Boundary value problems in complex analysis I

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Abstract

A systematic investigation of basic boundary value problems for complex partial differential equations of arbitrary order is started in these lectures restricted to model equations. In the first part [3] the Schwarz, the Dirichlet, and the Neumann problems are treated for the inhomogeneous Cauchy-Riemann equation. The fundamental tools are the Gauss theorem and the Cauchy-Pompeiu representation. The principle of iterating these representation formulas is introduced which will enable treating higher order equations. Those are studied in a second part of these lectures.

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1 Introduction

Complex analysis is one of the most influencial areas in mathematics. It has consequences in many branches like algebra, algebraic geometry, geometry, number theory, potential theory, differential equations, dynamical systems, integral equations, integral transformations, harmonic analysis, global analysis, operator theory and many others. It also has a lot of applications e.g. in physics. Classical ones are elasticity theory, fluid dynamics, shell theory, underwater acoustics, quantum mechanics etc.
In particular the theory of boundary value problems for analytic functions as the Riemann problem of linear conjugacy and the general Riemann-Hilbert problem has had a lot of influence and even has initiated the theory of singular integral equations and index theory.

Complex analysis is one of the main subjects in university curricula in mathematics. It is in fact a simply accessible theory with more relations to other subjects in mathematics than other topics. In complex analysis all structural concepts in mathematics are stressed. Algebraic, analytic and topological concepts occur and even geometry is involved. Also questions of ordering sets may be discussed in connection with complex analysis. Gauss, Cauchy, Weierstraß and Riemann were the main initiators of complex analysis and there was more

than a century of rapid development. Nowadays complex analysis is not anymore in the center of mathematicsl research. But there are still activities in this area and problems not yet solved. One of these subjects, complex methods for partial differential equations, will be presented in these lectures.
Almost everything in this course is elementary in the sense that the results are just consequences of the main theorem of culculus in the case of several variables, i.e. of the Gauss divergence theorem. Some nonelementary results will be used as properties of some singular integral operators. They will be just quoted and somebody interested in the background has to consult references given. Everything else is just combinatorics. Hierarchies of differential equations, of integral representation formulas, of kernel functions, of Green and Neumann functions arise by iterating processes leading from lower to higher order subjects. In this sense everything is evident. As Kronecker ones has expressed it, mathematics is the science where everything is evident. The beautiness of mathematics is partly reflected by esthetic formulas. All this will be seen below.

2 The complex Gauss theorems

In complex analysis it is convenient to use the complex partial differential operators ∂_z and $\partial_{\overline{z}}$ defined by the real partial differential operators ∂_x and ∂_y as

$$2\partial_z = \partial_x - i\partial_y \ , \ 2\partial_{\overline{z}} = \partial_x + i\partial_y \ . \tag{1}$$

Formally they are deducible by treating

$$z = x + iy \ , \ \overline{z} = x - iy \ , \ x, y \in \mathbb{R} \ ,$$

as independent variables using the chain rule of differentiation.
A complex-valued function $w = u + iv$ given by two real-valued functions u and v of the real variables x and y will be denoted by $w(z)$ although being rather a function of z and $\overline{z}$. In case when w is independent of $\overline{z}$ in an open set of the complex plane $\mathbb{C}$ it is an analytic function. It then is satisfying the Cauchy-Riemann system of first order partial differential equations

$$u_x = v_y \ , \ u_y = -v_x \ . \tag{2}$$

This is equivalent to

$$w_{\overline{z}} = 0 \tag{2'}$$

as follows from

$$2\partial_{\overline{z}} w = (\partial_x + i\partial_y)(u + iv) = \partial_x u - \partial_y v + i(\partial_x v + \partial_y u) \ . \tag{3}$$

In that case also

$$\begin{aligned} 2\partial_z w &= (\partial_x - i\partial_y)(u+iv) = \partial_x u + \partial_y v + i(\partial_x v - \partial_y u) \\ &= 2\partial_x w = -2i\partial_y w = 2w' \ . \end{aligned} \tag{4}$$

Using these complex derivatives the real Gauss divergence theorem for functions of two real variables being continuously differentiable in some regular domain, i.e. a bounded domain D with smooth boundary ∂D, and continuous in the closure $\overline{D} = D \cup \partial D$ of D, easily can be given in complex forms.

