# Bounds on the effective energy density of a special class $p$-dielectric

Gaetano Tepedino Aranguren, Javier Quintero C,
Eribel Marquina, José Soto

**Abstract.** This work gives lower and upper bounds on the effective energy density $\widetilde{W}$ of a two phase composites material composed by a periodical mixed of two nonlinear homogeneous isotropic dielectric materials in prescribed proportion. These bounds are given as a function of $\theta$, which is the volume fraction of the material with lowest dielectric constant in the mixture. The dielectric constant conductivity of the $k-$material are given respectively by

$$\nu_1(z) = \alpha_1|z|^{p-2}\ , \quad \nu_2(z) = \alpha_2|z|^{p-2},$$

where $0 < \alpha_1 < \alpha_2$ and $1 < p < \infty$.
For anisotropic composites the bounds are given in the form

$$\Phi(p, r, \theta) \leq \fint_{S_r} \tilde{W} \leq \Psi(p, r, \theta)\,,$$

where the functions $\Phi, \Psi$ reduce smoothly to the optimal lower and upper bound of the linear composite when $p \to 2$.

The method to obtain this bounds, in the case $p \neq 2$, follows a generalization of the Hashin–Shtrikman variational principles constructed from a comparison medium which is in general nonlinear and reduces to linear when $p = 2$.

**Resumen.** Este trabajo da cotas inferiores y superiores para la densidad de energía eficaz $\widetilde{W}$ de un material compuesto de dos fases, constituido por una mezcla periódica de dos materiales dieléctricos isotrópicos homogéneos no lineales en proporción prescrita. Estas cotas son dadas como una función de $\theta$, que es la fracción de volumen del material con constante dieléctrica más bajo en la mezcla. La constante dieléctrica de conductividad del $k$-material se dan, respectivamente, por

$$\nu_1(z) = \alpha_1|z|^{p-2}\ , \quad \nu_2(z) = \alpha_2|z|^{p-2},$$

donde $0 < \alpha_1 < \alpha_2$ y $1 < p < \infty$.
Para compuestos anisotrópicos los límites se dan en la forma

$$\Phi(p,r,\theta) \le \fint_{S_r} \tilde{W} \le \Psi(p,r,\theta)\,,$$

donde las funciones $\Phi, \Psi$ se reducen suavemente a la cota inferior y superior del compuesto lineal cuando $p \to 2$.

El método para obtener estas cotas, en el caso $p \neq 2$, sigue de una generalización de los principios variacionales de Hashin–Shtrikman construido a partir de un medio de comparación que es en general no lineal y se reduce al lineal cuando $p = 2$.

## 1 Introduction

In this work we will follow the $Y-$periodic microstructure of the mixture, been $Y$ the cell $\prod_{i=1}^{N}(0,a_i)$, $\{a_1, a_2, \ldots, a_N\} \subset (0,\infty)$, if $\theta_k$, for $k \in \{1,2\}$, is the proportion of material type $k$ in the mixed, $Y = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$ and $\chi_k$ is the characteristic function of the phase $Y_k$ which only contains material $k$, then $\theta_k = \fint_Y \chi_k$, $0 \le \theta_k \le 1$ and $\theta_1 + \theta_2 = 1$. Following the notation of [**T**.**Q**.**M**.**S**] , the *energy density* of the composite is the $Y-$periodic extension of the function $W : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ given by

$$\begin{array}{l} W(x,z) = \chi_1(x)W_1(z) + \chi_2(x)W_2(z)\,, \quad \text{where} \\ W_k(z) = \frac{\alpha_k}{p}|z|^p\,, \text{ and } 0 < \alpha_1 < \alpha_2, 1 < p < \infty\,. \end{array} \tag{1.1}$$

In [**T**.**Q**.**M**.**S**] has been proved that the *effective energy density* and its dual are given respectively by the variational principles:

$$\widetilde{W}(\xi) = \inf_{v\in V_p} \fint_Y W(x, v+\xi)\,dx\,, \quad \widetilde{W}^*(\eta) = \inf_{\sigma\in S_q} \fint_Y W^*(x, \sigma+\eta)\,dx\,, \tag{1.2}$$

where $V_p$ is the completion of $C^1_{per}(\overline{Y}, \mathbb{R}^n)$ under the $L_p-$norm and $S_q$ is the completion of $\mathfrak{N} = \{\sigma \in C^1_{per}(Y, \mathbb{R}^N) : \fint_Y \sigma = \theta$ and $div(\sigma) = 0$ in $Y\}$ under the $L_q-$norm. Notice that $v \in V_p \iff v = \nabla u$ for some $u \in K_p$. Here $K_p$ is the completion of $C^1_{per}(\overline{Y})$ under the norm $\|u\|_{1,p} = \left( \fint_Y |u|^p\,dx + \fint_Y |\nabla u|^p\,dx \right)^{1/p}$.

Also we have $V_p^{\perp} = S_q \oplus CV$, where $CV$ are the constants vector fields. See [**T**.**Q**.**M**.**S**] for a complete description of definitions and properties of these spaces.

## 2 Existence of $\widetilde{W}$, $\Gamma-$convergence and Homogenization

**Lemma 1** *The function* $W : \mathbb{R}^n \to \mathbb{R}^N \to \mathbb{R}$ *defined by (1.1) satisfies:*
(**1**)$\forall z \in \mathbb{R}^N : W(.,z)$ *is* $Y-$*periodic and measurable.*
(**2**)$\forall x \in \mathbb{R}^N : W(x,.)$ *is* $C^1(\mathbb{R}^n)$ *and strictly convex.*
(**3**)$\forall x, z \in \mathbb{R}^N : 0 \leq \frac{\alpha_1}{p}|z|^p \leq W(x,z) \leq \frac{\alpha_2}{p}|z|^p$.
(**4**)$\exists L \geq 0$ *such that* $\forall x, z_1, z_2 \in \mathbb{R}^N : |W(x,z_1)^{1/p} - W(x,z_2)^{1/p}| \leq L|z_1 - z_2|$

**Proof** *The items* (**1**) *to* (**3**) *are direct consequences of 1.1. In other hand let* $\beta_k = (\alpha_k)^{1/p}$, *then* $W(x,z_k)^{1/p} = \beta_1\chi_1(x)|z_k| + \beta_2\chi_2(x)|z_k|$ *and* $W(x,z_1)^{1/p} - W(x,z_2)^{1/p} = \beta_1\chi_1(x)(|z_1| - |z_2|) + \beta_2\chi_2(x)(|z_1| - |z_2|)$, *therefore* $|W(x,z_1)^{1/p} - W(x,z_2)^{1/p}| \leq L(|z_1| - |z_2|) \leq L|z_1 - z_2|$. □

**Lemma 2** *If* $W$ *is the function defned by 1.1, then* $\widetilde{W}$ *and* $\widetilde{W}^*$ *are given by 1.2.*

**Proof** *These are consequences of lema 1 and the articles* [**M**.**M**] , [**G**.**DA**] .□

**Lemma 3** *(Elementary Bounds)*
*If* $W$ *is defined by 1.1,* $\widetilde{W}$ *is defined by 1.2 and* $p^{-1} + q^{-1} = 1$, *then*

$$\forall \xi \in \mathbb{R}^N : \frac{1}{p}\left(\theta_1\alpha_1^{-q/p} + \theta_2\alpha_2^{-q/p}\right)^{-p/q}|\xi|^p \leq \widetilde{W}(\xi) \leq \frac{1}{p}(\theta_1\alpha_1 + \theta_2\alpha_2)|\xi|^p . \tag{2.1}$$

*These bounds are called The Elementary Lower and Upper Bounds on* $\widetilde{W}$.

