On the Poincaré series for a plane divisorial valuation

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Abstract

We introduce the Poincaré series for a divisorial valuation v and prove that this series is an equivalent algebraic datum to the dual graph of v. We give an explicit computation of the Poincaré series.

1 Introduction

Let (R, m) be a 2-dimensional local regular ring with the maximal ideal m and algebraically closed residue field K. Let F be the quotient field of R. We assume that R contains a coefficient field and we denote by X the scheme $SpecR$.

Let us consider divisorial valuations v of F, centered in R (valuations in this paper). If the center of v in X is a closed point, one can get a new center by blowing-up X at that point. We continue the sequence of quadratic transformations if the center remains closed and we stop whenever the center becomes a divisor. This sequence is determined by v and we will refer to it as the reduction process of the valuation. Associated to this process there is a dual graph that shows its complexity.

The reduction for valuations has applications in geometric problems. For instance, Spivakovsky ([5] and [6]) gives a description, for a surface, of the sandwiched singularities by blowing-up primary complete ideals for the maximal ideal m of R. These complete ideals are equivalent data to a finite number of valuations.

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Moreover, in the canonical desingularization process of a foliation over X (see [2] and [4]) the so-called dicritical divisors can be found. The dicritical divisors separate the foliation sheaves and play a significant role in the geometry of the foliation. One also has a finite number of valuations associated to this objects (see [3]).

In this paper, we characterize the valuations which have the same dual graph by means of an easy algebraic invariant: The Poincaré series of the graded ring $gr_v R$.

2 Preliminaries

Let v be a valuation centered in m, ϕ the value group of v and $\phi^+ = v(R \setminus \{0\})$ the value semigroup of v. For each $\alpha \in \phi^+$, we consider the ideals of R, $P_\alpha = \{f \in R | v(f) \geq \alpha\}$ and $P_{\alpha^+} = \{f \in R | v(f) > \alpha\}$. Then we have the following

Definition 1 *The algebra associated to v is defined to be the graded K-algebra,*

$$gr_v R = \bigoplus_{\alpha \in \phi^+} \frac{P_\alpha}{P_{\alpha^+}}.$$

Remark 1 Let $\Lambda = \{Q_i\}_{i \in I}$ be a sequence of elements of m where I is countable set. Let P'_α be the ideal generated by the set

$$\Lambda(\alpha) = \{ \prod_{j \in I_0 \subseteq I, I_0 \ finite} Q_j^{\gamma_j} | \gamma_j \in \mathbf{N}, \gamma_j > 0 \text{ and } \textstyle\sum_{j \in I} \gamma_j v(Q_j) \geq \alpha \ \}.$$

The two following conditions are equivalent:

i) Each v-ideal of R, a, namely, an ideal of R which is the contraction of some ideal of the valuation ring R_v of v, is generated by the set $\Lambda(v(a))$, where $v(a)$ is the value $min\{v(x) | x \in a\}$.

ii) For each $\alpha \in \phi^+$, $P'_\alpha = P_\alpha$.

Definition 2 *Any sequence $\Lambda = \{Q_i\}_{i \in I}$ satisfying the (equivalent) conditions of the remark 1 is called a generating sequence for v.*

2.1 The Hamburger-Noether expansion

Let v be a valuation and π the associated blowing-up sequence,

$$(\pi) : X^{(N+1)} \xrightarrow{\pi_{N+1}} X^{(N)} \longrightarrow \cdots \longrightarrow X^{(1)} \xrightarrow{\pi_1} X^{(0)} = X = SpecR.$$

In the sequel, we denote by P_i the center of π_{i+1} and by $\{x, y\}$ a regular system of parameters (rsp) of R. Some generators $\{x^{(1)}, y^{(1)}\}$ of the maximal ideal of the local ring $R_1 = \mathcal{O}_{X^{(1)}, P_1}$ can be obtained from one of the two following pairs of equalities $x^{(1)} = x$ and $x^{(1)}(y^{(1)} + \xi) = y$, $\xi \in K$ or $x^{(1)} y^{(1)} = x$ and $y = y^{(1)}$. Interchanging x and y, if necessary, and writing $a_{01} = \xi$, a rsp of R_1 will be $\{x, y^{(1)} = (y - a_{01}x)/x\}$. Let h_0 be the maximum of the non negative integers j such that for every $i \leq j$, the rsp of the local ring $R_i = \mathcal{O}_{X^{(i)}, P_i}$ is obtained through the first type of equality. Then some generators of the maximal ideal of the local ring R_i will be, inductively, $\{x^{(i)} =$

