Composition Operators on the Dirichlet space and related problems

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Abstract

In this paper we investigate the following question: when a bounded analytic function φ on the unit disk $\mathbb{D}$, fixing 0, is such that the family $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in the Dirichlet space $\mathcal{D}$?. We also consider the problem of characterizing the univalent, full self-maps φ of $\mathbb{D}$ in terms of the norm of the induced composition operator $C_\varphi : \mathcal{D} \to \mathcal{D}$. The first problem is analogous to a celebrated question asked by W. Rudin on the Hardy space setting that was answered recently ([3] and [14]). The second problem resembles a problem investigated by J. Shapiro in [13] about characterization of inner functions θ in the terms of $\|C_\theta\|_{H^2}$.

1 Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane. By a *self- map* of $\mathbb{D}$ we mean an analytic map $\varphi : \mathbb{D} \to \mathbb{C}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. The *composition operator* induced by φ is the linear transformation C_φ defined as $C_\varphi(f) := f \circ \varphi$ in the space of all analytic functions on $\mathbb{D}$.

The composition operators have been studied in many settings, and in particular in functional Banach spaces (cf. the books [4], [12], the survey of recent developments [7], and the references therein). The goal of this theory is to obtain characterizations of operator-theoretic properties of C_φ by function-theoretic properties of the symbol φ. Conversely, operator-theoretic properties of C_φ could suggest, or help to understand certain phenomena about function-theoretic properties of φ.

Recall that an analytic functional Banach space is a Banach space whose elements are analytic functions defined on a domain of $\mathbb{C}$ (or $\mathbb{C}^n$) such that the evaluation functionals are continuous.

Particular instances of functional Banach spaces are the Hardy space H^2, and the Bergman space A^2 of the unit disk. In these spaces, as a consequence of Littlewood's Subordination Principle [4], every self-map of $\mathbb{D}$ induces a bounded composition operator. Recently, there have been several articles that deal with the study of composition operators on the Dirichlet space: recall that if $dA(z) =$

$\frac{1}{\pi} dx\, dy = \frac{1}{\pi} r\, dr\, d\theta$, $(z = x + iy = re^{i\theta})$ denotes the normalized area Lebesgue measure on $\mathbb{D}$, the *Dirichlet space* $\mathcal{D}$ is the Hilbert space of analytic functions in $\mathbb{D}$ with a square integrable derivative and with norm given by

$$\|f\|_{\mathcal{D}} = \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) \right)^{1/2}.$$

It is well known that $\mathcal{D}$ is a functional Hilbert space, and that for each $w \in \mathbb{D}$ the function

$$K_w(z) = 1 + \log \frac{1}{1 - \overline{w}z},$$

is the *reproducing kernel* at w in the Dirichlet space, that is, for $f \in \mathcal{D}$ we have $f(w) = \langle f, K_w \rangle_{\mathcal{D}}$. It is easy to see that $\|K_w\|_{\mathcal{D}}^2 = \log \frac{1}{1-|w|^2}$.

A self-map of $\mathbb{D}$ does not induces, necessarily, a bounded composition operator on $\mathcal{D}$. An obvious necessary condition would be that $\varphi \in \mathcal{D}$ which, of course, is not always the case. Actually this condition is not sufficient. A necessary and sufficient condition for φ to induce a bounded composition operator on $\mathcal{D}$ is given in terms of *counting functions* and *Carleson measures* (see [8]).

The counting function $n_\varphi(w)$, $w \in \mathbb{D}$, associated to φ is defined as the cardinality of the set $\{z \in \mathbb{D} : \varphi(z) = w\}$ when the latter is finite and as $+\infty$ otherwise, with the usual rules of arithmetics for $\mathbb{R} \cup \{\pm\infty\}$.

We will make use of a change of variable formula for non-univalent functions: Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is a non-constant analytic function with counting function $n_\varphi(w)$. If $f : \mathbb{D} \to [0, \infty)$ is any Borel function, then

$$\int_{\mathbb{D}} f(\varphi(z)) |\varphi'(z)|^2 \, dA(z) = \int_{\mathbb{D}} f(w) n_\varphi(w) \, dA(w).$$

In particular one obtains that $\int_{\mathbb{D}} |\varphi'(z)|^2 \, dA(z) = \int_{\mathbb{D}} n_\varphi(w) \, dA(w)$. So, φ is in the Dirichlet space if and only if its counting function is an L^1 function.