**Proof** *Since the null vector field belongs to* $V_p$, *then from 1.2 we obtain*

$$\widetilde{W}(\xi) \leq \fint_Y W(x,\xi)\,dx = \theta_1\tfrac{\alpha_1}{p}|\xi|^p + \theta_2\frac{\alpha_2}{p}|\xi|^p = \frac{1}{p}(\theta_1\alpha_1 + \theta_1\alpha_2)|\xi|^p . \tag{2.2}$$

*In other hand since the null vector field belongs to* $S_q$, *then from 1.2 we get*

$$\widetilde{W}^*(\eta) \leq \fint_Y W^*(x,\eta)\,dx = \theta_1 W_1^*(\eta) + \theta_2 W_2^*(\eta) = \frac{1}{q}\left(\theta_1\alpha_1^{-q/p} + \theta_2\alpha_2^{-q/p}\right)|\eta|^q , \tag{2.3}$$

*from this, using a typical result of convex analysis, see for example* [**E**, **T**] , *we get*

$$\widetilde{W}(\xi) \geq \tfrac{1}{p}\left(\theta_1\alpha_1^{-q/p} + \theta_2\alpha_2^{-q/p}\right)^{-p/q}|\xi|^p ,$$

*from this last inequality and 2.2 we obtain 2.1.* □

**Lemma 4** *Under the same hypothesis of lemma 3 we have:*

$$\forall x,\xi \in \mathbb{R}^N : W(x,\xi) - W_1(\xi) = \chi_2(x)h(\xi), \;\; W_2(\xi) - W(x,\xi) = \chi_1(x)h(\xi) \tag{2.4}$$

$$\forall x,\eta \in \mathbb{R}^N : W_1^*(\eta) - W^*(x,\eta) = \chi_2(x)g(\eta), \;\; W^*(x,\eta) - W_2^*(\eta) = \chi_1(x)g(\eta) \tag{2.5}$$

$$\begin{array}{l} \forall \xi,\eta \in \mathbb{R}^N : h(\xi) = \frac{\alpha_2-\alpha_1}{p}|\xi|^p, \;\; h^*(\eta) = \frac{\beta}{q}|\eta|^q, \\ g(\eta) = \frac{\beta_1-\beta_2}{q}|\eta|^q, \;\; g^*(\xi) = \frac{\kappa}{p}|\xi|^p \end{array} \tag{2.6}$$

$$\textit{where } \beta_k^p\alpha_k^q = 1, \;\; \beta^p(\alpha_2-\alpha_1)^q = 1, \;\; \textit{and } \kappa^q(\beta_1-\beta_2)^p = 1. \tag{2.7}$$

**Proof** *From 1.1 we get* $W - W_1 = \chi_1 W_1 + \chi_2 W_2 - \chi_1 W_1 - \chi_2 W_1 = \chi_2(W_2 - W_1)$, *then* $h(z) = (W_2 - W_1)(z) = \frac{\alpha_2-\alpha_1}{p}|z|^p$ *which is a convex function, and* $h^*(z) = \frac{\beta}{q}|z|^q$ *where* $\beta^p(\alpha_2-\alpha_1)^q = 1$. *Moreover* $W_2 - W = \chi_1 W_2 + \chi_2 W_2 - \chi_1 W_1 - \chi_2 W_2 = \chi_1(W_2 - W_1)$.

*In other hand* $W^* = \chi_1 W_i^* + \chi_2 W_2^*$ *and* $W_k^*(z) = \frac{\beta_k}{q}|z|^q$ *where* $\beta_k^p\alpha_k^q = 1$. *Therefore* $W^* - W_2^* = \chi_2(W_2^* - W_1^*)$ *and* $W_1^* - W^* = \chi_2(W_1^* - W_2^*)$, *then* $g(z) = (W_1^* - W_2^*)(z) = \frac{\beta_1-\beta_2}{q}|z|^q$ *and* $g^*(z) = \frac{\kappa}{p}|z|^p$ *where* $\kappa^q(\beta_1-\beta_2)^p = 1$. □

**Lemma 5** *Under the same hypothesis of lemma 1 we have:*

$$\forall \xi \in \mathbb{R}^N : \widetilde{W}(\xi) = \inf_{\sigma \in L^q_{per}} \inf_{u \in K_p} \fint_Y \left[-\langle \nabla u + \xi, \sigma\rangle + \chi_1(x)h^*(\sigma) + W_2(\nabla u + \xi)\right] dx\,, \tag{2.8}$$

$$\forall \eta \in \mathbb{R}^N : \widetilde{W}^*(\eta) = \inf_{\zeta \in L^p_{per}} \inf_{\sigma \in S_q} \fint_Y \left[-\langle \sigma + \eta, \zeta\rangle + \chi_2(x)g^*(\zeta) + W_1^*(\sigma + \eta)\right] dx\,. \tag{2.9}$$

**Proof** *These variational principles are consequences of theorem 3 of the article* [**T.Q.M.S**] *and the lemma 4.* □

**Lemma 6** *Let* $\varphi$ *is a* $Y-$*periodic solution of* $\Delta\varphi = \chi_k - \theta_k$ *in* $Y$, $H$ *its Hessian matrix,* $\delta \in \mathbb{R}$, $r > 0$, $\eta \in \mathbb{R}^N$ *and* $u = \delta\langle\nabla\varphi, \eta\rangle$, *then*

$$u \in K_p, \;\; \nabla u = \delta H\eta, \;\; \fint_Y H\chi_k = \fint_Y H^2, \;\; \fint_Y \langle \nabla u, \eta\rangle\chi_k = \delta\fint_Y |H\eta|^2\,, \tag{2.10}$$

$$\fint_{S_r}\fint_Y |H\eta|^2 = \frac{r^2}{N}\theta_1\theta_2\,. \tag{2.11}$$

**Proof** *See for example the theorem 4 of* [**T.Q.M.S**] . □

**Lemma 7** *If $2 \le p < \infty$, $r > 0$ and $H$ as in the lemma 6, then*

$$\mathcal{G}(p,r) = \fint_{S_r}\fint_{Y} |H\eta|^p \le \frac{r^2}{N}\theta_1\theta_2 + (p-2)N^{\frac{p+1}{2}}\theta_1\theta_2\mathcal{Z}(p)(r^2 + r^{p+1}), \qquad (2.12)$$

*where $\mathcal{Z}(p) = \mathcal{C}^{p+1}(p+1)$ and $\mathcal{C}$ is the Calderon-Zygmund-Stein constant given in* [**T**.**Q**.**M**] *.* □

**Proof** *Using the corollary 3 of* [**T**.**Q**.**M**] *there is $t \in [2,p]$ such that*

$$\mathcal{G}(p,r) \le \mathcal{G}(2,r) + (p-2)N^{\frac{t+1}{2}} r^{t+1}\mathcal{C}^{t+1}(t+1)\theta_1\theta_2(\theta_1^t + \theta_2^t). \qquad (2.13)$$

*Clearly $\mathcal{C}^{t+1}(t+1) \le \mathcal{C}^{p+1}(p+1) = \mathcal{Z}(p)$, $\theta_k^t \le \theta_k^2 \le \theta_k$ then $\theta_1^t + \theta_2^t \le 1$ and $N^{(t+1)/2} \le N^{(p+1)/2}$, then using 2.11 and 2.13 we get 2.12.* □

**Lemma 8** *Given $N \in \mathbb{N}, 2 \le p < \infty, p^{-1} + q^{-1} = 1, 0 < \epsilon < 1$ and $s \ge 0$, then $\forall x, y \in \mathbb{R}^N$:*

$$\frac{1}{p}|x+y|^p \le \frac{1}{p}\left[1 + (p-1)(p-2)2^{p-2}\right]|x|^p + \langle x, y\rangle|x|^{p-2} + \frac{1}{q}(p-1)2^{p-2}\epsilon^{-sp/2}|y|^p. \qquad (2.14)$$

