On almost pseudo concircular Ricci symmetric manifolds

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Abstract. The object of the present paper is to study almost pseudo concircular Ricci symmetric manifolds and its decomposibility. Among others it is shown that in a decomposable almost pseudo concircular Ricci symmetric manifold one of the decompositions is Einstein and the other decomposition is concircular Ricci symmetric. The totally umbilical hypersurfaces of almost pseudo concircular Ricci symmetric manifolds are also studied.

Resumen. El objetivo del presente trabajo es estudiar variedades simétricas de Ricci casi–pseudo concirculares y su decomponibilidad. Entre otros, se demuestra que en una variedad de Ricci simétrica descomponible casi–seudo concircular una de las descomposiciones es Einstein y la otra descomposición es concircular simétrica Ricci. También se estudian las hiper–superficies totalmente umbilicales de variedades de Ricci concirculares seudo–simétricas.

1 Introduction

As an extended class of pseudo Ricci symmetric manifolds introduced by Chaki [1], recently Chaki and Kawaguchi [2] introduced the notion of almost pseudo Ricci symmetric manifolds. A Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = [A(X) + B(X)]S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X), \quad (1)$$

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where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and A, B are nowhere vanishing 1-forms such that $g(X,\rho) = A(X)$ and $g(X,\mu) = B(X)$ for all X and ρ, μ are called the basic vector fields of the manifold. The 1-forms A and B are called associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$. The almost pseudo Ricci symmetric manifolds have also been studied by Shaikh, Hui and Bagewadi [18].

If, in particular, $B = A$ then (1) reduces to

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X), \tag{2}$$

which represents a pseudo Ricci symmetric manifold [1].

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [26]. The interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$, which is defined by [26]

$$\begin{aligned}\tilde{C}(Y,Z,U,V) &= R(Y,Z,U,V) \\ &- \frac{r}{n(n-1)}\big[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\big],\end{aligned} \tag{3}$$

where R and r denotes the curvature tensor and the scalar curvature of the manifold respectively.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at each point of the manifold and let

$$P(Y,V) = \sum_{i=1}^{n} \tilde{C}(Y,e_i,e_i,V), \tag{4}$$

then from (3), we get

$$P(Y,V) = S(Y,V) - \frac{r}{n}g(Y,V). \tag{5}$$

The tensor P is called the concircular Ricci tensor [4], which is a symmetric tensor of type (0,2). The present paper deals with a type of non-flat Riemannian manifold (M^n, g), $n > 2$ (the condition $n > 2$ is assumed throughout the paper), whose concircular Ricci tensor P is not identically zero and satisfies the condition

$$(\nabla_X P)(Y,Z) = [A(X) + B(X)]P(Y,Z) + A(Y)P(X,Z) + A(Z)P(Y,X), \tag{6}$$

where A, B and ∇ has the same meaning as before. Such a manifold is called *almost pseudo concircular Ricci symmetric manifold* and an n-dimensional manifold of this kind is denoted by $A(P\tilde{C}RS)_n$.

The paper is organized as follows. Section 2 is devoted to the study of some basic results of $A(P\tilde{C}RS)_n$. In this section, we investigate the nature of scalar curvature of $A(P\tilde{C}RS)_n$ and it is shown that in an $A(P\tilde{C}RS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ.

In 1970 Pokhariyal and Mishra [16] were introduced new tensor fields, called W_2 and E tensor fields, in a Riemannian manifold and studied their properties. According to them a W_2-curvature tensor on a manifold (M^n, g), $n > 2$, is defined by [16]

$$\begin{aligned} W_2(X,Y,Z,U) &= R(X,Y,Z,U) \\ &+ \frac{1}{n-1}\big[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)\big]. \end{aligned} \tag{7}$$

In this connection it may be mentioned that Pokhariyal and Mishra ([16], [17]) and Pokhariyal [12] introduced some new curvature tensors defined on the line of Weyl projective curvature tensor.

The W_2-curvature tensor was introduced on the line of Weyl projective curvature tensor and by breaking W_2 into skew-symmetric parts the tensor E has been defined. Rainich conditions for the existence of the non-null electrovariance can be obtained by W_2 and E, if we replace the matter tensor by the contracted part of these tensors. The tensor E enables to extend Pirani formulation of gravitational waves to Einstein space ([14], [15]). It is shown that [16] except the vanishing of complexion vector and property of being identical in two spaces which are in geodesic correspondence, the W_2-curvature tensor possesses the properties almost similar to the Weyl projective curvature tensor. Thus we can very well use W_2-curvature tensor in various physical and geometrical spheres in place of the Weyl projective curvature tensor.

