in section 1 the power-series of

$$\mu_\infty(F,\Phi) = \arcsin\left(\frac{\varepsilon}{2R}\right) \ .$$

In order to treat internally tangent and externally tangent circles at the same time we allow (unless otherwise stated) R to be negative. In a suitable coordinate-system such a circle can be represented by the equation

$$F(x,y) = x^2 + y^2 - 2Ry = 0 \ ,$$

where R is either positive or negative.
Then we get for mutually tangent circles

$$\mu_\infty(F_1,F_2) = \arcsin\left(\frac{\varepsilon}{2R_1}\right) - \arcsin\left(\frac{\varepsilon}{2R_2}\right) \ .$$

If we measure an angle of tangent circles in this way, such an angle is determined up to congruence, as will be shown in the next theorem.

Theorem 1
Let F_1, F_2 and F_3, F_4 represent two pairs of tangent circles with corresponding radii R_1, R_2 and R_3, R_4 respectively. Suppose $|R_1| \le |R_2|$ and $|R_3| \le |R_4|$. Then we have $\mu_\infty(F_1,F_2) = \mu_\infty(F_3,F_4)$ if and only if $R_1 = R_3$ and $R_2 = R_4$.

Proof
We only treat the case that all radii are positive.

By comparing the coefficients of the power-series of the left and right hand sides of

$$\arcsin\left(\frac{\varepsilon}{2R_1}\right) - \arcsin\left(\frac{\varepsilon}{2R_2}\right) = \arcsin\left(\frac{\varepsilon}{2R_3}\right) - \arcsin\left(\frac{\varepsilon}{2R_4}\right)$$

we find that

$$\frac{1}{R_1^{2n+1}} - \frac{1}{R_2^{2n+1}} = \frac{1}{R_3^{2n+1}} - \frac{1}{R_4^{2n+1}}\ , \text{ for all } n \geq 0\ ,$$

which can be rewritten as

$$\left(\frac{R_3}{R_1}\right)^{2n+1} = \frac{1-(R_3/R_4)^{2n+1}}{1-(R_1/R_2)^{2n+1}}\ .$$

So $\lim_{n\to\infty}\left(\frac{R_3}{R_1}\right)^{2n+1} = 1$, from which we get $R_1 = R_3$, and therefore $R_2 = R_4$ as well.

This measure for angles of tangent circles does not leave much room for further exploration of interesting results. For instance, even the bisectrix of an angle cannot be a circle, and therefore does not fit in the set of geometrical objects considered in this section. Instead of taking the power-series as an absolute measure for this kind of angles we could consider less rigourous measures by taking only a Taylor-polynomial of a certain degree (see section 1). Especially the Taylor-polynomial of degree 1 yields some elegant results. For the sake of simplicity we introduce a new measure. We write $\mu_1(F_1,F_2) = \frac{1}{2}\mu_1^*(F_1,F_2)\varepsilon$, and we define as a new measure the real number $\mu_1^*(F_1,F_2)$. From previous results we see that

$$\mu_1^*(F_1,F_2) = \frac{1}{R_1} - \frac{1}{R_2}$$

If we consider Φ as a circle with radius ∞ we have

$$\mu_1^*(F,\Phi) = \frac{1}{R}\ .$$

Now this definition leaves much more room for angles to be equal. For instance, if F_1, F_2 are tangent circles we can find a circle F such that $\mu_1^*(F_1,F_2) = \mu_1^*(F,\Phi)$.

Theorem 2
$\mid \mu_1^*(F_1,F_2) \mid = \mid \mu_1^*(\Phi,F) \mid$, with $|R| = \left|\frac{R_1R_2}{R_1-R_2}\right|$.

Proof
A straightforward calculation, where we have to take some care of the sign of $\mu_1^*(F_1,F_2)$.

Theorem 3
The bisectrix of an angle of two tangent circles is a circle.

Proof
Let $R = \dfrac{2R_1R_2}{R_1+R_2}$, and let F represent the corresponding circle. Then it is easy to

see that $\mu_1^*(F_1, F) = \mu_1^*(F, F_2)$.

The special case $R_2 = -R_1$ yields $R = \infty$, and therefore a straight line as a bisectrix, as was to be expected.
It is easy to see that the bisectrix is given by the equation

$$R_2 F_1(x, y) + R_1 F_2(x, y) = 0 \ .$$

Next we will consider a triangle formed by three externally tangent circles. Things become a bit more complicated now, since only one of the meeting points can be chosen as the origin of a coordinate system. Besides, it is unclear in this situation which radii to choose positive and which negative. Therefore we slightly change the notation in this case. We take the radii R_1, R_2, R_3 to be positive. In some coordinate system the circles are given by

$$F_1(x, y) = 0\,,\ \ F_2(x, y) = 0\,,\ \ F_3(x, y) = 0\ .$$

Theorem 4
The three bisectrices of a triangle formed by three externally tangent circles intersect at two points, one inside the triangle, the other outside.
Proof
Since the circles are externally tangent the bisectrices are given by the three equations

$$\begin{array}{c} R_2 F_1(x, y) - R_1 F_2(x, y) = 0 \\ R_3 F_2(x, y) - R_2 F_3(x, y) = 0 \\ R_1 F_3(x, y) - R_3 F_1(x, y) = 0 \end{array}$$