**Proof** *If $p > 2$ the function $F : \mathbb{R}^N \to \mathbb{R}$ given as $F(x) = \frac{1}{p}|x|^p$ is of class $C^2(\mathbb{R}^N)$, then given $x, y \in \mathbb{R}^N$ there is $t \in [0,1]$ such that $F(x+y) = F(x) + \langle \nabla F(x), y\rangle + \frac{1}{2}\langle A(z)y, y\rangle$ where $z = x + ty$ and $A(z)$ is the Hessian matrix of $F$ at $z$. We have $\nabla F(x) = x|x|^{p-2}$ and $A(z) = I|z|^{p-2} + (p-2)B(z)|z|^{p-4}$, where $I$ is the identity matrix and $B(z)$ is the matrix $((z_i, z_j))$, clearly the greatest eigenvalue of $B(z)$ is $|z|^2$, the greatest eigenvalue of $A(z)$ is $(p-1)|z|^{p-2}$, then $\frac{1}{2}\langle A(z)y, y\rangle \le \frac{1}{2}|y|^2|z|^{p-2}$. Using a standard inequality gives $|y|^2|z|^{p-2} = (\epsilon^{-s}|y|^2)(\epsilon^s|z|^{p-2}) \le \frac{2}{p}\epsilon^{-sp/2}|y|^p + \frac{p-2}{p}\epsilon^{sp/p-2}|z|^p \le \frac{2}{p}\epsilon^{-sp/2}|y|^p + \frac{p-2}{p}\epsilon^{sp/p-2}2^{p-1}(|x|^p + |y|^p) = \frac{p-2}{p}\epsilon^{sp/p-2}2^{p-1}|x|^p + \frac{2}{p}(\epsilon^{-sp/2} + (p-2)\epsilon^{sp/p-2}2^{p-2})|y|^p$, then*

$$\begin{aligned}\frac{1}{2}\langle A(z)y, y\rangle \le \quad & \frac{(p-1)}{p}(p-2)\epsilon^{sp/p-2}2^{p-2} + \\ & + \frac{(p-1)}{p}\left(\epsilon^{-sp/2} + (p-2)\epsilon^{sp/p-2}2^{p-2}\right)|y|^p,\end{aligned}$$

*then*

$$\begin{aligned}\frac{1}{p}|x+y|^p \quad \le \quad & \frac{1}{p}\left(1 + (p-1)(p-2)\epsilon^{sp/p-2}2^{p-2}\right)|x|^p + \\ & + \frac{1}{q}\left(\epsilon^{-sp/2} + (p-2)\epsilon^{sp/p-2}2^{p-2}\right)|y|^p + \\ & + \langle x, y\rangle|x|^{p-2}.\end{aligned}$$

*Since* $0 < \epsilon < 1, s > 0$ *and* $p > 2$, *then* $\epsilon^{sp/p-2} \leq 1, 2^{p-2} > 1, 1 \leq \epsilon^{-sp/2}$ *and* $\epsilon^{-sp/2} + (p-2)\epsilon^{sp/p-2}2^{p-2}2^{p-2} < \epsilon^{-sp/2} + (p-2)2^{p-2} < \epsilon^{-sp/2}2^{p-2}(p-2)\epsilon^{-sp/2}2^{p-2} = (p-1)\epsilon^{-sp/2}2^{p-2}$. *From this we obtain 2.14. We notice that in the special case* $p = 2$ *we have* $\frac{1}{2}|x+y|^2 = \frac{1}{2} + \langle x, y\rangle + \frac{1}{2}|y|^2 \leq \frac{1}{2}|x|^2 + \langle x, y\rangle + \frac{1}{2}\epsilon^{-s}|y|^2$, *therefore the inequality 2.14 is also true when* $p = 2$. □

**Lemma 9** *Given* $\gamma_1 \in \mathbb{R}, \gamma_2 > 0, \eta \in \mathbb{R}^N$ *and* $T_2 : K_2 \to \overline{\mathbb{R}}$ *defined as*

$T_2(u) = \displaystyle⨍_Y \left[\gamma_1\langle \nabla u, \eta\rangle \chi_k + \frac{\gamma_2}{2}|\nabla u|^2\right] dx$, *then* $\inf_{u\in K_2} = T_2(\widehat{u})$, *where* $\widehat{u} = -\frac{\gamma_1}{\gamma_2}\langle \nabla\varphi, \eta\rangle$ *and* $\varphi$ *is the* $Y-$*periodic solution of* $\Delta\varphi = \chi_k - \theta_k$ *in* $Y$.

**Proof** *Clearly* $T_2$ *is a proper strictly convex function and* $G^1-$*differentiable on the reflexive Banach space* $K_2$. *By the used of the Poincarè inequality we can prove that* $\lim_{||u||\to\infty} T_2(u) = +\infty$, *therefore there is an unique minimizer* $\widehat{u} \in K_2$ *which satisfies* $\forall u \in K_2 : DT_2(\widehat{u}, u) = 0$. *Since* $DT_2(\widehat{u}, u) = ⨍_Y [\gamma_1\langle \nabla u, \eta\rangle\chi_k + \gamma_2\langle \nabla u, \nabla\widehat{u}\rangle] = ⨍_Y \langle \nabla u, \gamma_1\eta\chi_k + \gamma_2\nabla\widehat{u}\rangle$, *then* $⨍_Y div\,(\gamma_1\eta\chi_k + \gamma_2\nabla\widehat{u})u\,dx = 0$, *hence* $\gamma_2\delta\widehat{u} = -div\,(\gamma_1\eta\chi_k)$ *in* $Y$, *from here we get the expected result.* □

**Lemma 10** *Given* $\gamma_1 \in \mathbb{R}, \gamma_2 > 0, \xi \in \mathbb{R}^N$ *and* $M_2 : S_2 \to \overline{\mathbb{R}}$ *defines as*

$M_2(\sigma) = ⨍_Y \left[\gamma_1\langle \sigma, \xi\rangle\chi_k + \frac{\gamma_2}{2}|\sigma|^2\right] dx$, *then* $\inf_{\sigma\in S_2} M_2(\sigma) = M_2(\widehat{\sigma})$, *where* $\widehat{\sigma} = \frac{\gamma_1}{\gamma_2}(H\xi - (\chi_k - \theta_k)\xi)$ *and* $H$ *is the Hessian matrix of the* $Y-$*periodic solution of* $\Delta\varphi = \chi_k - \theta_k$ *in* $Y$.

**Proof** *Clearly* $M_2$ *is a proper strictly convex function which is* $G^1$-*differentiable and coercive (Poincarè inequality) on the reflexive Banach space* $S_2$, *therefore there is an unique minimizer* $\widehat{\sigma}$ *which satisfies* $\forall\sigma \in S_2 : DM_2(\widehat{\sigma}, \sigma) = 0$, *since* $DM_2(\widehat{\sigma}, \sigma) = ⨍_Y [\gamma_1\langle\sigma, \xi\rangle\chi_k + \gamma_2\langle\sigma, \widehat{\sigma}\rangle]\,dx = ⨍_Y \langle\sigma, \gamma_1\xi\chi_k + \gamma_2\widehat{\sigma}\rangle\,dx$, *then* $(\gamma_1\xi\chi_k + \gamma_2\widehat{\sigma}) \in S_2^{\perp} = V_2 \oplus CV$, *then there is* $u \in K_2$ *and* $c \in \mathbb{R}^N$ *such that* $\gamma_1\xi\chi_k + \gamma_2\widehat{\sigma} = \nabla u + c$. *Since* $div(\widehat{\sigma}) = 0$ *in* $Y$ *we have* $\Delta u = div(\gamma_1\xi\chi_k)$, *that is* $u = \gamma_1\langle\nabla\varphi, \xi\rangle$ *where* $\varphi$ *is the* $Y-$*periodic solution of* $\Delta\varphi = \chi_k - \theta_k$ *in* $Y$. *Since* $\nabla u = \gamma_1 H\xi$, *then* $\gamma_1\xi\chi_k + \gamma_2\widehat{\sigma} = \gamma_1 H\xi + c$. *From the fact* $⨍_Y \widehat{\sigma} = \theta$, *we obtain* $\gamma_1\xi\theta_k = c$ *and* $\widehat{\sigma} = \frac{\gamma_1}{\gamma_2}(H\xi - (\chi_k - \theta_k)\xi)$. □