$x, y^{(i)} = (y^{(i-1)} - a_{0i}x)/x\}$ for every $i \leq h_0$. And if we put $z_1 = (y^{(h_0-1)} - a_{0h_0}x)/x$ one has that a rsp of $\mathcal{O}_{X^{(h_0+1)},P_{h_0+1}}$ will be $\{x/z_1, z_1\}$.

Following with the same procedure that Campillo uses in [1, Chap. II] for algebroid curves, we find a finite sequence of positive integers $h_0, h_1, \ldots, h_{s_g}$, a finite set of subindices $\{s_0, s_1, \ldots, s_g\}$, $s_0 = 0$, positive integers $k_1, k_2, \ldots\ldots, k_g$ with $2 \leq k_i \leq h_{s_i}$ and a collection of expressions (1), called Hamburger-Noether expansion of the valuation v in the rsp $\{x, y\}$. The data h_j, $0 \leq j \leq s_g$, s_i and k_i, $0 \leq i \leq g$ are independent of the rsp.

$$\begin{array}{rcl} y & = & a_{01}x + a_{02}x^2 + \cdots + a_{0h_0}x^{h_0} + x^{h_0}z_1 \\ x & = & z_1^{h_1}z_2 \\ \vdots & & \vdots \\ z_{s_1-2} & = & z_{s_1-1}^{h_{s_1-1}}z_{s_1} \\ z_{s_1-1} & = & a_{s_1k_1}z_{s_1}^{k_1} + \cdots + a_{s_1h_{s_1}}z_{s_1}^{h_{s_1}} + z_{s_1}^{h_{s_1}}z_{s_1+1} \\ \vdots & & \vdots \\ z_{s_g-1} & = & a_{s_gk_g}z_{s_g}^{k_g} + \cdots + a_{s_gh_{s_g}}z_{s_g}^{h_{s_g}} + z_{s_g}^{h_{s_g}}u. \end{array} \qquad (1)$$

The change of row in the above expansion is related to the change of position of the blowing-up center. u is one of the regular parameters of $\mathcal{O}_{X^{(N)},P_N}$.

2.2 The dual graph

Following the results of Spivakovsky [5], for each valuation v there exists a collection of non-negative integers:

$g \in \mathbf{N}$

$m_i \in \mathbf{N}$, for $1 \leq i \leq g+1$, $m_i \geq 2$ for $i \leq g$, and, $m_{g+1} = 1$;

$a_j^{(i)} \in \mathbf{N}$, for $1 \leq i \leq g$, $1 \leq j \leq m_i$, $a_1^{(g+1)} > 0$

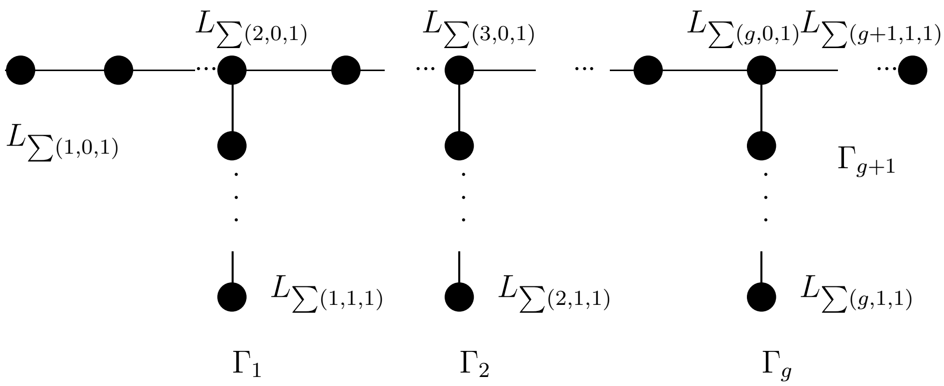
such that, if we denote

$$\sum(i, m, a) = \sum_{k=1}^{i-1}\sum_{p=1}^{m_k} a_p^{(k)} + \sum_{p=1}^{m} a_p^{(i)} + a$$