Gauss Theorem *(real form) Let $(f,g) \in C^1(D;\mathbb{R}^2) \cap C(\overline{D};\mathbb{R}^2)$ be a differentiable real vector field in a regular domain $D \subset \mathbb{R}^2$ then*

$$\int_D (f_x(x,y) + g_y(x,y))dxdy = -\int_{\partial D} (f(x,y)dy - g(x,y)dx) \ . \tag{5}$$

Remark The two-dimensional area integral on the left-hand side is taken for $div(f,g) = f_x + g_y$. The boundary integral on the right-hand side is just the one dimensional integral of the dot product of the vector (f,g) with the outward normal vector $\nu = (\partial_s y, -\partial_s x)$ on the boundary ∂D with respect to the arc length parameter s. This Gaus Theorem is the main theorem of calculus in $\mathbb{R}^2$.

Gauss Theorems *(complex form) Let $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ in a regular domain D of the complex plane $\mathbb{C}$ then*

$$\int_D w_{\overline{z}}(z)dxdy = \frac{1}{2i}\int_{\partial D} w(z)dz \tag{6}$$

and

$$\int_D w_z(z)dxdy = -\frac{1}{2i}\int_{\partial D} w(z)d\overline{z} \ . \tag{6'}$$

Proof Using (3) and applying (5) shows

$$\begin{aligned} 2\int_D w_{\overline{z}}(z)dxdy &= \int_D (u_x(z) - v_y(z))dxdy + i\int_D (v_x(z) + u_y(z))dxdy \\ &= -\int_{\partial D} (u(z)dy + v(z)dx) - i\int_{\partial D} (v(z)dy - u(z)dx) \\ &= i\int_{\partial D} w(z)dz \end{aligned}$$

This is formula (6). Taking complex conjugation and observing

$$\overline{\partial_{\overline{z}} w} = \partial_z \overline{w}$$

and replacing $\overline{w}$ by w leads to (6′).

Remark Formula (6) contains the Cauchy theorem for analytic functions

$$\int_{\gamma} w(z)dz = 0$$

as particular case. If γ is a simple closed smooth curve and D the inner domain bounded by γ then this integral vanishes as (2′) holds.

3 Cauchy-Pompeiu representation formulas

As from the Cauchy theorem the Cauchy formula is deduced from (6) and (6′) representation formulas can be deduced.

Cauchy-Pompeiu representations *Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then using $\zeta = \xi + i\eta$ for $z \in D$*

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int_{D} w_{\overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z} \tag{7}$$

and

$$w(z) = -\frac{1}{2\pi i} \int_{\partial D} w(\zeta) \frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi} \int_{D} w_{\zeta}(\zeta) \frac{d\xi d\eta}{\overline{\zeta - z}} \tag{7′}$$

hold.

Proof Let $z_0 \in D$ and $\varepsilon > 0$ be so small that

$$\overline{K_\varepsilon(z_0)} \subset D \ , \ K_\varepsilon(z_0) = \{z : | z - z_0 | < \varepsilon\} \ .$$

Denoting $D_\varepsilon = D \setminus \overline{K_\varepsilon(z_0)}$ and applying (6) gives

$$\frac{1}{2i} \int_{\partial D_\varepsilon} w(\zeta) \frac{d\zeta}{\zeta - z_0} - \int_{D_\varepsilon} w_{\overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z_0} = 0 \ .$$

Introducing polar coordinates

$$\int_{\overline{K_\varepsilon(z_0)}} w_{\overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z_0} = \int_0^{\varepsilon} \int_0^{2\pi} w_{\overline{\zeta}}(z_0 + te^{i\varphi}) e^{-i\varphi} d\varphi dt$$

and it is seen that

$$\int\limits_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z_0} = \int\limits_{D_\varepsilon} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z_0} + \int\limits_{\overline{K_\varepsilon(z_0)}} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z_0}$$

exists and hence

$$\lim_{\varepsilon\to 0}\int\limits_{D_\varepsilon} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z_0} = \int\limits_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z_0} \ .$$