## 3 An Upper Bound on $\widetilde{W}$ when $2 \leq p < \infty$

**Theorem 1** *Given* $W$ *and* $\widetilde{W}$ *as 1.1 and 1.2, then if* $2 \leq p < \infty$ *and* $p^{-1}+q^{-1} = 1$ *we have*

$$\forall r > 0, t > 0 : \fint_{S_r} (W^0-\widetilde{W})^*(\eta) \leq \mathcal{F}_1(p,\theta,r,t) = ar^q - br^2 t - cr^2 t^p + d(r^2+r^{p+1})t^p, \tag{3.1}$$

$$\text{where } W^0(\xi) = \tfrac{C^0}{p}|\xi|^p, \;\; C^0 = \alpha_2[1+(p-1)(p-2)2^{p-2}] \tag{3.2}$$

$$\text{and } a = \tfrac{\beta}{q\theta_1^{p-1}}, \;\; b = \tfrac{\theta_2^{2-2/p}}{N\alpha_2\theta_1^{2/p}}, \;\; c = \tfrac{(p-1)2^{p-2}\theta_2}{Nq\theta_1^{p-1}\alpha_2^{p-1}}, \;\; d = \tfrac{(p-1)(p-2)2^{p-2}\theta_1\theta_2 N^{(p+1)/2}\mathcal{Z}(p)}{\theta_1^{p-1}\alpha_2^{p-1}}. \tag{3.3}$$

**Proof** *We will use the variational principle 2.8 where* $h$ *is given by 2.6. Given* $\eta \in \mathbb{R}^N$ *and* $\sigma(x) = \chi_1(x)\eta$ *we have*

$$\forall \xi\eta \in \mathbb{R}^N : \widetilde{W}(\xi) \leq \theta_1\langle\xi,\eta\rangle + \theta_1 h^*(\eta) + \inf_{u\in K_p} \fint_Y [-\langle\nabla u,\eta\rangle\chi_1(x) + W_2(\nabla u+\xi)]\, dx. \tag{3.4}$$

*Using lemma 8 with* $x = \xi, y = \nabla u$ *and* $s = 2-4/p$ *we get*

$$\begin{aligned} W_2(\nabla u+\xi) \leq & \frac{\alpha_2}{p}\left[1+(p-1)(p-2)2^{p-2}\right] + \\ & + \alpha_2\langle\xi,\nabla u\rangle|\xi|^{p-2} + \frac{\alpha_2}{q}(p-1)2^{p-2}\epsilon^{2-p}|\nabla u|^p, \end{aligned}$$

*substituting this into 3.4 we obtain*

$$\forall \xi,\eta \in \mathbb{R}^N : \widetilde{W}(\xi) - W^0(\xi) \leq -\theta_1\langle\xi,\eta\rangle + \theta_1 h^*(\eta) + \inf_{u\in K_p} T_p(u), \tag{3.5}$$

*where* $T_p(u) = \fint_Y [-\langle\nabla u,\eta\rangle\chi_1 + \frac{\alpha_2}{q}(p-1)2^{p-2}\epsilon^{2-p}|\nabla u|^p]\, dx.$

*It is known that* $\inf_{u\in K_2} T_2(u) = T_2(\tilde{u})$ *where* $\tilde{u} = \frac{1}{\alpha_2}\langle\nabla\varphi,\eta\rangle$ *being* $\varphi$ *the* $Y-$*periodic solution of* $\Delta\varphi = \chi_1 - \theta_1$*, then* $\forall t > 0$*:*

$$\begin{aligned} &\inf_{u\in K_p} T_p(u) \leq T_p(t(\theta_1\theta_2)^{1-p/2}\tilde{u}) = \\ &\quad \leq \fint_Y \left[-t(\theta_1\theta_2)^{1-2/p}\langle\nabla\tilde{u},\eta\rangle\chi_1 + \frac{\alpha_2}{q}(p-1)2^{p-2}\epsilon^{2-p}t^p(\theta_1\theta_2)^{p-2}|\nabla\tilde{u}|^p\right] dx. \end{aligned}$$

*If $H$ is the Hessian matrix of $\varphi$, then using the lemma 6 and replacing 2.10 into the last inequality we obtain that $\forall t > 0$:*

$$\inf_{u\in K_p} T_p(u) \leq \fint_Y \left[ -\frac{t}{\alpha_2}(\theta_1\theta_2)^{1-2/p}|H\eta|^2 + \frac{p-1}{q}2^{p-2}\epsilon^{2-p}t^p\frac{(\theta_1\theta_2)^{p-2}}{\alpha_2^{p-1}}|H\eta|^p \right] dx\,, \tag{3.6}$$

*choosing $\epsilon = \theta_1\theta_2$ and replacing 3.6 into 3.5 we get $\forall \xi, \eta \in \mathbb{R}^N, \forall t > 0$:*

$$\begin{aligned}(\widetilde{W} - W^0)(\xi) \leq &- \theta_1\langle \xi, \eta\rangle + \theta_1 h^*(\eta) - t\frac{(\theta_1\theta_2)^{1-2/p}}{\alpha_2}\fint_Y |H\eta|^2\, dx + \\ &+ \frac{p-1}{q}2^{p-2}\frac{t^p}{\alpha_2^{p-1}}\fint_Y |H\eta|^p\, dx\,,\end{aligned}$$

*replacing $\eta \rightleftharpoons \eta/\theta_1$, adding $\langle \xi, \eta\rangle$ to both sides of the last result and taking sup over $\xi \in \mathbb{R}^N$ we obtain $\forall \eta \in \mathbb{R}^N, \forall t > 0$:*

$$(W^0 - \widetilde{W})^*(\eta) \leq \theta_1 h^*(\eta/\theta_1) - \frac{t\theta_2^{1-2/p}}{\alpha_2\theta_1^{1+2/p}}\fint_Y |H\eta|^2\, dx + \frac{p-1}{q}2^{p-2}\frac{t^p}{\alpha_2^{p-1}\theta_1^p}\fint_Y |H\eta|^p\, dx\,. \tag{3.7}$$

*Given $r > 0$, integrating both sides of 3.7 over $S_r$ and using 2.11 of lemma 6 and 2.12 of lemma 7 we obtain 3.1, 3.2 and 3.3.* □

**Theorem 2** *Under the same hypothesis of theorem 1, given $\mathcal{F}_1$ and $C^0$ by 3.1 and 3.2, then:*

$$\forall r > 0: \fint_{S_r} \widetilde{W} \leq \frac{1}{p}\left[C^0(p) - (qA_1(p))^{1-p}\right] r^p\,, \quad \text{where} \tag{3.8}$$

$$A_1(p) = \inf_{r>0}\inf_{t>0} r^{-q}\mathcal{F}_1(p, r, t)\,. \tag{3.9}$$

**Proof** *Since $\forall \xi \in \mathbb{R}^N, \forall r > 0 : W^0(r\xi) = r^p W^0(p)$ and $\widetilde{W}(r\xi) = r^p\widetilde{W}(\xi)$, then $(W^0 - \widetilde{W})(r\xi) = r^p(W^0 - \widetilde{W})\xi)$ and $(W^0 - \widetilde{W}^*(r\eta) = r^q(W^0 - \widetilde{W})^*(\eta)$, hence $(W^0 - \widetilde{W})^*(\eta) = r^{-q}(W^0 - \widetilde{W})^*(\eta)$ and by 3.1 we obtain*

$$\begin{aligned}\fint_{S_1}(W^0 - \widetilde{W})^*(\eta)\, ds(\eta) &= r^q\fint_{S_1}(W^0 - \widetilde{W})^*(r\eta)\, ds(\eta) \\ &= r^{-q}\fint_{S_r}(W^0 - \widetilde{W})^* \leq r^{-q}\mathcal{F}_1(p, r, t)\,,\end{aligned}$$

*therefore* $\displaystyle\fint_{S_1}(W^0-\widetilde{W})^*\leq A_1(p)$, *where* $A_1$ *is given by 3.9.*

*In other hand,* $\forall\xi,\eta\in\mathbb{R}^N:(W^0-\widetilde{W}^*(\eta)\geq\langle\xi,\eta\rangle-W^0(\xi)+\widetilde{W}(\xi)$. *Let* $\eta\neq\theta, r>0$ *and* $\xi=r\eta/|\eta|$, *we get* $(W^0-\widetilde{W})^*(\eta)\geq r|\eta|-W^0(r|\eta/|\eta|)+\widetilde{W}(r\eta/|\eta|)=r|\eta|-\frac{C^0}{p}r^p+r^p\widetilde{W}(\eta/|\eta|)$, *integrating over* $S_1$ *we obtain* $A_1(p)\geq r-\frac{C^0}{p}r^p+r^p\displaystyle\fint_{S_1}\widetilde{W}$, *from this* $\displaystyle\fint_{S_1}\widetilde{W}\leq\frac{C^0}{p}-r^{1-p}+r^{-p}A_1(p)$, *then* $\displaystyle\fint_{S_1}\widetilde{W}\leq\frac{C^0}{p}\inf_{r>0}\{-r^{1-p}+r^{-p}A_1(p)\}=\frac{C^0}{p}+\widehat{r}^{1-p}+A_1(p)\widehat{r}^{-p}$ *where* $\widehat{r}=qA_1(p)$, *the substitution gives the estimation 3.9 with* $r=1$, *then* $\displaystyle\fint_{S_r}\widetilde{W}(\xi)\,ds(\xi)=\fint_{S_1}\widetilde{W}(r\xi)\,ds(\xi)=$

$$r^p\fint_{S_1}\widetilde{W}(\xi)\,ds(\xi)\leq\frac{1}{p}\left[C^0-(qA_1(p))^{1-p}\right]r^p.\qquad\square$$