The W_2-curvature tensor have also been studied by various authors in different structures such as De and Sarkar [5], Matsumoto, Ianus and Mihai [9], Pokhariyal ([13], [14], [15]), Shaikh, Jana and Eyasmin [19], Shaikh, Matsuyama and Jana [20], Taleshian and Hosseinzadeh [22], Tripathi and Gupta [23], Venkatesha, Bagewadi and Kumar [24], Yildiz and De [28] and many others.

Section 3 deals with the W_2-curvature tensor of an $A(P\tilde{C}RS)_n$. It is proved that in a W_2-conservative $A(P\tilde{C}RS)_n$, the vector field μ and ξ are co-directional, where ξ is defined by $B(QX) = g(QX,\mu) = D(X) = g(X,\xi)$. In section 4, we study decomposable $A(P\tilde{C}RS)_n$ and it is shown that in a decomposable $A(P\tilde{C}RS)_n$ one of the decompositions is concircular Ricci flat and the other decomposition is concircular Ricci symmetric.

Recently Özen and Altay [10] studied the totally umbilical hypersurfaces of weakly and pseudosymmetric spaces. Again Özen and Altay [11] also studied the totally umbilical hypersurfaces of weakly concircular and pseudo concircular symmetric spaces. In this connection it may be mentioned that Shaikh, Roy and Hui [21] studied the totally umbilical hypersurfaces of weakly conharmonically symmetric spaces. Section 5 deals with the study of totally umbilical hypersurfaces of $A(P\tilde{C}RS)_n$. It is proved that the totally umbilical hypersurface of an $A(P\tilde{C}RS)_n$ is also an $A(P\tilde{C}RS)_n$.

2 Some basic results of $A(P\tilde{C}RS)_n$

Let Q be the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S. Then $g(QX, Y) = S(X, Y)$ for all vector fields X, Y.
Using (5) in (6), we get

$$\begin{aligned} (\nabla_X S)(Y,Z) - \frac{dr(X)}{n} g(Y,Z) &= [A(X)+B(X)][S(Y,Z) - \frac{r}{n} g(Y,Z)] \quad (8) \\ &+ A(Y)[S(X,Z) - \frac{r}{n} g(X,Z)] \\ &+ A(Z)[S(Y,X) - \frac{r}{n} g(Y,X)]. \end{aligned}$$

Setting $Y = Z = e_i$ in (8) and then taking summation over i, $1 \leq i \leq n$, we obtain

$$A(QX) = \frac{r}{n} A(X), \tag{9}$$

i.e.

$$S(X,\rho) = \frac{r}{n} g(X,\rho). \tag{10}$$

This leads to the following:

Proposition 2.1. *In an $A(P\tilde{C}RS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ.*

Also from (6), we get

$$(\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z) = B(X)P(Y,Z) - B(Y)P(X,Z). \tag{11}$$

Contracting (11) over Y and Z and using (5), we get

$$\frac{n-2}{2n} dr(X) = B(QX) - \frac{r}{n} B(X). \tag{12}$$

If the scalar curvature r is constant, then

$$dr(X) = 0. \tag{13}$$

By virtue of (13), (12) yields

$$B(QX) = \frac{r}{n}B(X), \tag{14}$$

i.e.

$$S(X, \mu) = \frac{r}{n}g(X, \mu). \tag{15}$$

In the other way, we assume that the concircular Ricci tensor of this manifold is Codazzi type [7] then we have

$$(\nabla_X P)(Y, Z) - (\nabla_Y P)(X, Z) = 0. \tag{16}$$

Using (16) in (11), we get

$$B(X)P(Y, Z) - B(Y)P(X, Z) = 0. \tag{17}$$

Contracting (17) over Y and Z and using (5), we also obtain

$$B(QX) = \frac{r}{n}B(X). \tag{18}$$

This leads to the following:

Proposition 2.2. *In an* $A(P\tilde{C}RS)_n$*, if*
(i) *the scalar curvature is constant or*
(ii) *the concircular Ricci tensor is Codazzi type*
then $\frac{r}{n}$ *is an eigenvalue of the Ricci tensor* S *corresponding to the eigenvector* μ*.*