where $1 \leq i \leq g+1$, $0 \leq m \leq m_i - 1$ and $1 \leq a \leq a_{m+1}^{(i)}$, the above set of numbers exhausts the set $\{1, 2, \ldots, N+1\}$. The dual graph associated to v, $G(v) = \bigcup_{i=1}^{g+1}\Gamma_i$ will be the one that appears at the top of the next page.

Moreover, there exist some values $\{\bar{\beta}_i\}_{i=0}^{g+1}$, such that, the set $\{\bar{\beta}_i\}_{i=0}^{g}$ generates the semigroup ϕ^+. These values can be obtained from the dual graph and vice-versa (see [5] and [6]).

This dual graph is weighted by a dynamical form. That is to say, $G(v)$ is obtained as a limit graph $G(v) = lim_{i\to\infty}G_i$ where the weights of the vertices of $G(v)$ and of their homologous ones in some G_i's can be different (see [6, chap. 5]). There exists an equivalent statical form, such that, the weight of each vertex is the "age" of the corresponding divisor. This dual graph can be reconstructed easily from the Hamburger-Noether expansion for v. (See [3]).



Definition 3 *Let C_i, $0 < i \leq g$, be an analytically irreducible curve of X. We say that C_i has $(i-1)$-Puiseux exponents if the total transform of C_i in $X_{\sum(i,0,1)}$ is a divisor with normal crossings and the strict transform meets $L_{\sum(i,0,1)}$ transversely in one point.*

Definition 4 *Let $\mathcal{C}$ be the family of analytically irreducible curves, whose strict transforms in $X^{(N)}$ are smooth and meets L_N transversely. Let C_i be a curve with $(i-1)$-Puiseux exponents. C_i is said to have maximal contact with $\mathcal{C}$ if the strict transform of C_i in $X^{(N)}$ meets $L^{(N)}_{\sum(i,i,1)}$ (necessarily transversely) and no other exceptional curves. We denote by $L_j^{(N)}$ the strict transform of L_j in $X^{(N)}$.*

Above discussion does not take into account the case $a_1^{(g+1)} = 0$. In that case, the curves in $\mathcal{C}$ will be required to be transverse not only to L_N but also to $L_{\sum(g,m_{g-1},0)}$.

Proposition 1 *a) Any generating sequence for a divisorial valuation contains a subsequence $\{Q_i\}_{0\leq i\leq g}$ such that each curve C_{Q_i} whose equation is given by Q_i has $(i-1)$-Puiseux exponents.*

b) If $a_1^{(g+1)} = 0$ then $\{Q_i\}_{0\leq i\leq g}$ is a minimal generating sequence of v. If $a_1^{(g+1)} > 0$, every generating sequence, $\{Q_i\}_{0\leq i\leq g+1}$, contains a subsequence $\{Q_i\}_{0\leq i\leq g}$ as in a) and the curve $C_{Q_{g+1}}$ belongs to $\mathcal{C}$.

c) $\{\bar\beta_i = v(Q_i)\}_{0\leq i\leq g}$ is a minimal system of generators for the semigroup ϕ^+.

(See [6, chap. 8])

Proposition 2 *A set $\{Q_i\}_{i\in I}$ where $Q_i \in m$, is a generating sequence of v if, and only if, the K-algebra gr_vR is generated by the images of Q_i in gr_vR.*

Proof. "Only if" is obvious.