**Corollary 1** *Under the same hypothesis of theorem 1, if* $\widetilde{W}$ *is isotropic, then*

$$\forall\xi\in\mathbb{R}^N:\widetilde{W}(\xi)\leq\frac{1}{p}\left[C^0-(qA_1(p))^{1-p}\right]|\xi|^p\,,\tag{3.10}$$

*where* $C^0$ *and* $A_1$ *are given by 3.2 and 3.9.*

**Observation-(1)**: In the limit case $p=2$ we have $\mathcal{F}_1(2,r,t)=\frac{\beta}{2\theta_1}r^2-\frac{\theta_2}{\alpha_2N\theta_1}r^2t+\frac{\theta_2}{2\theta_1N\alpha_2}=\frac{\beta}{2}\theta_1^{-1}r^2+(\frac{t^2}{2}-t)\alpha_2^{-1}\theta_1^{-1}\theta_2N^{-1}r^2$, where $\beta=(\alpha_2-\alpha_1)^{-1}$, then $A_1(2)=\inf\limits_{r>0}\inf\limits_{t>0}r^{-2}\mathcal{F}_1(2,r,t)=\frac{1}{2\theta_1}(\beta-\frac{\theta_2}{\alpha_2N})$, therefore

$$\tfrac{1}{2}\left[C^0-(2A_1(2))^{-1}\right]=\tfrac{1}{2}\left[\alpha_2-\theta_1\left(\frac{1}{\alpha_2-\alpha_1}-\frac{\theta_2}{\alpha_2N}\right)^{-1}\right],$$

which is the *optimal upper bound of the linear composite.*

## 4 An Upper Bound On $\widetilde{W}$ when $1<p\leq2$

***Theorem 3*** *Given* $W$ *and* $\widetilde{W}$ *as 1.1 and 1.2, then if* $1<p\leq2$ *and* $p^{-1}+q^{-1}=1$ *we have*

$$\forall r>0,\forall t>0:\ \fint_{S_r}(W_2-\widetilde{W})^*\leq\mathcal{F}_2(p,r,t)=ar^q-btr^2+ct^pr^p\,,\tag{4.1}$$

$$\textit{where}\ \ a=\frac{\beta}{\theta_1^{q-1}},\ \ b=\frac{\theta_2}{\alpha_2N\theta_1},\ \ c=\frac{(\theta_1\theta_2)^{p/2}N^{-p/2}}{p\alpha_2^{p-1}\theta_1^p}\,.\tag{4.2}$$

***Proof*** *Given $u \in K_p$ and $\xi \in \mathbb{R}^N$, since $\displaystyle\fint_Y \langle \nabla u, \xi\rangle = 0$, using the Jensen inequality and the inequality $(|a|+|b|)^{p/2} \leq |a|^{p/2} + |b|^{p/2}$ when $1 < p \leq 2$, we have*

$$\fint_Y |\nabla u+\xi|^p \leq \left(\fint_Y |\nabla u+\xi|^2\right)^{p/2} = \left(|\xi|^2 + \fint_Y |\nabla u|^2\right)^{p/2} \leq |\xi|^p + \left(\fint_Y |\nabla u|^2\right)^{p/2},$$

*substituting this result into 3.4 we obtain $\forall \xi, \eta \in \mathbb{R}^N$:*

$$\begin{aligned}&(\widetilde{W} - W_2)(\xi) \leq -\theta_1\langle \xi, \eta\rangle + \theta_1 h^*(\eta) + \inf_{u\in K_p} M_p(u),\\ &\text{where } M_p(u) = \tfrac{\alpha_2}{p}\left(\fint_Y |\nabla u|^2\right)^{p/2} - \fint_Y \langle \nabla u, \eta\rangle \chi_1 .\end{aligned} \tag{4.3}$$

*It is known that $\inf_{u\in K_2} M_2(u) = M_2(\widehat{u})$ where $\widehat{u} = \frac{1}{\alpha_2}\langle \nabla\varphi, \eta\rangle$ being $\varphi$ the $Y-$periodic solution of $\nabla\varphi = \chi_1 - \theta_1$ in $Y$. Therefore $\forall t > 0 : \inf_{u\in K_p} M_p(u) \leq M_p(t\widehat{u})$ and by the same arguments used in the proof of theorem 1 we obtain*

$$\inf_{u\in K_p} \leq \frac{t^p}{p\alpha_2^{p-1}}\left(\fint_Y |H\eta|^2\right)^{p/2} - \frac{t}{\alpha_2}\fint_Y |H\eta|^2,$$

*substituting this into 4.3 we get*

$$\begin{aligned}&\forall \xi, \eta \in \mathbb{R}^N, \forall t > 0 :\\ &(\widetilde{W} - W_2)(\xi) \leq -\theta_1\langle \xi, \eta\rangle + \theta_1 h^*(\eta) + \frac{t^p}{p\alpha_2^{p-1}}\left(\fint_Y |H\eta|^2\right)^{p/2} - \frac{t}{\alpha_2}\fint_Y |H\eta|^2 .\end{aligned} \tag{4.4}$$

*Replacing $\eta \rightleftharpoons \eta/\theta_1$ in 4.4, adding $\langle \xi, \eta\rangle$ to both sides of that result and taking sup over $\xi \in \mathbb{R}^N$ we obtain*

$$\begin{aligned}&\forall \xi \in \mathbb{R}^N, \forall t > 0 :\\ &(W_2 - \widetilde{W})^*(\eta) \leq \theta_1 h^*(\eta/\theta_1) + \frac{t^p}{p\alpha_2^{p-1}\theta_1^p}\left(\fint_Y |H\eta|^2\right)^{p/2} - \frac{t}{\alpha_2\theta_1^2}\fint_Y |H\eta|^2 .\end{aligned} \tag{4.5}$$

*Given $r > 0$ the Jensen inequality gives $\displaystyle\fint_{S_r}\left(\fint_Y |H\eta|^2\right)^{p/2} \leq \left(\fint_{S_r}\fint_Y |H\eta|^2\right)^{p/2}$.*

*Hence integrating 4.5 over $S_r$ and using 2.11 we get 4.1 and 4.2.* □

***Theorem 4*** *Under the same hypothesis of theorem 3, given $\mathcal{F}_2$ by 4.1and 4.2, then:*

$$\fint_{S_r} \widetilde{W} \leq \frac{1}{p}\left[\alpha_2 - (qA_2(p))^{1-p}\right] r^p\,, \tag{4.6}$$

$$\text{where } \; A_2(p) = \inf_{r>0}\inf_{t>0} r^{-q}\mathcal{F}_2(p,r,t)\,. \tag{4.7}$$

***Proof*** *Similar to the proof of theorem 2.* □

***Corollary 2*** *Under the same hypothesis of theorem 3, if $\widetilde{W}$ is isotropic, then*

$$\forall \xi \in \mathbb{R}^N: \;\; \widetilde{W}(\xi) \leq \frac{1}{p}\left[\alpha_2 - (qA_2(p))^{1-p}\right] r^p\,, \tag{4.8}$$