Definition 2.1. *A Riemannian manifold is said to admit cyclic parallel concircular Ricci tensor if*

$$(\nabla_X P)(Y, Z) + (\nabla_Y P)(Z, X) + (\nabla_Z P)(X, Y) = 0. \tag{19}$$

By virtue of (6), (19) yields

$$\lambda(X)P(Y, Z) + \lambda(Y)P(X, Z) + \lambda(Z)P(X, Y) = 0, \tag{20}$$

where $\lambda(X) = 3A(X) + B(X) = g(X, \sigma)$ for all X.
Contracting (20) over Y and Z, we get

$$\lambda(QX) = \frac{r}{n}\lambda(X), \tag{21}$$

i.e.

$$S(X, \sigma) = \frac{r}{n}g(X, \sigma). \tag{22}$$

This leads to the following:

Proposition 2.3. *In an $A(P\tilde{C}RS)_n$ with cyclic parallel concircular Ricci tensor, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector σ defined by $g(X,\sigma) = \lambda(X) = 3A(X) + B(X)$ for all X.*

In terms of local coordinates, the relation (20) can be written as

$$\lambda_i P_{jk} + \lambda_j P_{ki} + \lambda_k P_{ji} = 0. \tag{23}$$

Next, we consider a lemma, which is as follows:

Lemma 2.1. (Walker's Lemma) [25] *If a_{ij} and b_i are numbers satisfying $a_{ij} = a_{ji}$, $a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$ for i, j, $k = 1, 2, \cdots, n$, then either all a_{ij} are zero or all the b_i are zero.*

By virtue of Lemma 2.1, it follows from (23) that either $\lambda_k = 0$ or $P_{ij} = 0$. Also by definition of $A(P\tilde{C}RS)_n$, $P_{ij} \neq 0$ and hence $\lambda_k = 0$, i.e.

$$3A_k + B_k = 0. \tag{24}$$

This leads to the following:

Proposition 2.4. *In an $A(P\tilde{C}RS)_n$ with cyclic parallel concircular Ricci tensor, the 1-forms A and B are related in the form* (24).

We assume that $A(P\tilde{C}RS)_n$ is conformally flat. In a conformally flat Riemannian manifold, the following condition holds [6]

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}\big[dr(X)g(Y,Z) - dr(Y)g(X,Z)\big]. \tag{25}$$

From (5) and (25) we find

$$(\nabla_X P)(Y,Z) - (\nabla_Y P)(X,Z) = \frac{2-n}{2(n-1)}\big[dr(X)g(Y,Z) - dr(Y)g(X,Z)\big]. \tag{26}$$

If the concircular Ricci tensor is Codazzi type then from (26) we have

$$dr(X)g(Y,Z) - dr(Y)g(X,Z) = 0 \tag{27}$$

Contracting on Y and Z in (27) we obtain that the scalar curvature r is constant. Conversely, from (26), if the manifold is of constant curvature then the concircular Ricci tensor of this manifold is Codazzi type. Thus we can state the following theorem:

Theorem 2.1. *In a conformally flat $A(P\tilde{C}RS)_n$, the concircular Ricci tensor of this manifold is Codazzi type if and only if the scalar curvature of this manifold is constant.*

3 W_2-curvature tensor of an $A(P\tilde{C}RS)_n$

Let us consider a W_2-flat Riemannian manifold. Then from (7), we have

$$R(X,Y,Z,U) + \frac{1}{n-1}\big[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)\big] = 0. \qquad (28)$$

Contracting (28) over X and U, we get $P(Y,Z) = 0$ and hence the manifold is not $A(P\tilde{C}RS)_n$. Thus we can state the following:

Theorem 3.1. *There does not exist W_2-flat $A(P\tilde{C}RS)_n$.*

From (7), we obtain

$$\begin{aligned}(div\ W_2)(X,Y)Z &= (div\ R)(X,Y)Z \\ &+ \frac{1}{2(n-1)}\big[dr(Y)g(X,Z) - dr(X)g(Y,Z)\big],\end{aligned} \qquad (29)$$

where 'div' denotes the divergence.
Again it is known that in a Riemannian manifold, we have

$$(div\ R)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \qquad (30)$$

Consequently by virtue of (30), (29) takes the form

$$\begin{aligned}(div\ W_2)(X,Y)Z &= (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &+ \frac{1}{2(n-1)}\big[dr(Y)g(X,Z) - dr(X)g(Y,Z)\big].\end{aligned} \qquad (31)$$