Conversely, suppose that $\bar Q_i = Q_i + P_{\bar\beta_i^+}$ generates gr_vR. We are going to prove that if $\alpha \in \phi^+$ then $P_\alpha = P'_\alpha$. The inclusion $P'_\alpha \subseteq P_\alpha$ is trivial. To prove the opposite inclusion, we remark that if $\alpha, \beta \in \phi^+$ and $\alpha < \beta$, one has $P'_\beta \subseteq P'_\alpha$ and $P_\beta \subseteq P_\alpha$. Let $f \in P_\alpha$ be such that $v(f) = \beta_0 \geq \alpha$, then $f \in P_{\beta_0}$ and $f + P_{\beta_0^+} \in gr_vR$ is a homogeneous element. Since $f + P_{\beta_0^+} = \sum_{\gamma\in M'\subseteq M} A_\gamma \prod \bar Q_i^{\gamma_i}$ for some M', one has $f - \sum_{\gamma\in M'\subseteq M} A_\gamma \prod Q_i^{\gamma_i} \in P_{\beta_0^+}$, and obviously, $f \in P'_{\beta_0} + P_{\beta_0^+}$. We can write

$f + f_0 \in P_{\beta_0^+}$, for some $f_0 \in P'_{\beta_0}$ and similarly, $f + f_0 \in P'_{\beta_0} + P_{\beta_1^+}$ for some $\beta_1 \in \phi^+, \beta_1 > \beta_0$. Iterating this process, there exist $\beta_0 < \beta_1 < \cdots < \beta_i < \cdots$, with $\beta_i \in \phi^+$, such that $f \in P'_{\beta_0} + P_{\beta_i^+}$ for each i, therefore

$$f \in \bigcap_{i=0}^{\infty}(P'_{\beta_0} + P_{\beta_i^+}).$$

Now, we prove that $\bigcap_{i=0}^{\infty}(P'_{\beta_0} + P_{\beta_i^+}) = \bigcap_{i=0}^{\infty}(P'_{\beta_0} + m^i)$. Since v is a divisorial valuation, v has a minimal generating sequence $\{G_i\}_{i \in I}$ and if $f \in P_\beta$ with $\beta \in \phi^+$ then,

$$f = \sum_{\gamma \in M_0 \subseteq M} A_\gamma \prod G_i^{\gamma_i}$$

for $M = \{\gamma = (\gamma_0, \ldots, \gamma_k) | k \in \mathbf{N}, \sum_{j=0}^{k} \gamma_j \bar{\beta}_j \geq \beta\}$.

If $\mu_\beta = min\{\sum_{j=0}^{k} \gamma_j | (\gamma_0, \ldots, \gamma_k) \in M\}$, one has that, $f \in m^{\mu_\beta}$, and moreover $\mu_{\beta'} > \mu_\beta$ if $\beta' > \beta$. Thus $\bigcap_{i=0}^{\infty}(P'_{\beta_0} + P_{\beta_i^+}) \subseteq \bigcap_{i=0}^{\infty}(P'_{\beta_0} + m^i)$. The opposite inclusion is easy, because R is a noetherian domain and P_β is a m-primary ideal for all β.

To complete the proof, consider the quotient ring R/P'_{β_0} and set $m + P'_{\beta_0} = \bar{m}$. One has, in this ring,

$$\bigcap_{i=0}^{\infty}(P'_{\beta_0} + m^i) = \bigcap_{i=0}^{\infty} \bar{m}^i = \bar{0} = P'_{\beta_0}.$$

Then, $\bigcap_{i=0}^{\infty}(P'_{\beta_0} + P_{\beta_i^+}) = P'_{\beta_0}$ and $f \in P'_{\beta_0} \subseteq P'_\alpha$.

3 The Poincaré series

3.1

Let v be a valuation and (1) the Hamburger-Noether expansion of v with respect to a rsp of R, $\{x, y\}$. The set $\mathcal{C}$ of Definition 4 is the set of analytically irreducible curves f of genus g whose Hamburger-Noether expansion is the same as (1), except for the last row, which looks like

$$z_{s_g - 1} = a_{s_g k_g} z_{s_g}^{k_g} + \cdots + a_{s_g h_{s_g}} z_{s_g}^{h_{s_g}} + \cdots$$

for a suitable (and obvious) basis of the maximal ideal of $R/(f)$.