In two recent papers, [9] and [10], M. Martín and D. Vukotić, studied composition operators on the Dirichlet space. In this article, based on the results in those works, we consider related questions. In Section 1, we investigate the analogous on the Dirichlet space to a problem proposed by W. Rudin in the context of Hardy spaces: When a bounded analytic function φ on the unit disk $\mathbb{D}$ fixing 0 is such that $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in $\mathcal{D}$? In Section 2 we consider the problem of characterizing the univalent, full self-maps φ of $\mathbb{D}$ in terms of the norm of $\|C_\varphi\|_{\mathcal{D}}$. This problem, is analogous to a question asked and answered by J. Shapiro in [13] about inner functions in the Hardy space setting.

We will denote as $\mathcal{D}_0$ the subspace of $\mathcal{D}$ of those function that vanish at 0, and we will use the notation $\|C_\varphi : \mathcal{H} \to \mathcal{H}\|$ to denote the norm of the induced composition operator on a Hilbert space $\mathcal{H}$.

2 Orthogonal functions in the Dirichlet space.

The problem of describing the isometric composition operators acting on Hilbert spaces of analytic functions has been studied in several contexts. Namely, it was proved by Nordgren in [11] that the composition operator C_φ induced on H^2 by an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$ is an isometry on H^2 if and only if $\varphi(0) = 0$ and φ is an inner function (see also [4, p. 321]. In the Bergman space A^2 it is a straightforward consequence of Schwarz Lemma that φ induces an isometric composition operator if and only if φ is a rotation.

Recently, M. Martin and D. Vukotić showed in [10] that in $\mathcal{D}$, the isometric composition operators are those induced by univalent full self-maps of the disk that fix the origin (a self-map of $\mathbb{D}$ is said to be a *full map* if $A[\mathbb{D} \setminus \varphi(\mathbb{D})] = 0$).

W. Rudin in 1988 (MSRI conference) proposed the following problem: If φ is a bounded analytic on the unit disk $\mathbb{D}$ such that $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in H^2, does φ must be a constant multiple of an inner function? C. Sundberg [14] and C. Bishop [3] solved independently the problem. In fact, they showed that there exists a function φ which is not an inner function and the family $\{\varphi^n\}$ is orthogonal in H^2.

As asserted by M. Martín y D. Vukotić in [10], their characterization of the isometric composition operators acting on $\mathcal{D}$ can be interpreted as follows: the univalent full maps of the disk that fix the origin are the Dirichlet space counterpart of the inner functions that fix the origin for the composition operators on H^2. Now, we propose the following question: When a function $\varphi \in H^\infty(\mathbb{D}) \cap \mathcal{D}_0$ is such that the family $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in $\mathcal{D}$? (since $\mathcal{D} \cap H^\infty(\mathbb{D})$ is an algebra, $\{\varphi^n\}$ is in $\mathcal{D}$ for all n).

We will answer this question when n_φ is essentially bounded, that is, when there is a constant C so that $n_\varphi(w) \leq C$ for all w except those in a set of measure zero. Actually, we shall need the following possibly weaker hypothesis: that the functions $\int_0^{2\pi} e^{ik\theta} n_\varphi(re^{i\theta})d\theta$ belong to $L^2[0, 1]$ for every positive entire k.

Our result is analogous to a characterization given by P. Bourdon in [2] in the context of H^2: the functions that satisfy the hypotheses of the Rudin's problem are characterized as those maps φ such that their Nevanlinna counting function N_φ is essentially radial. Our assumption that n_φ is essentially bounded is clearly stronger that assuming that $\varphi \in \mathcal{D}$ and it possibly can be weakened. The proof relies on the techniques of the proof given in [2].