*where $A_2$ is given by 4.6.*

**Observation-(2)**: *In the limit case $p = 2$ the formula 4.8 gives the optimal upper bound of the linear composite.*

## 5 A Lower Bound On $\widetilde{W}$ when $2 \leq p < \infty$

***Theorem 5*** *Given $W$ and $\widetilde{W}$ as 1.1 and 1.2, then if $2 \leq p < \infty$ and $p^{-1}+q^{-1} = 1$ we have*

$$\forall r > 0, \forall t > 0 : \fint_{S_r} (W_1 - \widetilde{W})^* \leq \mathcal{F}_3(p,r,t) = ar^p - btr^2 + dt^q r^q\,, \tag{5.1}$$

$$\text{where } \;\; a = \frac{\kappa}{p}\theta_2^{1-p}, \;\; b = (1-N^{-1})\theta_1\theta_2^{-1}\beta_1^{-1}, \;\; d = (1-N^{-1})^{q/2}\theta_1^{q/2}\theta_2^{-q/2}\frac{\beta_1^{1-q}}{q}\,, \tag{5.2}$$

*and $\kappa, \beta_1$ are given by 2.7*

***Proof*** *We choose the variational principal 2.9, given $\xi \in \mathbb{R}^N$ we take $\zeta = \chi_2\xi$, then $\forall \xi, \eta \in \mathbb{R}^N$:*

$$\widetilde{W}^*(\eta) \leq -\langle \eta, \xi\rangle\theta_2 + \theta_2 g^*(\xi) + \inf_{\sigma\in S_q} \fint_Y \left[-\langle\sigma,\xi\rangle\chi_2 + \frac{\beta_1}{q}|\sigma+\eta|^q\right] dx\,. \tag{5.3}$$

*Since $1 < q \leq 2$ and $\fint_Y \langle\sigma,\eta\rangle = 0$, then $\fint_Y |\sigma+\eta|^q \leq \left(\fint_Y |\sigma+\eta|^2\right)^{q/2} = \left(|\eta|^2 + \fint_Y |\sigma|^2\right)^{q/2} \leq |\eta|^q + \left(\fint_Y |\sigma|^2\right)^{q/2}$, hence*

$$\forall \eta, \xi \in \mathbb{R}^N : \widetilde{W}^*(\eta) \leq W_1^*(\eta) - \langle\eta,\xi\rangle\theta_2 + \theta_2 g^*(\xi) + \inf_{\sigma\in S_q} M_q(\sigma)\,, \tag{5.4}$$

*where* $M_q(\sigma) = \frac{\beta_1}{q}\left(\fint_Y |\sigma|^2\right)^{q/2} - \fint_Y \langle\sigma,\xi\rangle\chi_2$. *We know that* $\inf_{\sigma\in S_2} M_2(\sigma) = M_2(\widehat{\sigma})$, *where* $\widehat{\sigma} = -\frac{1}{\beta_1}\left(H\xi - (\chi_2-\theta_2)\xi\right)$, *being* $H$ *the Hessian matrix of the* $Y-$*periodic solution of* $\Delta\varphi = \chi_2 - \theta_2$, *then*

$$\inf_{\sigma\in S_q} M_q(\sigma) \le M_q(t\widehat{\sigma}) = \frac{\beta_1}{q}t^q\left(\fint_Y |\widehat{\sigma}|^2\right)^{q/2} - t\fint_Y \langle\widehat{\sigma},\xi\rangle\chi_2\,. \tag{5.5}$$

*We have* $\langle\widehat{\sigma},\xi\rangle = -\frac{1}{\beta_1}\langle H\xi,\xi\rangle\chi_2 + \frac{1}{\beta_1}(\chi_2-\theta_2)|\xi|^2\chi_2$ *and* $\fint_Y \langle\widehat{\sigma},\xi\rangle\chi_2 = -\frac{1}{\beta_1}\fint_Y |H\xi|^2 + \frac{1}{\beta_1}\theta_1\theta_2|\xi|^2$. *Also* $|\widehat{\sigma}|^2 = \frac{1}{\beta_1^2}\left(|H\xi|^2 - 2(\chi_2-\theta_2)\langle H\xi,\xi\rangle + (\chi_2-\theta_2)^2|\xi|^2\right)$ *and* $\fint_Y |\widehat{\sigma}|^2 = \frac{1}{\beta_1^2}\left(\theta_1\theta_2|\xi|^2 - \fint_Y |H\xi|^2\right)$. *Hence*

$$\inf_{\sigma\in S_q} M_2(\sigma) \le \frac{\beta_1^{1-q}}{q}t^q\left(\theta_1\theta_2|\xi|^2 - \fint_Y |H\xi|^2\right)^{q/2} + \frac{t}{\beta_1}\fint_Y |H\xi|^2 - \frac{t}{\beta_1}\theta_1\theta_2|\xi|^2\,. \tag{5.6}$$

*Substituting 5.5 5.6 into 5.4 we get*

$$\begin{aligned}\widetilde{W}^*(\eta) - W_1^*(\eta) \le -\langle\eta,\xi\rangle\theta_2 + \theta_2 g^*(\xi) + \frac{\beta_1^{1-q}}{q}t^q\left(\theta_1\theta_2|\xi|^2 - \fint_Y |H\xi|^2\right)^{q/2}\\ + \tfrac{t}{\beta_1}\fint_Y |H\xi|^2 - \tfrac{t}{\beta_1}\theta_1\theta_2|\xi|^2\,.\end{aligned} \tag{5.7}$$

*Replacing* $\xi \rightleftharpoons \xi/\theta_2$ *in 5.7, adding* $\langle\xi,\eta\rangle$ *to both side of the last result and taking sup over* $\eta\in\mathbb{R}^N$ *we get*

$$\begin{aligned}(W_1^* - \widetilde{W}^*)^*(\xi) \le \frac{\kappa\theta_2^{1-p}}{p}|\xi|^p + \frac{\beta_1^{1-q}t^q}{q}\left(\theta_1\theta_2^{-1}|\xi|^2 - \theta_2^{-2}\fint_Y |H\xi|^2\right)^{q/2}\\ + \tfrac{t}{\beta_1}\theta_2^{-2}\fint_Y |H\xi|^2 - \tfrac{t}{\beta_1}\theta_1\theta_2|\xi|^2\,,\end{aligned} \tag{5.8}$$

*integrating over* $S_r$ *we finally get 5.1 and 5.2.* □

***Theorem 6*** *Under the same hypothesis of theorem 5 we have*

$$\fint_{S_1} \widetilde{W}^* \leq \frac{1}{q}\left[\beta_1 - (pA_3(p))^{1-q}\right], \quad \fint_{S_1} \widetilde{W} \geq \frac{1}{p}\left[\beta_1 - (pA_3(p))^{1-q}\right]^{1-p}, \tag{5.9}$$

*where* $A_3(p) = \inf_{r>0}\inf_{t>0} r^{-p}\mathcal{F}_(p,r,t)$ *being* $\mathcal{F}_3$ *given by 5.1.*

***Proof*** *By the same homogeneity properties using in the proof of the theorem 4 we have* $\fint_{S_1}(W_2^* - \widetilde{W}^*)^* \leq r^{-p}\mathcal{F}_3(p,r,t)$, *then* $\fint_{S_1}(W_2^* - \widetilde{W}^*)^* \leq A_3(p)$. *Since* $(W_1^* - \widetilde{W}^*)^* \geq \langle\xi,\eta\rangle - W_1^*(\eta) + \widetilde{W}^*(\eta)$, *fixing* $\xi \neq \theta$ *and* $r > 0$, *let* $\eta = r\xi/|\xi|$, *we obtain* $(W_1^* - \widetilde{W}^*)^*(\xi) \geq r|\xi| - r^q W_1^*(\xi/|\xi|) + r^q\widetilde{W}^*(\xi/|\xi|)$, *then* $A_3(p) \geq \fint_{S_1}(W_1^* - \widetilde{W}^*)^* \geq r - r^q\frac{\beta_1}{q} + r^q\fint_{S_1}\widetilde{W}^*$ *and* $\fint_{S_1}\widetilde{W}^* \leq r^{-q}A_3(p) - r^{1-q} + \frac{\beta_1}{q}$, *hence* $\fint_{S_1}\widetilde{W}^* \leq \inf_{r>0}\left(r^{-q}A_3(p) - r^{1-q}\right) + \frac{\beta_1}{q}$, *that is*

$$\fint_{S_1}\widetilde{W}^* \leq \frac{1}{q}\left[\beta_1 - (pA_3(p))^{1-q}\right] = B_3(p)\,. \tag{5.10}$$