The curves $C_i, 0 \leq i \leq g$, of maximal contact with $\mathcal{C}$, will have a Hamburger-Noether expansion whose first $s_i - 1$ rows coincide with those of (1) and whose last row will be $z_{s_i - 1} = a_{s_i k_i} z_{s_i}^{k_i} + \cdots$.

If $f \in R$, $v(f) = min_{g \in \mathcal{C}}(f, g)$, where (f, g) denotes the intersection multiplicity between the algebroid curves C_f and C_g, whose equations are given by f and g, respectively (see [6]). It follows that the Hamburger-Noether expansion of v yields the value $v(f)$. Indeed: put $z_{s_g} = t$ in the Hamburger-Noether expansion of v, then the curves of $\mathcal{C}$ have a Hamburger-Noether expansion similar to (1) with $u =$

$a_0t + a_1t^2 + \cdots; a_i \in K$. Thus, $v(f) = min\mu_t[f(x(t,u), y(t,u))]$, where μ_t is the natural discrete valuation of the field $K((t))$ and, $x(t,u)$ and $y(t,u)$ are obtained by reverse substitution in the Hamburger-Noether expansion.

Therefore, if $f(x(t,u), y(t,u)) = \sum_{i\geq\beta} A_i(u)t^i$, and $A_i(u) = \sum A_{ji}u^j$ we have $v(f) = \beta+\gamma$ if, and only if, $A_{j,\beta+d} = 0$ for $j = 0, 1, \ldots, \gamma-d-1$; $d = 0, 1, \ldots, \gamma-1$, and, moreover, some value $A_{j,\beta+\gamma-j}, j = 0, 1, \ldots, g$, does not vanish.

Proposition 3 *Let v be a valuation. Then,*

a) $gr_vR = \bigoplus_{\alpha\in\mathbf{N}} P_\alpha/P_{\alpha^+}$.

b) $dim_K P_\alpha/P_{\alpha^+} < \infty$, *for all α in* $\mathbf{N}$. *(As K-vector space).*

c) Assume that the number of elements for a generating sequence of v is $r+1$. Then there exist a K-algebra graduation for the polynomial ring $S = K[X_0, \ldots, X_r]$ and a 0-degree epimorphism $\psi : S \longrightarrow gr_vR$ of graded algebras.

The proof is obvious.

Definition 5 *Let v be a valuation. We define the Poincaré series of gr_vR to be*

$$H_{gr_vR}(t) = \sum_{\alpha=0}^{\infty} dim_K(P_\alpha/P_{\alpha^+})t^\alpha.$$

Remark 2 Consider the epimorphism $\psi : S \longrightarrow gr_vR$ of Proposition 3. If $J = Ker\psi$ then J is a homogeneous S-ideal $J = \bigoplus_{\alpha\in\mathbf{N}} J_\alpha$. For each $r \in \mathbf{N}$, such that there exists a generating sequence for v with $r+1$ elements, consider the exact sequence of graded algebras,

$$0 \longrightarrow J \longrightarrow S \xrightarrow{\psi} gr_vR \longrightarrow 0$$

that allow us to write the Poincaré series of gr_vR in relation to the Poincaré series of the graded rings of the K-algebras J and S, as follows

$$H_{gr_vR}(t) = H_S(t) - H_J(t).$$

If r_0 denotes the minimum of the r's satisfying the above condition, the Poincaré series of S is $H_S(t) = 1/(\prod_{i=0}^{r_0}(1-t^{\bar{\beta}_i}))$.