Theorem 2.1. *Let φ be a self-map on $\mathbb{D}$ fixing 0 such that n_φ is essentially bounded. The family $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in $\mathcal{D}$ if and only if there is a function $g : [0, 1) \to [0, \infty)$ such that for almost every $r \in [0, 1)$, $n_\varphi(re^{i\theta}) = g(r)$ for almost every $\theta \in [0, 2\pi]$ (this is, n_φ is essentially radial).*

Proof. Suppose that n_φ is essentially radial. Let n, m be nonnegative integers such that $n > m$. Then we have

$$\begin{aligned}
\langle \varphi^n, \varphi^m \rangle_{\mathcal{D}} &= nm \int_{\mathbb{D}} \varphi(z)^{n-1} \overline{\varphi(z)^{m-1}} |\varphi'(z)|^2 \, dA(z) \\
&= nm \int_{\mathbb{D}} w^{n-1} \overline{w^{m-1}} n_\varphi(w) \, dA(w) \\
&= nm \int_0^1 r^{n+m-1} \left[\frac{1}{\pi} \int_0^{2\pi} e^{i(n-m)\theta} n_\varphi(re^{i\theta}) \, d\theta \right] dr \\
&= nm \int_0^1 r^{n+m-1} g(r) \left[\frac{1}{\pi} \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta \right] dr \\
&= 0.
\end{aligned}$$

Conversely, if the family $\{\varphi^n : n = 0, 1, 2 \dots\}$ is orthogonal in $\mathcal{D}$ and k is any positive integer, then for each integer $n > k$, we have

$$\begin{aligned}
0 = \langle \varphi^n, \varphi^{n-k} \rangle_{\mathcal{D}} &= n(n-k) \int_{\mathbb{D}} \varphi(z)^{n-1} \overline{\varphi(z)^{n-k-1}} |\varphi'(z)|^2 \, dA(z) \\
&= n(n-k) \int_{\mathbb{D}} w^{n-1} \overline{w^{n-k-1}} n_\varphi(w) \, dA(w) \\
&= n(n-k) \int_0^1 r^{2n-k-1} \left[\frac{1}{\pi} \int_0^{2\pi} e^{ik\theta} n_\varphi(re^{i\theta}) \, d\theta \right] dr.
\end{aligned}$$

Since n_φ is essentially bounded, the functions $f_k(r) := \int_0^{2\pi} e^{ik\theta} n_\varphi(re^{i\theta}) \, d\theta$ are in $L^2[0,1]$, and the preceding chain of equalities shows that they are orthogonal in $L^2[0,1]$ to the maps $\left\{ r \mapsto r^{2n-k-1} : n > k \right\}$. The linear span of this latter set is dense in $L^2[0,1]$ (cf. [2]), and so $f_k(r) = 0$ for almost every $r \in [0,1]$. Taking complex conjugates, we see that $\int_0^{2\pi} e^{ij\theta} n_\varphi(re^{i\theta}) \, d\theta = 0$ for all $j \neq 0$, and almost every $r \in [0,1]$. Thus that $\theta \mapsto n_\varphi(re^{i\theta})$ is essentially constant for almost every r. □

Proposition 2.2. *Suppose that φ is a self-map with counting function essentially bounded, and essentially radial. Then φ is a constant multiple of a full self-map of $\mathbb{D}$.*

Proof. Suppose that φ is not constant. If the range of φ contains a point in the circle $S_r = \left\{ re^{i\theta} : \theta \in [0, 2\pi] \right\}$, $\varphi(\mathbb{D})$ contains an arc because this is an open subset of $\mathbb{D}$. In this arc $n_\varphi \geq 1$, and so the range of φ may omit only a θ-zero-measure subset of S_r because n_φ is essentially constant on S_r.

Thus the range of φ contain almost every point in the disk $\{z : |z| < \|\varphi\|_\infty\}$. □

3 Composition Operators vs. Full Maps

In the Hardy space, J. Shapiro [13] characterized, in terms of their norms, those composition operators C_φ whose symbol is an *inner function.* In fact, J. Shapiro showed:

1. If $\varphi(0) = 0$ then φ is inner if and only if $\|C_\varphi : H_0^2 \to H_0^2\| = 1$, where H_0^2 is the subspace of functions in H^2 vanishing at 0, and

2. If $\varphi(0) \neq 0$ then φ is inner if and only if $\|C_\varphi : H^2 \to H^2\| = \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}$.