*In other hand, since* $\widetilde{W}^*(\eta) \geq \langle\xi,\eta\rangle - \widetilde{W}$, *taking* $\eta \neq \theta$, $r > 0$ *and* $\xi = r\eta/|\eta|$, *we get* $\widetilde{W}^*(\eta) \geq r|\eta| - r^p\widetilde{W}(\eta/|\eta|)$, *integrating over* $S_1$ *and using 5.10 we obtain* $B(p) \geq r - r^p\fint_{S_1}\widetilde{W}$ *and* $\fint_{S_1}\widetilde{W} \geq r^{1-p} - r^{-p}B(p)$, *then* $\sup_{r>0}\left(r^{1-p} - r^{-p}B(p)\right) \leq \fint_{S_1}\widetilde{W}$. *From here and 5.10 we obtain 5.9.* □

***Corollary 3*** *Under the same hypothesis of theorem 5 we have*

$$\forall r > 0: \quad \fint_{S_r}\widetilde{W}^* \leq \frac{1}{q}\left[\beta_1 - (pA_3(p))^{1-q}\right] r^q, \quad \fint_{S_r}\widetilde{W} \geq \frac{1}{p}\left[\beta_1 - (pA_3(p))^{1-q}\right]^{1-p} r^p, \tag{5.11}$$

***Corollary 4*** *Under the same hypothesis of theorem 5, if* $\widetilde{W}$ *is isotropic, then*

$$\forall \xi,\eta \in \mathbb{R}^N: \quad \widetilde{W}^*(\eta) \leq \frac{1}{q}\left[\beta_1 - (pA_3(p))^{1-q}\right]|\eta|^q, \quad \widetilde{W}(\xi) \geq \frac{1}{p}\left[\beta_1 - (pA_3(p))^{1-q}\right]^{1-p}|\xi|^p\,. \tag{5.12}$$

## 6 Lower Bound on $\widetilde{W}$, when $1 < p \leq 2$

***Theorem 7*** *Given $W$ and $\widetilde{W}$ as 1.1 and 1.2, then if $1 \leq p \leq 2$ and $p^{-1}+q^{-1}=1$ we have*

$$\forall r>0, t>0: \fint_{S_r} (W_0^* - \widetilde{W}^*)^* \leq \mathcal{F}_4(p,r,t) = ar^p - br^2 t + dr^2 t^q + ct^q(r^2 + r^{q+1}) \tag{6.1}$$

$$\text{where } W_0^*(\eta) = \frac{C_0^*}{q}|\eta|^q, \ \ C_0^* = \beta_1\left(1+(q-1)(q-2)2^{q-2}\right) \ \text{ and} \tag{6.2}$$

$$a = \frac{\theta_2^{1-p}\kappa}{p}, \ \ b = (1-\frac{1}{N})\beta_1^{2-2/q}\theta_2^{-2/q}, \ \ c = \frac{\beta_1^q}{q}(q-1)2^{2q-2}\theta_2^{-1}(1+\frac{1}{N}), \tag{6.3}$$

*$d = \frac{\beta_1^{-q}}{q}(q-1)(q-2)2^{2q-2}\theta_2^{-q}N^{(q+1)/2}\mathcal{Z}(q)$ and $\mathcal{Z}(p) = \mathcal{C}(q+1)^{q+1}$ is the Stein-constants.*

*We choose the variational principle 4.3, then $\forall \eta \in \mathbb{R}^N$:*

$$\widetilde{W}(\eta) \leq -\langle \eta, \xi\rangle\theta_2 + g^*(\xi)\theta_2 + \inf_{\sigma\in S_q} \fint_Y \left[-\langle \sigma,\xi\rangle\chi_2 + \frac{\beta_1}{q}|\sigma+\eta|^q\right] dx\,. \tag{6.4}$$

*Since $q \geq 2$ using 2.14 with $x=\eta$, $y=\sigma$ and $s = 2-4/q$ we get*

$$\frac{1}{q}|\sigma+\eta|^q \leq \frac{1}{q}\left[1+(q-1)(q-2)2^{q-2}\right]|\eta|^q + \langle \eta,\sigma\rangle|\eta|^{q-2} + \frac{1}{q}(q-1)2^{q-2}\epsilon^{2-q}|\sigma|^q\,,$$

*substituting this inequality into 6.4 we get*

$$\forall \xi,\eta \in \mathbb{R}^N : \widetilde{W}(\eta) - W_0(\eta) \leq -\langle \eta,\xi\rangle\theta_2 + g^*(\eta)\theta_2 + \inf_{\sigma in S_q} M_q(\sigma)\,, \tag{6.5}$$

*where $M_q(\sigma) = \fint_Y \left[-\langle\sigma,\xi\rangle\chi_2 + \frac{1}{q}(q-1)(q-2)2^{q-2}\epsilon^{2-q}|\sigma|^q\right] dx$. Since $\inf_{\sigma\in S_2} M_2(\sigma) = M_2(\widehat{\sigma})$, where $\widehat{\sigma} = -\frac{1}{\beta_1}\left[H\xi - (\chi_2-\theta_2)\right]$, being $H$ the Hessian matrix of the $Y-$periodic solution of $\Delta\varphi = \chi_2 - \theta_2$, then $\forall t>0$:*

$$\begin{aligned}\inf_{\sigma\in S_q} M_q(\sigma) &\leq M_q(t(\theta_1\theta_2)^{1-2/q}\widehat{\sigma}) \\ &= -t(\theta_1\theta_2)^{1-2/q}\fint_Y \langle\widehat{\sigma},\xi\rangle\chi_2 + \frac{\beta_1}{q}(q-1)2^{q-2}\epsilon^{2-q}t^q(\theta_1\theta_1)^{q-2}\fint_Y |\widehat{\sigma}|^q\,.\end{aligned} \tag{6.6}$$

*In other hand* $\langle\widehat{\sigma},\xi\rangle = -\frac{1}{\beta_1}\left[\langle H\xi,\xi\rangle - (\chi_2-\theta_2)|\xi|^2\right]$ *and* $\fint_Y \langle\widehat{\sigma},\xi\rangle\chi_2 = -\frac{1}{\beta_1}\fint_Y |H\xi|^2$. *Therefore from here, 6.6, taking* $\epsilon = \theta_1\theta_1$ *and using an standard inequality, we get*

$$\inf_{\sigma\in S_q} M_q(\sigma) \leq \quad -t\beta_1^{-1}(\theta_1\theta_2)^{2-2/q}|\xi|^2 + t(\theta_1\theta_2)^{1-2/q}\beta_1^{-1}\fint_Y |H\xi|^2 + \\ +t^q\frac{\beta_1^{-q}}{q}2^{2q-3}(\theta_1\theta_2^q+\theta_1^q\theta_2)|\xi|^q + t^q\frac{\beta_1^{-1}}{q}(q-1)2^{2q-3}\fint_Y |H\xi|^q\,,$$

*substituting this inequality into 6.4 we obtain that* $\forall\xi,\eta\in\mathbb{R}^N$*:*

$$\begin{aligned}\widetilde{W}^*(\eta) - W_0^*(\eta) \leq &- \langle\xi,\eta\rangle\theta_2 + g^*(\xi)\theta_2 - t\beta_1^{-1}(\theta_1\theta_2)^{2-2/q}|\xi|^2 \\ &+ t(\theta_1\theta_2)^{1-2/q}\beta_1^{-1}\fint_Y |H\xi|^2 + \frac{t^q\beta_1^{-1}}{q}(q-1)2^{2q-2}|\xi|^2 + \\ &+ \frac{t^q\beta_1^{-q}}{q}(q-1)2^{2q-3}\fint_Y |H\xi|^q\,,\end{aligned} \tag{6.7}$$

*replacing* $\xi \rightleftharpoons \xi/\theta_2$*, adding* $\langle\xi,\eta\rangle$ *to both side of the result and taking sup over* $\eta\in\mathbb{R}^N$*, we finally get 6.1, 6.2 and 6.3.* □

***Theorem 8*** *Under the same hypothesis of theorem 7 we have*

$$\forall r>0: \fint_{S_r}\widetilde{W}^* \leq \frac{1}{q}\left[C_0^* - (pA_4(p))^{1-q}\right]r^q,\ \fint_{S_r}\widetilde{W} \geq \frac{1}{p}\left[C_0^* - [pA_4(p))^{1-q}\right]^{1-p} r^p\,, \tag{6.8}$$

*where* $A_4 = \inf_{r>0}\inf_{t>0}\mathcal{F}_4(p,r,t)$.