Theorem 1 *Let v be a valuation and $\{\bar{\beta}_i\}_{0\leq i\leq g+1}$ the values of 2.2. Define $e_i = g.c.d.(\bar{\beta}_0, \ldots, \bar{\beta}_i)$; $i = 0, \ldots, g+1$ and $N_i = e_{i-1}/e_i$. Then:*

If $a_1^{(g+1)} \neq 0$, one has the equality

$$H_{gr_vR}(t) = \frac{1}{1-t^{\bar{\beta}_0}} \prod_{i=1}^{g} \frac{1-t^{N_i\bar{\beta}_i}}{1-t^{\bar{\beta}_i}} \frac{1}{1-t^{\bar{\beta}_{g+1}}}$$

and if $a_1^{(g+1)} = 0$, then

$$H_{gr_vR}(t) = \frac{1}{1-t^{\bar{\beta}_0}} \prod_{i=1}^{g-1} \frac{1-t^{N_i\bar{\beta}_i}}{1-t^{\bar{\beta}_i}} \frac{1}{1-t^{\bar{\beta}_g}}.$$

Proof. Consider the Hamburger-Noether expansion (1) for v. For each $\alpha \in \phi^+$ we define the K-vector spaces homomorphism, $\gamma_\alpha : P_\alpha \longrightarrow K[u]$ by

$$\gamma_\alpha(f) = \sum_{i+j=\alpha, A_{ji}\neq 0} A_{ji}u^j$$

where $f \in P_\alpha$, $x(t,u)$ and $y(t,u)$ are as in 3.1 and,

$$f(x(t,u), y(t,u)) = \sum_{i\geq\beta} A_i(u)t^i$$

and $A_i(u) = \sum_j A_{ij}u^j$.

By 3.1 if $v(f) = \alpha$, then, $\gamma_\alpha(f) \neq 0$ and if $v(f) > \alpha$, $\gamma_\alpha(f) = 0$. Therefore, $Ker\gamma_\alpha = P_{\alpha^+}$ and obviously,

$$\bar{\gamma}_\alpha : P_\alpha/P_{\alpha^+} \longrightarrow K[u]$$

is a vector space monomorphism.

Let $\{Q_i\}_{0\leq i\leq g+1}$ be a generating sequence for v such that $v(Q_i) = \bar{\beta}_i \in \mathbf{Z}$. This sequence is minimal if $a_1^{(g+1)} \neq 0$. Put $\gamma_{\bar{\beta}_i}(Q_i) = \bar{A}_i(u) \in K[u]$ and $\bar{Q}_i = Q_i + P_{\bar{\beta}_i^+} \in gr_v R$. Moreover, for all $\alpha \in \mathbf{N}$, P_α/P_{α^+} is the homogeneous component of degree α of $K[\bar{Q}_0, \bar{Q}_1, \ldots, \bar{Q}_{g+1}] = gr_v R$, therefore it is generated by

$$\{\prod_{i=0}^{g+1} \bar{Q}_i^{v_i} | \sum_{i=0}^{g+1} v_i\bar{\beta}_i = \alpha\}$$

as a K-vector space. Since P_α/P_{α^+} is isomorphic to $\bar{\gamma}_\alpha(P_\alpha/P_{\alpha^+})$ as a vector space, this space will be generated by

$$\{\prod_{i=0}^{g+1}[\bar{A}_i(u)]^{v_i} \mid \sum_{i=0}^{g+1} v_i\bar{\beta}_i = \alpha\},$$

hence the dimension of (P_α/P_{α^+}) and $\bar{\gamma}_\alpha(P_\alpha/P_{\alpha^+})$ as K-vector spaces coincides.

Now, we take an index $i \in \mathbf{N}, 0 \leq i \leq g$. Then, $v(Q_i)$ and hence $\gamma_{\bar{\beta}_i}(Q_i)$ are constant under the change of variables $u \to u+ct$ for any $c \in K$, hence $\bar{A}_i(u) = A_i \in K$. $\bar{A}_{g+1}(u) \in K[u]$ and so the dimension of the space generated by the products,

$$\mathcal{A} = \{\prod_{i=0}^{g} A_i^{v_i}[\bar{A}_{g+1}(u)]^{v_{g+1}} \mid \sum_{i=0}^{g+1} v_i\bar{\beta}_i = \alpha\}$$

is equal to $dim_K(P_\alpha/P_{\alpha^+})$.