Let us consider a W_2-conservative $A(P\tilde{C}RS)_n$. Then we have [8]

$$(div\ W_2)(X,Y)Z = 0 \qquad (32)$$

and hence (31) yields

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)]. \qquad (33)$$

By virtue of (5), (11) and (12), it follows from (33) that

$$\begin{aligned}B(X)P(Y,Z) - B(Y)P(X,Z) &= -\frac{1}{n-1}\Big[\big\{B(QX) - \frac{r}{n}B(X)\big\}g(Y,Z) \\ &- \big\{B(QY) - \frac{r}{n}B(Y)\big\}g(X,Z)\Big].\end{aligned} \qquad (34)$$

Putting $Z=\mu$ in (34) we obtain

$$B(X)B(QY)-B(Y)B(QX)=0. \tag{35}$$

Let $B(QX)=g(QX,\mu)=D(X)=g(X,\xi)$ for all X. Then from (35), we get

$$B(X)D(Y)=B(Y)D(X), \tag{36}$$

which implies that the vector field μ and ξ are co-directional. This leads to the following:

Theorem 3.2. *In a W_2-conservative $A(P\tilde{C}RS)_n$, the vector field μ and ξ are co-directional.*

From (10), we have

$$P(X,\rho)=0. \tag{37}$$

Setting $Z=\rho$ in (34) and using (37), we get

$$A(Y)D(X)-A(X)D(Y)=\frac{r}{n}\big\{A(Y)B(X)-A(X)B(Y)\big\}. \tag{38}$$

Let the scalar curvature $r\neq 0$. Then (38) implies that the vector fields ρ and ξ are co-directional if and only if the vector fields ρ and μ are co-directional.
Thus we can state the following:

Theorem 3.3. *In a W_2-conservative $A(P\tilde{C}RS)_n$ with non-zero scalar curvature, the vector fields ρ and ξ are co-directional if and only if the vector fields ρ and μ are co-directional.*

4 Decomposable $A(P\tilde{C}RS)_n$

A Riemannian manifold (M^n,g) is said to be decomposable manifold [27] if it can be expressed as $M_1^p\times M_2^{n-p}$ for $2\leq p\leq n-2$, that is, in some coordinate neighbourhood of the Riemannian manifold (M^n,g), the metric can be expressed as

$$ds^2=g_{ij}dx^idx^j=\bar{g}_{ab}dx^adx^b+\overset{*}{g}_{\alpha\beta}\,dx^\alpha dx^\beta, \tag{39}$$

where $\bar{g}_{ab}$ are functions of $x^1,x^2,\cdots,x^p(p<n)$ denoted by $\bar{x}$ and $\overset{*}{g}_{\alpha\beta}$ are functions of $x^{p+1},x^{p+2},\cdots,x^n$ denoted by $\overset{*}{x}$; $a,b,c,\cdots$ run from 1 to p and $\alpha,\beta,\gamma,\cdots$ run from $p+1$ to n. The two parts of (39) are the metrics of $M_1^p(p\geq 2)$ and $M_2^{n-p}(n-p\geq 2)$ which are called the decompositions of the decomposable manifold $M^n=M_1^p\times M_2^{n-p}(2\leq p\leq n-2)$.

Let (M^n, g) be a decomposable Riemannian manifold such that $M^n = M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-2$. Here throughout this section each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by a 'star' is assumed to be from M_2.

Let $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{U}$, $\bar{V} \in \chi(M_1)$ and $\overset{*}{X}$, $\overset{*}{Y}$, $\overset{*}{Z}$, $\overset{*}{U}$, $\overset{*}{V} \in \chi(M_2)$, $\chi(M_i)$ being the Lie algebra of smooth vector fields on M_i, $i = 1, 2$. Let R, $\bar{R}$ and $\overset{*}{R}$ (resp. S, $\bar{S}$ and $\overset{*}{S}$) be the curvature tensor (resp. Ricci tensor) of the manifold M, M_1 and M_2 respectively. Also let P, $\bar{P}$ and $\overset{*}{P}$ be the concircular Ricci tensor of M, M_1 and M_2 respectively. Then we have the following relations [27]:

$$\begin{aligned}
&R(\overset{*}{X}, \bar{Y}, \bar{Z}, \bar{U}) = 0 = R(\bar{X}, \overset{*}{Y}, \bar{Z}, \overset{*}{U}) = R(\bar{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),\\
&(\nabla_{\overset{*}{X}} R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_{\bar{X}} R)(\bar{Y}, \overset{*}{Z}, \bar{U}, \overset{*}{V}) = (\nabla_{\overset{*}{X}} R)(\bar{Y}, \overset{*}{Z}, \bar{U}, \overset{*}{V}),\\
&R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = \bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}); \quad R(\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}) = \overset{*}{R}(\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),\\
&S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}); \quad S(\overset{*}{X}, \overset{*}{Y}) = \overset{*}{S}(\overset{*}{X}, \overset{*}{Y}),\\
&(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}); \quad (\nabla_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}) = (\overset{*}{\nabla}_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}),\\
&P(\bar{X}, \bar{Y}) = \bar{P}(\bar{X}, \bar{Y}); \quad P(\overset{*}{X}, \overset{*}{Y}) = \overset{*}{P}(\overset{*}{X}, \overset{*}{Y}),\\
&(\nabla_{\bar{X}} P)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} P)(\bar{Y}, \bar{Z}); \quad (\nabla_{\overset{*}{X}} P)(\overset{*}{Y}, \overset{*}{Z}) = (\overset{*}{\nabla}_{\overset{*}{X}} P)(\overset{*}{Y}, \overset{*}{Z}),\\
&r = \bar{r} + \overset{*}{r},
\end{aligned}$$

where r, $\bar{r}$, and $\overset{*}{r}$ are the scalar curvature of M, M_1, M_2 respectively.

Let us consider a Riemannian manifold (M^n, g) which is decomposable $A(P\tilde{C}RS)_n$. Then $M^n = M_1^p \times M_2^{n-p}$, $(2 \leq p \leq n-2)$.
Now from (6), we find

$$(\nabla_{\bar{X}} P)(\bar{Y}, \bar{Z}) = [A(\bar{X}) + B(\bar{X})]P(\bar{Y}, \bar{Z}) + A(\bar{Y})P(\bar{X}, \bar{Z}) + A(\bar{Z})P(\bar{Y}, \bar{X}), \quad (40)$$

$$(\nabla_{\overset{*}{X}} P)(\overset{*}{Y}, \overset{*}{Z}) = [A(\overset{*}{X}) + B(\overset{*}{X})]P(\overset{*}{Y}, \overset{*}{Z}) + A(\overset{*}{Y})P(\overset{*}{X}, \overset{*}{Z}) + A(\overset{*}{Z})P(\overset{*}{Y}, \overset{*}{X}), \quad (41)$$

$$[A(\overset{*}{X}) + B(\overset{*}{X})]P(\bar{Y}, \bar{Z}) = 0, \quad (42)$$

$$A(\overset{*}{Z})P(\bar{Y}, \bar{X}) = 0. \quad (43)$$

From (42) and (43), it follows that either M_1 is concircular Ricci flat, i.e., Einstein or $A = 0$, $B = 0$ on M_2 and hence from (41), we have $(\nabla_{\overset{*}{X}} P)(\overset{*}{Y}, \overset{*}{Z}) = 0$, i.e., M_2 is concircular Ricci symmetric. Similarly it can be easily shown that either M_2 is Einstein or M_1 is concircular Ricci symmetric. Thus we can state the following:

Theorem 4.1. *Let (M^n, g) be a Riemannian manifold such that $M^n = M_1^p \times M_2^{n-p}$, $(2 \leq p \leq n-2)$. If (M^n, g) is a $A(P\tilde{C}RS)_n$, then either*
(1) M_1 (resp. M_2) is Einstein or
(2) M_2 (resp. M_1) is concircular Ricci symmetric.