We are going to investigate the analogous questions on the Dirichlet space.

In [9] M. Martín and D. Vukotić calculated the norm of a composition operator $C_\varphi : \mathcal{D} \to \mathcal{D}$ induced by a univalent full map φ of $\mathbb{D}$. They obtain

$$\|C_\varphi : \mathcal{D} \to \mathcal{D}\| = \sqrt{\frac{L + 2 + \sqrt{L(4+L)}}{2}}, \tag{3.1}$$

where $L = \log \frac{1}{1-|\varphi(0)|^2}$, and show that it is an upper bound for the norms of composition operators, acting on the Dirichlet space, induced by univalent symbols.

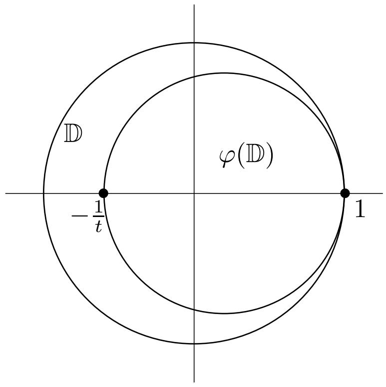
The fact that univalent full maps of the disk that fix 0 are the Dirichlet space counterpart of inner functions that fix the origin (for composition operators on H^2) [10], lead us to investigate if the equality in the equation (3.1) characterizes the univalent full maps of the disk among the univalent self-maps of $\mathbb{D}$.

In addition, the main result in [10] says that $\varphi(0) = 0$ and φ is a univalent full self-map of the disk if and only if C_φ is an isometry on $\mathcal{D}$, and hence on $\mathcal{D}_0$, so in particular its restriction to $\mathcal{D}_0$ has norm 1. Is the converse true?

It is easy to see that this is not true. In fact, let φ_t, $t \geq 1$, be the linear fractional transformation given by

$$\varphi_t(z) = \frac{2z}{(1-t)z + (1-z)}, \quad z \in \mathbb{D}.$$

One easily sees that $\varphi_t(\mathbb{D}) \subset \mathbb{D}$, $\varphi_t(0) = 0$, $\varphi_t(1) = 1$, and $\varphi_t(-1) = -1/t$ (see figure). If $t > 1$ clearly φ_t is not a full map, but a calculation in [1, Cor. 6.1] shows that $\|C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0\| = 1$ when φ is a linear fractional self-map of $\mathbb{D}$ with a boundary fixed point.



Nevertheless, we have the following results, analogous to the results in [13].

Theorem 3.1. *Suppose that φ is a univalent, holomorphic self-map of $\mathbb{D}$, with n_φ essentially radial and $\varphi(0) = 0$. Then φ is a full map if and only if*

$$\|C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0\| = 1.$$

Proof. One direction follows from Proposition 2.2. For the converse, suppose that φ is a univalent holomorphic self-map of $\mathbb{D}$, with n_φ essentially radial, $\varphi(0) = 0$, and that φ is not a full map.

We are going to show that the restriction of C_φ to $\mathcal{D}_0$ has norm < 1. We have that $\varphi(\mathbb{D})$ is contained in the disk $D(0,\rho) = \{z : |z| < \|\varphi\|_\infty = \rho\}$ and $A[D(0,\rho) \setminus \varphi(\mathbb{D})] = 0$ with $0 < \rho < 1$ (cf. proof of Proposition 2.2.)