Let us observe that if $\sum_{i=0}^{g} v_i\bar{\beta}_i = \alpha = \sum_{i=0}^{g} v'_i\bar{\beta}_i$, then $\prod_{i=0}^{g}(Q_i)^{v_i} \equiv c\prod_{i=0}^{g}(Q_i)^{v'_i}$ mod P_{α^+} for a suitable $c \in K$. To show it, we note that $Q_i(x(t,u), y(t,u)) = A_i t^{\bar{\beta}_i} + M_i(t,u)$, $0 \neq A_i \in K$, $M_i(t,u) \in K[[t,u]]$, $ord(M_i(t,u)) > \bar{\beta}_i$. Then, applying 3.1

$$v(\prod_{i=0}^{g}(Q_i)^{v_i} - c\prod_{i=0}^{g}(Q_i)^{v'_i}) = \mu_t[At^\alpha + M(t,u) - c(A't^\alpha + M'(t,u))] > \alpha$$

for $c = A/A'$ ($u \in \{a_0t + a_1t^2 + \cdots\}$). The data $A, A' \in K, A \neq 0 \neq A'$ and $M(t,u), M'(t,u) \in K[[t,u]]$, $ord(M(t,u)) > \alpha < ord(M'(t,u))$ are the ones got by substituting and computing.

Consider the set

$$H_\alpha = \{v^* \in \mathbf{N} |\ \text{there exists a } (g+1)\text{-uple of non negative integers}$$

$$(v_0, \ldots, v_g) \text{ such that } \textstyle\sum_{i=0}^{g} v_i\bar{\beta}_i + v^*\bar{\beta}_{g+1} = \alpha\ \}.$$

Since the K-vector space generated by $\mathcal{A}$ and the vector space generated by

$$\{[\bar{A}_{g+1}(u)]^{v_{g+1}} | \sum_{i=0}^{g+1} v_i\bar{\beta}_i = \alpha\}$$

coincide, and its dimension is $card(H_\alpha)$, one has that

$$H_{gr_vR}(t) = \sum_{\alpha \in \mathbf{N}} card(H_\alpha)t^\alpha.$$

In order to give an explicit computation of $H_{gr_vR}(t)$ we define

$$h_{\alpha,a} = \begin{cases} 0 & \text{if } a \notin H_\alpha \\ 1 & \text{if } a \in H_\alpha \end{cases}$$

and then $card(H_\alpha) = \sum_{a \in \mathbf{N}} h_{\alpha,a}$, therefore,

$$H_{gr_vR}(t) = \sum_{\alpha \in \mathbf{N}} card(H_\alpha)t^\alpha = \sum_{\alpha \in \mathbf{N}} (\sum_{a \in \mathbf{N}} h_{\alpha,a})t^\alpha = \sum_{a \in \mathbf{N}} \sum_{\alpha \in \mathbf{N}} h_{\alpha,a}t^\alpha.$$

The second equality holds because for each $\alpha \in \mathbf{N}$ the values $h_{\alpha,a}$ vanish all but a finite number of a.

Thus, one can write

$$H_{gr_vR}(t) = \sum_{\alpha \in \mathbf{N}} h_{\alpha,0}t^\alpha + \sum_{\alpha \in \mathbf{N}} h_{\alpha,1}t^\alpha + \cdots$$

and since $\{\bar{\beta}_0, \ldots, \bar{\beta}_g\}$ generates the semigroup of values ϕ^+, the following equalities hold

$$h_{\alpha,0} = \begin{cases} 0 & \text{if } \alpha \notin \phi^+ \\ 1 & \text{if } \alpha \in \phi^+ \end{cases}$$

$$h_{\alpha,a} = \begin{cases} 0 & \text{if } \alpha - a\bar{\beta}_{g+1} \notin \phi^+ \\ 1 & \text{if } \alpha - a\bar{\beta}_{g+1} \in \phi^+. \end{cases}$$