***Corollary 5*** *Under the same hypothesis of theorem 7, when* $\widetilde{W}$ *is isotropic we have* $\forall\xi,\eta\in\mathbb{R}^N$*:*

$$\begin{aligned}\widetilde{W}(\xi) &\geq \frac{1}{p}\left[C_0^* - (pA_4(p))^{1-q}\right]^{1-p}|\xi|^p \\ \widetilde{W}^*(\eta) &\leq \frac{1}{q}\left[C_0^* - (pA_4(p))^{1-q}\right]|\eta|^q\,.\end{aligned}$$

## 7 Summary and Conclusions

*Given* $1 < p < \infty, p^{-1} + q^{-1} = 1, 1 < \theta_k < 1, \theta_1 + \theta_2 = 1, \theta = \theta_1$, *we obtain* $C^0(p) > 0$ *and* $C_0(p) > 0$ *such that*

$$\frac{1}{p}\left[C_0(p) - \theta_1^{q(p-1)}\left(pA(p,\theta)\right)^{1-q}\right]^{1-p} r^p \leq \fint_{S_r} \widetilde{W} \leq \frac{1}{p}\left[C^0(p) - \theta_2^{p(q-1)}\left(qB(p.\theta)\right)^{1-p}\right] r^p$$

$$C_0(p) = \begin{cases} \alpha_2[1 + (p-1)(p-2)2^{p-1}] & if \quad p \geq 2 \\ \alpha_2 & if \quad 1 < p < 2 \end{cases},$$

$$A(\theta, p) = \inf_{(x,y)\in\mathbb{R}_0^+\times\mathbb{R}_0^+} x^{-p}\mathcal{F}(\theta, p, x, y)$$

$$C^0(p) = \begin{cases} \beta_1 & if \quad p \geq 2 \\ \beta_1[1 + (q-1)(q-2)2^{q-1}] & if \quad 1 < p < 2 \end{cases},$$

$$B(\theta, p) = \inf_{(x,y)\in\mathbb{R}_0^+\times\mathbb{R}_0^+} x^{-q} \gg (\theta, p, x, y)$$

*where the functions* $\mathcal{F}, \gg$ *are given in this work.*

*In the isotropic case we have*

$$\frac{1}{p}\left[C_0(p) - \theta_1^{q(p-1)}\left(pA(p,\theta)\right)^{1-q}\right]^{1-p} |\xi|^p \leq \widetilde{W}(\xi) \leq \frac{1}{p}\left[C^0(p) - \theta_2^{p(q-1)}\left(qB(p,\theta)\right)^{1-p}\right] |\xi|^p .$$

*Here we have* $\theta_1 = \theta$ *and* $\theta_2 = 1 - \theta$, *where* $1 \leq \theta \leq 1$ *is the volume fraction of the first material.*

*In the limit case* $p = 2$, *the left and right hand sides of the last inequality give the optimal lower and upper bound of the linear composite.*

**Acknowledgement**: *The authors; want to thank Vicenzo Constenzo Alvarez of the Universidad Simón Bolívar, Department of Physics, for having revised this paper and Oswaldo Araujo of the University of the Andes, Faculty of Science, Department of Mathematics who helped this paper to be published in these Bulletin. Likewise, we thank Mr. Antonio Vizcaya P. for transcription it.*

## References


[1] **J. Ash and R. Jones**. *Calderon-Zygmund and Beyond. Conference in Honor of Stephen Vagi Retirement. De Paul University, Chicago, Illinois, 2006.*

[2] **B.Bergman**. *Bulk Physical Properties of a Composite Media. lectures Notes. L'Ecole d'ete' de Analyse Numerique, 1983.*

[3] **A. Braides**. *Omogeneizazione di Integrali non Coercive. Estratto de Ricerche di Matematica, Volume XXXII, pp.347, 1983.*

[4] **R.Burridge, S.Childress, G.Papanicolaou**. *Macroscopic Properties of Disordered Media. Spronger-Verlag, New York, 1982.*

[5] **I.Ekland,R.Temam**. *Convex Analysis and variational Problems. North Holland, 1976.*

[6] **G.Dell'Antonio**. *Non-linear Electrostatic in Inhomogeneous Media. Preprint, Dipartamento di Matematica, Universita' di Roma, La Sapienza, Italia, 1987.*

[7] **K.Lurie, A.Cherkaev**. *Exact Estimates of Conductivity of Composites Formed by two Isotropic Conducting Media Taken in Prescribed Proportion. Proc. Royal Soc. Edimburg, 29A, pp.71-87,1984.*

[8] **Hashin, Shtrikman**. *A Variational Approach to the Theory of the Effective Magnetic Permeability of Multi-phase Materials. J.Appl.Phys,33,pp.3125-3131, 1962.*

[9] **M.Milkis**. *Effective Dielectric Constants of a Non-Linear Composites Material. SIAM, J.Appl Math, Vol 43, N.3, Oct,1983.*

[10] **Yves-Patricck Pellegrini**. *Self-consistent Effective-medium Approximation for Strongly Nonlinear Media. Phis.Rev B. Vol. 64, 2001.*

[11] **Sundaram Thangavelu** . *Harmonic Analysis on the Heisemberg Group. Springer, 1998.*

[12] **Sundaram Thangavelu** . *An Introduction to the Uncertainty Principle. Springer, 2004.*

[13] **Dong-Hau Kuo, Wun-Ku Wang**. *Dielectric Properties of three Ceramic Epoxy Composites. Materials Chemistry and Physics. Volume 85. Issue 1, 2004, pages 201-206.*

[14] **S.Talbot, J.Willis**. *Variational Principles for Inhomogeneous Non-linear Media. IMA, Journal of Applied Mathematics, 35,pp. 39-54, 1985.*

[15] **G.Tepedino, J. Quintero, E. Marquina**. *An application of the Theory of Calderon-Zygmund to the Sciences of the Materials. Journal of Mathematical Control Science and Applications (JMCSA). Vol.5, N.1, June 2012, pp, 11-18.)*

[16] **G.Tepedino**. *Bounds on the Effective Energy Density of Nonlinear Composites. Doctoral Thesis, Courant Institute of Mathematical Sciences, 1988.*

[17] **G.Tepedino, J. Quintero, E. Marquina**. *Bounds on the Effective Energy Density of a More General Class Willis Dielectric Composites. To appear.*

[18] **Hung T. Vo and Frank G. Shi** *Towards Model-Based Engineering of Optoelectronic Packing Materials: Dielectric Constant Modeling. Microelectronics Journal. Volume 33. Issues 5-6, 2002, pages 409-415.*

[19] **Tom Wolf**. *Lectures in Harmonic Analysis. AMS University Lecture Series, Vol 29, 2003.*

[20] **J.Willis**. *Variational and Related Methods for the Overall Properties of Composites. Advanced in Applied Mechanics. Academic Press, Vol. 21, 1981.*

[21] **Zhou C., Neese B., Zhang Q., Bauur F.** *Relaxor Ferroelectric Poly(Vinylidene fluoride-trifluororthylene-chlofluoroethylene) Terpolymer for High Energy Density Storage Capacitors. Dielectrics and Electrical Insulation, IEEE Transaction . Vol 13, Issue 5, 2006.*



*Gaetano Tepedino Aranguren,*
*Departamento de Matemáticas*
*Facultad de Ciencias,*
*Universidad de los Andes*
*Mérida, Venezuela.*

*Javier Quintero C.*
*Área de Matemática,*
*Universidad Nacional Abierta de Mérida.*

*Eribel Marquina.*
*Área de Matemática,*
*UNEFA, Mérida.*

*José Soto,*
*Departamento de Matemáticas*
*Facultad de Ciencias,*
*Universidad de los Andes*
*Mérida, Venezuela.*