Let $f \in \mathcal{D}_0$ and define

$$g(r) := \frac{1}{\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta.$$

Since $|f'|^2$ is subharmonic in $\mathbb{D}$ then g is monotone increasing for $0 \le r < 1$. The change of variable formula gives

$$\begin{aligned} \|C_\varphi f\|_{\mathcal{D}}^2 &= \int_{\mathbb{D}} |f'(\varphi(z))|^2 \, |\varphi'(z)|^2 \, dA(z) \\ &= \int_{\varphi(\mathbb{D})} |f'(w)|^2 \, dA(w) \\ &= \int_0^\rho g(r) \, r \, dr. \end{aligned}$$

and thus:

$$\begin{aligned}
\|f\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(w)|^2\, dA(w) &= \int_0^\rho g(r)\, r\, dr + \int_\rho^1 g(r)\, r\, dr \\
&\geq \int_0^\rho g(r)\, r\, dr + \frac{1-\rho^2}{2} g(\rho) \\
&= \int_0^\rho g(r)\, r\, dr + \frac{(1-\rho^2)/2}{\rho^2/2}(\rho^2/2) g(\rho) \\
&\geq \int_0^\rho g(r)\, r\, dr + \frac{(1-\rho^2)/2}{\rho^2/2} \int_0^\rho g(r)\, r\, dr \\
&= \left(1 + \frac{(1-\rho^2)/2}{\rho^2/2}\right) \int_0^\rho g(r)\, r\, dr \\
&= \left(1 + \frac{(1-\rho^2)/2}{\rho^2/2}\right) \|C_\varphi f\|_{\mathcal{D}}^2,
\end{aligned}$$

for each $f \in \mathcal{D}_0$. It yields the desired result: the restriction of C_φ to $\mathcal{D}_0$ has norm $\leq \nu = \left(1 + \frac{(1-\rho^2)/2}{\rho^2/2}\right)^{-1/2} < 1$. □

In the next theorem, we consider the case $\varphi(0) \neq 0$. The proof follows nearly the one in [13, Th. 5.2]).

Theorem 3.2. *Suppose that φ is a univalent, holomorphic self-map of $\mathbb{D}$ with n_φ essentially radial and $\varphi(0) \neq 0$. Then φ is a full map if and only if*

$$\|C_\varphi : \mathcal{D} \to \mathcal{D}\| = \sqrt{\frac{L + 2 + \sqrt{L(4+L)}}{2}},$$

where $L = -\log(1 - |\varphi(0)|^2)$.

Proof. The necessity follows as in [9, Th. 1]. For the converse, suppose that φ is a univalent, holomorphic self-map of $\mathbb{D}$ with n_φ essentially radial, such that $\varphi(0) = p \neq 0$, and that φ is not a full map. We want to show that the norm of C_φ is strictly less that $\sqrt{\frac{L+2+\sqrt{L(4+L)}}{2}}$, where $L = -\log(1 - |p|^2)$.

For this we consider α_p, the standard automorphism of $\mathbb{D}$ that interchanges p with the origin; i.e.,

$$\alpha_p := \frac{p - z}{1 - \overline{p}z}, \quad z \in \mathbb{D}.$$

Put $\varphi_p := \alpha_p \circ \varphi$. Then $\varphi_p(0) = 0$. Since this function is a univalent, self map of $\mathbb{D}$ with counting function essentially radial, but not a full map, Theorem 3.1 affirms that the restriction of the operator C_{φ_p} to $\mathcal{D}_0$ has norm $\nu < 1$.

Because α_p is self-inverse, $\varphi = \alpha_p \circ \varphi_p$, and so, for each $f \in \mathcal{D}$:

$$C_\varphi f = C_{\varphi_p}(f \circ \alpha_p) = C_{\varphi_p} g + f(p),$$

where $g = f \circ \alpha_p - f(p)$.

The function $C_{\varphi_p} g$ belong to $\mathcal{D}_0$ and thus:

$$\begin{aligned} \|C_\varphi f\|_{\mathcal{D}} =& \|C_{\varphi_p} g\|_{\mathcal{D}}^2 + |f(p)|^2 \\ &\leq \nu^2 \|g\|_{\mathcal{D}}^2 + |f(p)|^2 \\ &= \nu^2 \|(C_{\alpha_p} f) - f(p)\|_{\mathcal{D}}^2 + |f(p)|^2. \end{aligned} \tag{3.2}$$