Now, the Poincaré series can be written as follows

$$H_{gr_vR}(t) = \sum_{\rho\in\phi^+} t^{\rho} + \sum_{\rho\in\phi^+} t^{\rho+\bar\beta_{g+1}} + \cdots + \sum_{\rho\in\phi^+} t^{\rho+(a-1)\bar\beta_{g+1}} + \cdots$$

and making the change $\rho = \alpha - (a-1)\bar\beta_{g+1}$ in the a-th term, we obtain:

$$(\sum_{\rho\in\phi^+} t^{\rho})(1 + t^{\bar\beta_{g+1}} + \cdots + t^{(a-1)\bar\beta_{g+1}} + \cdots) = H_{\phi^+}(t)(1/1 - t^{\bar\beta_{g+1}}).$$

The right hand side of this equality follows from the fact that the first series is the Poincaré series of a curve in $\mathcal{C}$, that we denote by $H_{\phi^+}(t)$, and the second one is a geometric series.

Let $C \in \mathcal{C}$ be an analytically irreducible curve with semigroup of values ϕ^+, and maximal contact values $\{\bar\beta_i\}_{i=0}^{g}$. If $\alpha \in \phi^+$, there exists a unique expression of α, $\alpha = i_0\bar\beta_0 + \cdots + i_g\bar\beta_g$ such that its indices satisfy the conditions,

$$i_0 \geq 0,\ i_j < N_j, i \leq j \leq g. \qquad (2)$$

(See [1, 4.3.9]).

Then,

$$H_{\phi^+} = \sum_{\{i_0,\ldots,i_g,\ satisfy\ (2)\}} t^{i_0\bar\beta_0+\cdots+i_g\bar\beta_g} =$$

$$= (\sum_{i_0\in\mathbf{N}} t^{i_0\bar\beta_0})(\sum_{0\leq i_1<N_1} t^{i_1\bar\beta_1})\cdots(\sum_{0\leq i_g<N_g} t^{i_g\bar\beta_g}) =$$

$$= \frac{1}{1-t^{\bar\beta_0}}\,\frac{1-t^{N_1\bar\beta_1}}{1-t^{\bar\beta_1}}\cdots\frac{1-t^{N_g\bar\beta_g}}{1-t^{\bar\beta_g}}.$$

Finally, the theorem is completed in the case $a_1^{(g+1)} \neq 0$, and in the remaining case it suffices to simplify the formula considering that $\bar\beta_{g+1} = N_g\bar\beta_g$ according to [6, 8.13].

Theorem 2 *Let v be a valuation. The dual graph and the Poincaré series $H_{gr_vR}(t)$ are equivalent data.*

Proof. From the dual graph is easy to compute the data $\{\bar\beta_i\}_{i=0}^{r_0}$ and from them the Poincaré series $H_{gr_vR}(t)$.

Conversely, we are going to obtain the values $\bar\beta_i$ from $H_{gr_vR}(t)$. If $H_{gr_vR}(t) = \sum_{i=0}^{\infty} a_i t^i$, one has $\phi_v^+ = \{j | a_j \neq 0\}$. Let $\{\bar\beta_i\}_{i=0}^{g}$ be the minimal system of generators for ϕ_v^+ and write,

$$H_g(t) = \frac{1}{1-t^{\bar\beta_0}}\prod_{i=1}^{g-1}\frac{1-t^{N_i\bar\beta_i}}{1-t^{\bar\beta_i}}\,\frac{1}{1-t^{\bar\beta_{g+1}}} \text{ and } H_{\phi^+}(t) = \frac{1}{1-t^{\bar\beta_0}}\prod_{i=1}^{g-1}\frac{1-t^{N_i\bar\beta_i}}{1-t^{\bar\beta_i}}.$$

If $H_g(t) = H_{gr_vR}(t)$, one has $a_1^{(g+1)} = 0$. If $H_g(t) \neq H_{gr_vR}(t)$ we have the equality $H_{gr_vR}(t) = H_{\phi^+}(t)\ \frac{1}{1-t^{\beta_{g+1}}}$ therefore

$$t^{\bar\beta_{g+1}} = 1 - (H_{\phi^+}(t)/H_{gr_vR}(t)).$$

Thus, we obtain $\bar{\beta}_{g+1}$, that together with $\{\bar{\beta}_i\}_{i=0}^{g}$ are some data which show the dual graph of v.

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