5 Totally umbilical hypersurfaces of $A(P\tilde{C}RS)_n$

Let $(\bar{V}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{U, y^\alpha\}$. Let (V, g) be a hypersurface of $(\bar{V}, \bar{g})$ defined in a locally coordinate system by means of a system of parametric equation $y^\alpha = y^\alpha(x^i)$, where Greek indices take values $1, 2, \cdots, n$ and Latin indices take values $1, 2, \cdots, (n+1)$. Let N^α be the components of a local unit normal to (V, g). Then we have

$$g_{ij} = \bar{g}_{\alpha\beta} y_i^\alpha y_j^\beta, \tag{44}$$

$$\bar{g}_{\alpha\beta} N^\alpha y_j^\beta = 0, \quad \bar{g}_{\alpha\beta} N^\alpha N^\beta = e = 1, \tag{45}$$

$$y_i^\alpha y_j^\beta g^{ij} = \bar{g}^{\alpha\beta} - N^\alpha N^\beta, \quad y_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}. \tag{46}$$

The hypersurface (V, g) is called a totally umbilical hypersurface ([3],[6]) of $(\bar{V}, \bar{g})$ if its second fundamental form Ω_{ij} satisfies

$$\Omega_{ij} = H g_{ij}, \quad y_{i,j}^\alpha = g_{ij} H N^\alpha, \tag{47}$$

where the scalar function H is called the mean curvature of (V, g) given by $H = \frac{1}{n} \sum g^{ij} \Omega_{ij}$. If, in particular, $H = 0$, i.e.,

$$\Omega_{ij} = 0, \tag{48}$$

then the totally umbilical hypersurface is called a totally geodesic hypersurface of $(\bar{V}, \bar{g})$.
The equation of Weingarten for (V, g) can be written as $N_{,j}^\alpha = -\frac{H}{n} y_j^\alpha$. The structure equations of Gauss and Codazzi ([3],[6]) for (V, g) and $(\bar{V}, \bar{g})$ are respectively given by

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\delta} B_{ijkl}^{\alpha\beta\gamma\delta} + H^2 G_{ijkl}, \tag{49}$$

$$\bar{R}_{\alpha\beta\gamma\delta} B_{ijk}^{\alpha\beta\gamma} N^\delta = H_{,i}\, g_{jk} - H_{,j}\, g_{ik}, \tag{50}$$

where R_{ijkl} and $\bar{R}_{\alpha\beta\gamma\delta}$ are curvature tensors of (V, g) and $(\bar{V}, \bar{g})$ respectively, and

$$B^{\alpha\beta\gamma\delta}_{ijkl} = B^{\alpha}_{i} B^{\beta}_{j} B^{\gamma}_{k} B^{\delta}_{l}, \quad B^{\alpha}_{i} = y^{\alpha}_{i}, \quad G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}. \tag{51}$$

Also we have ([3],[6])

$$\bar{S}_{\alpha\delta} B^{\alpha}_{i} B^{\delta}_{j} = S_{ij} - (n-1)H^2 g_{ij}, \tag{52}$$

$$\bar{S}_{\alpha\delta} N^{\alpha} B^{\delta}_{i} = (n-1)H_{,i} \ , \tag{53}$$

$$\bar{r} = r - n(n-1)H^2, \tag{54}$$

where S_{ij} and $\bar{S_{\alpha\delta}}$ are the Ricci tensors of (V, g) and $(\bar{V}, \bar{g})$ respectively and r and $\bar{r}$ are the scalar curvatures of (V, g) and $(\bar{V}, \bar{g})$ respectively.
By virtue of (52) and (54), we have

$$\bar{P}_{\alpha\delta} B^{\alpha}_{i} B^{\delta}_{j} = P_{ij}, \tag{55}$$

where P_{ij} and $\bar{P_{\alpha\delta}}$ are the concircular Ricci tensors of (V, g) and $(\bar{V}, \bar{g})$ respectively.
In terms of local coordinates the relation (6) can be written as

$$P_{ij,k} = (A_k + B_k)P_{ij} + A_i P_{jk} + A_j P_{ki}. \tag{56}$$

Let $(\bar{V}, \bar{g})$ be an $A(P\tilde{C}RS)_n$. Then we get

$$\bar{P}_{\alpha\beta,\gamma} = (A_\gamma + B_\gamma)\bar{P}_{\alpha\beta} + A_\alpha \bar{P}_{\gamma\beta} + A_\beta \bar{P}_{\alpha\gamma}, \tag{57}$$

where A and B are nowhere vanishing 1-forms.
Multiplying both sides of (57) by $B^{\alpha\beta\gamma}_{ijk}$ and then using (55), we obtain the relation (56). Hence we can state the following:

Theorem 5.1. *The totally umbilical hypersurface of an $A(P\tilde{C}RS)_n$ is also an $A(P\tilde{C}RS)_n$.*

Corollary 5.1. *The totally geodesic hypersurface of an $A(P\tilde{C}RS)_n$ is also an $A(P\tilde{C}RS)_n$.*

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