Domains with linear growth

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In [6] Small and Warfield prove that if R is a prime algebra over a field k, finitely generated as an algebra and of Gelfand-Kirillov dimension one, then R satisfies a polynomial identity and so is a finite module over its centre. This has proved to be a very useful result, cf. [11]. However, the proof is indirect and it is not obvious why the result holds. Smith [10] has appealed for a more direct proof, even in the case that R is a domain [9]. The intention of this note is to give such a proof that only depends on simple growth calculations. For details concerning Gelfand-Kirillov dimension readers are referred to [3]. However, Bergman has shown that an algebra with Gelfand-Kirillov dimension one actually has linear growth [1], [3, Theorem 2.5] and one only needs this concept for the proof.

Let A be an affine k-algebra and let V be a finite dimensional k-subspace that contains 1 and a set of algebra generators of A. The *growth function of A (relative to V)* is the function f given by $f(n) = \dim(V^n)$. We say that the growth of A is *bounded by a linear polynomial* if there exists a constant c such that $f(n) \leq cn$, for all n. This notion is independent of the choice of V, cf. [3, Lemma 1.1].

We need one standard result from the theory of finite dimensional division algebras.

Proposition 1 *Let D be a division algebra with centre k and let F be a maximal subfield of D. If $\dim({}_kF)$ is finite then $\dim({}_kD)$ is finite.*

Proof. [7, Proposition 9.5.2]. □

First, we see that an affine division algebra with linear growth is in fact a finite dimensional algebra; the proof is based on [4, Theorem 2].

Theorem 2 *Let D be an affine division algebra with centre k and suppose that the growth of D is bounded by a linear polynomial. Then D is finite dimensional over k.*

Proof. Let F be a maximal subfield of D. If $\dim(_F D)$ is finite then, by a version of the Artin-Tate Lemma [5, C(i)], F is affine over k and so is finite dimensional over k by the commutative Nullstellensatz. Thus $\dim(_k D)$ is finite as required. By this remark and the result above, we may assume that both $\dim(_F D)$ and $\dim(_k F)$ are infinite. Let N be any finite dimensional k-subspace of D and let V be a finite dimensional generating subspace of D. If $NV \subseteq FN$ then $D = \cup NV^n \subseteq FN$, and so $D = FN$, which is impossible since $\dim(_F D)$ is infinite. We now claim that there are elements $w_0, w_1, \ldots$ such that $w_i \in V^i$ and $\{w_i\}$ is a left F-linearly independent set. To see this, set $w_0 = 1$ and suppose that $w_0, w_1, \ldots w_{i-1}$ have been chosen. Set

$$N = w_0 V^{i-1} + w_1 V^{i-2} + \ldots w_{i-1} k \subseteq V^{i-1}.$$

Since $NV \nsubseteq FN$, choose $w_i \in NV \backslash FN$. Note that $w_i \in V^i$. Now $Fw_i \cap FN = 0$, for otherwise $w_i \in FN$. Thus $Fw_0 + Fw_1 \ldots + Fw_i$ is a direct sum. Next, choose $f_1, f_2, \ldots \in F$ linearly independent over k. Then the elements $f_i w_j$ are linearly independent over k.

Suppose that $\dim(V^n) \leq cn$, for some c. Let m be such that $f_1, \ldots f_{c+1} \in V^m$. Then, for $1 \leq i \leq c+1$ and $1 \leq j \leq n$,

$$f_i w_j \in V^{m+n},$$

and so

$$c(m+n) \geq \dim(V^{m+n}) \geq (c+1)n,$$

for all n. Thus $cm \geq n$, for all n, a contradiction.   □

There is an old conjecture that any division algebra that is affine over its centre is finite dimensional over its centre. In the case that the centre is uncountable this is known to be true, but the countable case seems to be very difficult. The above theorem is a first (tiny) step in the direction of a general proof, which suggests that it might be possible to prove the conjecture under the assumption of polynomial growth.

Theorem 3 *Let R be an affine k-algebra that is a domain with growth bounded by a linear polynomial. Then R satisfies a polynomial identity and hence is a finite module over its centre.*

Proof. If R is algebraic over k then R is a division algebra and so is finite dimensional over its centre by the previous result; hence R satisfies a polynomial identity. Otherwise, choose $t \in R$ transcendental over k and set $S = k[t]$. Then, by a result of Borho and Kraft [2, Satz 6.7] or [3, Theorem 4.12], $S^* = k[t]\backslash 0$ is a right Ore set in R and $D = RS^{*-1}$ is the quotient division ring of R. Also, D is finite dimensional as a right vector space over $F = k(t)$. Thus,

$$R \subseteq D \cong End(D_D) \subseteq End(D_F) \cong M_n(F),$$

where $n = \dim(D_F)$; and so R satisfies a polynomial identity since it is isomorphic to a subring of matrices over a commutative ring. It follows from [8] that R is a finite module over its centre.   □

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