Since $\langle h, 1\rangle_{\mathcal{D}} = h(0)$ for each $h \in \mathcal{D}$,

$$\langle C_{\alpha_p} f, f(p)\rangle_{\mathcal{D}} = \overline{f(p)} C_{\alpha_p} f(0) = |f(p)|^2,$$

and we obtain

$$\begin{aligned} \|(C_{\alpha_p} f) - f(p)\|_{\mathcal{D}}^2 &= \|C_{\alpha_p} f\|_{\mathcal{D}}^2 - 2\Re\langle C_{\alpha_p} f, f(p)\rangle_{\mathcal{D}} + |f(p)|^2 \\ &= \|C_{\alpha_p} f\|_{\mathcal{D}}^2 - 2|f(p)|^2 + |f(p)|^2 \\ &= \|C_{\alpha_p} f\|_{\mathcal{D}}^2 - |f(p)|^2. \end{aligned}$$

This identity and Equation (3.2) yield,

$$\|C_\alpha f\|_{\mathcal{D}}^2 \leq \nu^2 \|C_{\alpha_p} f\|_{\mathcal{D}}^2 + (1 - \nu^2)|f(p)|^2.$$

We know from [10, Th. 1] that $\|C_{\alpha_p} : \mathcal{D} \to \mathcal{D}\| = (L + 2 + \sqrt{L(4+L)})/2$, and we have the following estimate for $|f(p)|$:

$$|f(p)| \leq \|f\|_{\mathcal{D}} \|K_p\|_{\mathcal{D}} = \sqrt{1+L}\|f\|_{\mathcal{D}}.$$

Then

$$\|C_\alpha f\|_{\mathcal{D}}^2 \leq \left[\nu^2 \left(\frac{L + 2 + \sqrt{L(4+L)}}{2}\right) + (1 - \nu^2)(1 + L)\right] \|f\|_{\mathcal{D}}^2,$$

and $\delta = \left[\nu^2 \left(\frac{L+2+\sqrt{L(4+L)}}{2}\right) + (1 - \nu^2)(1 + L)\right] < 1$ because $p \neq 0$ and $L > 0$. □

3.1 The essential norm

Recall that the essential norm of an operator T in a Hilbert space $\mathcal{H}$ is defined as $\|T\|_e := \inf\{\|T - K\| : K \text{ is compact }\}$; that is, the essential norm of T is the norm of the equivalence class of T in the Calkin algebra. It is well known

[4] that in any Hilbert space of analytic functions containing every power of $f(z) \equiv z$, we have

$$\|C_\varphi\|_e = \lim_n \|C_\varphi R_n\|, \tag{3.3}$$

where R_n denotes the orthogonal projection of $\mathcal{H}$ onto $z^n\mathcal{H}$.

In [13], it is proved that a self-map $\varphi : \mathbb{D} \to \mathbb{D}$ is inner if and only if the essential norm of C_φ in the Hardy space is equal to $\sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}$. Because of the analogies discussed here between inner functions and univalent full-maps, one might ask: Are full-maps characterized by the fact that the essential norm of C_φ in the Dirichlet space is equal to $\sqrt{\frac{L+2+\sqrt{L(4+L)}}{2}}$? where $L = \log \frac{1}{1-|\varphi(0)|^2}$. The answer is not, in fact *every univalent full-map has essential norm equal to 1 in the Dirichlet space*:

Theorem 3.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ a univalent full-map, then $\|C_\varphi\|_e = 1$ in the Dirichlet space.*

Proof. Suppose first that $\varphi(0) = 0$, then ([10]) C_φ is an isometry and equation 3.3 gives:

$$\|C_\varphi\|_e = \lim_n \{ \sup_{\|f\|=1} \|C_\varphi R_n f\|\} = \lim_n \|R_n\| = 1.$$

If $\varphi(0) = p \neq 0$, then the function $\varphi_p := \alpha_p \circ \varphi$ is a univalent full-map fixing the origin and then for every function $f \in \mathcal{D}$ with $\|f\| = 1$ we have that $\|C_\varphi R_n f\| = \|C_{\alpha_p} R_n f\|$. Thus, $\|C_\varphi\|_e = \|C_{\alpha_p}\|_e$.

But in [5, Cor. 5.9], it is proved that the essential norm of any composition operator induced by an automorphism of $\mathbb{D}$ is equal to 1 and the result follows. $\square$

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