It is obvious that the first two bisectrices intersect at two points, one inside the triangle, one outside. Suppose the first two bisectrices pass through a point (x_0, y_0). From the first equation we get

$$F_1(x_0, y_0) = \frac{R_1}{R_2} F_2(x_0, y_0)\ ,$$

and the second equation yields

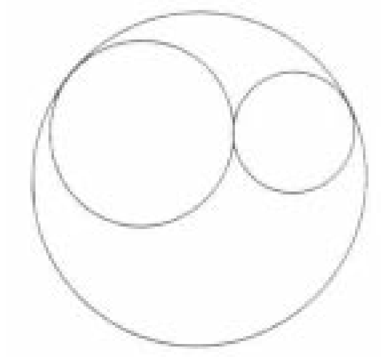
$$F_3(x_0, y_0) = \frac{R_3}{R_2} F_2(x_0, y_0)\ ,$$

Substitute (x_0, y_0) in the third equation:

$$R_1 F_3(x_0, y_0) - R_3 F_1(x_0, y_0) = R_1 \frac{R_3}{R_2} F_2(x_0, y_0) - R_3 \frac{R_1}{R_2} F_2(x_0, y_0) = 0\ ,$$

and we find that (x_0, y_0) is on the third bisectrix too.

Another way to create triangles with tangent circles is to take two externally tangent circles, which are both internally tangent to a third (bigger) circle:



In this situation we can distinguish two triangles. In each triangle the bisectrix of each angle is bisectrix of a corresponding angle in the other triangle too.
By a calculation similar to the one above we get:

Theorem 5
In the above situation the three bisectrices have two points in common, one in each triangle.

Of course $|\mu_1^*(F_1, F_2)|$ is invariant under any isometry of the plane. But there are more transformations which leave $|\mu_1^*(F_1, F_2)|$ invariant. We will show that some circle inversions do. We start with a simple situation. Let $F(x, y) = x^2+y^2-2yR = 0$ be an equation for a circle with a fixed radius $R > 0$.
Let $F_1(x, y) = x^2 + y^2 - 2yR_1 = 0$ and $F_2(x, y) = x^2 + y^2 - 2yR_2 = 0$ represent two more circles, where R_i may be either positive or negative. so all three circles are tangent at the origin. We consider inversion with respect to the first, fixed, circle. Then the images of F_1 and F_2 are again circles, tangent at the origin. If we denote these images by F_1' and F_2' respectively, it is a straightforward calculation, to show that their radii are given by

$$R_i' = \frac{R\,R_i}{2R_i - R}\ .$$

For the angle between F_1' and F_2' we get

$$\begin{aligned}
\mu_1^*(F_1', F_2') &= \frac{1}{R_1'} - \frac{1}{R_2'} \\
&= \frac{2R_1 - R}{R\,R_1} - \frac{2R_2 - R}{R\,R_2} \\
&= -\frac{1}{R_1} + \frac{1}{R_2} \\
&= -\mu_1^*(F_1, F_2)\ .
\end{aligned}$$

So we have proved:

Theorem 6
If we have three circles, tangent at the same point, inversion with respect to any one of these leaves the absolute value of the measure of the angle between the other two invariant.

In particular does this mean that in such a situation the image of a bisectrix the bisectrix of the images is. This sheds a new light on theorems 4 and 5. If, in the situation of theorem 5, we apply inversion with respect to the biggest circle we get the situation of theorem 4.
We also remark that if we have two tangent circles, and if we apply inversion with respect to one of them, this circle of inversion is bisectrix of the angle of the other circle and its image. Or, inversion with respect to a bisectrix transposes one side of the angle into the other one.
In the general case circle inversion is not an angle-preserving transformation.

4 Additional remarks

In the introduction of section 1 we suggested $\mu(F_1, F_2) = \dfrac{l(\varepsilon)}{\varepsilon}$, a more or less geometric way of defining a measure for curvilineal angles. However, we proceeded in a different, analytic way, and in order to do some arithmetic we only considered curves satisfying some differentiability conditions. We conclude this paper with two remarks about the function $\dfrac{l(\varepsilon)}{\varepsilon}$.

1. Instead of $l(\varepsilon)$, the length of an arc, one might consider $L(\varepsilon)$, the area of the region between a curve F, Φ, and a circle with center at the origin and radius ε. We could use the function $\dfrac{L(\varepsilon)}{\varepsilon^2}$ as a measure for the angle (F, Φ). But this would not produce essentially new results since $L'(\varepsilon) = l(\varepsilon)$.

2. The function $\mu(F, \Phi) = \dfrac{l(\varepsilon)}{\varepsilon}$ can be applied to a larger class of curves than the one we considered in section 2. For instance we consider a curve defined by $F(x) = x^\alpha$, where $\alpha \in \mathbb{R}\backslash\mathbb{N}$, $\alpha > 1$. A simple calculation shows that

$$\mu(F, \Phi) = \arcsin\left(\frac{y}{\varepsilon}\right) ,$$

where y is the solution of $y^{2/\alpha} + y^2 = \varepsilon^2$.
In this case we do not get a polynomial (or power series) in ε, but for instance

$$\mu(F, \Phi) = \varepsilon^{\alpha-1} + o(\varepsilon^{\alpha-1}) , \quad \varepsilon \to 0 ,$$

or

$$\mu(F, \Phi) = \varepsilon^{\alpha-1} + \frac{1-3\alpha}{6}\varepsilon^{3\alpha-3} + o(\varepsilon^{3\alpha-3}) , \quad \varepsilon \to 0 .$$

If we take the part of the well-known curve $y^2 = x^3$ where $y \geq 0$, this yields

$$\mu(F, \Phi) = \sqrt{\varepsilon} + o(\sqrt{\varepsilon}) , \quad \varepsilon \to 0 ,$$

and

$$\mu(F, \Phi) = \sqrt{\varepsilon} - \frac{7}{12}\varepsilon\sqrt{\varepsilon} + o(\varepsilon\sqrt{\varepsilon}) , \quad \varepsilon \to 0 ,$$

respectively.

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