Once again using polar coordinates

$$\int\limits_{\partial D_\varepsilon} w(\zeta)\frac{d\zeta}{\zeta - z_0} = \int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z_0} - \int\limits_{\partial K_\varepsilon(z_0)} w(\zeta)\frac{d\zeta}{\zeta - z_0} \ ,$$

where

$$\int\limits_{\partial K_\varepsilon(z_0)} w(\zeta)\frac{d\zeta}{\zeta - z_0} = i\int\limits_0^{2\pi} w(z_0 + \varepsilon e^{i\varphi})d\varphi \ ,$$

is seen to give

$$\lim_{\varepsilon\to 0}\int\limits_{\partial D_\varepsilon} w(\zeta)\frac{d\zeta}{\zeta - z_0} = \int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z_0} - 2\pi i w(z_0) \ .$$

This proves (7). Formula (7′) can be either deduced similarly or by complex conjugation as in the preceding proof.

Definition 1 *For $f \in L_1(D;\mathbb{C})$ the integral operator*

$$Tf(z) = -\frac{1}{\pi}\int\limits_D f(\zeta)\frac{d\xi d\eta}{\zeta - z} \ , \ z \in \mathbb{C} \ ,$$

is called Pompeiu operator.

The Pompeiu operator, see [10], is investigated in detail in connection with the theory of generalized analytic functions in Vekua's book [12], see also [1]. Its differentiability properties are important here in the sequal. For generalizations, see e.g. [2], for application [6, 3].

Theorem 1 *If $f \in L_1(D;\mathbb{C})$ then for all $\varphi \in C_0^1(D;\mathbb{C})$*

$$\int\limits_D Tf(z)\varphi_{\overline{z}}(z)dxdy + \int\limits_D f(z)\varphi(z)dxdy = 0 \qquad (8)$$

Here $C_0^1(D;\mathbb{C})$ denotes the set of complex-valued functions in D being continuously differentiable and having compact support in D, i.e. vanishing near the boundary.

Proof From (7) and the fact that the boundary values of φ vanish at the boundary

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \varphi(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{D} \varphi_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} = (T\varphi_{\overline{\zeta}})(z)$$

follows. Thus interchanging the order of integration

$$\int\limits_D Tf(z)\varphi_{\overline{z}}(z)dxdy = -\frac{1}{\pi}\int\limits_D f(\zeta)\int\limits_D \varphi_{\overline{z}}(z)\frac{dxdy}{\zeta - z}d\xi d\eta = -\int\limits_D f(\zeta)\varphi(\zeta)d\xi d\eta$$

Formula (8) means that

$$\partial_{\overline{z}}Tf = f \tag{9}$$

in distributional sense.

Definition 2 *Let $f, g \in L_1(D;\mathbb{C})$. Then f is called generalized (distributional) derivative of g with respect to $\overline{z}$ if for all $\varphi \in C_0^1(D;\mathbb{C})$*

$$\int\limits_D g(z)\varphi_{\overline{z}}(z)dxdy + \int\limits_D f(z)\varphi(z)dxdy = 0\ .$$

This derivative is denoted by $f = g_{\overline{z}} = \partial_{\overline{z}}g$.

In the same way generalized derivatives with respect to z are defined. In case a function is differentiable in the ordinary sense it is also differentiable in the distributional sense and both derivatives coincide.

Sometimes solutions to differential equations in distributional sense can be shown to be differentiable in the classical sense. Then generalized solutions become classical solutions to the equation. An example is the Cauchy-Riemann system $(2')$, see [12, 1].

More delecate is the differentiation of Tf with respect to z. For $z \in \mathbb{C} \setminus \overline{D}$ obviously Tf is analytic and its derivative

$$\partial_z Tf(z) = \Pi f(z) = -\frac{1}{\pi}\int\limits_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}\ . \tag{10}$$

That this holds in distributional sense also for $z \in D$ almost everywhere when $f \in L_p(D;\mathbb{C}), 1 < p$, and the integral on the right-hand side is understood as a

Cauchy principal value integral

$$\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta-z)^2} = \lim_{\varepsilon\to 0} \int_{D\setminus \overline{K_\varepsilon(z)}} f(\zeta)\frac{d\xi d\eta}{(\zeta-z)^2}$$

is a deep result of Calderon-Zygmund [7].

With respect to boundary value problems a modification of the Cauchy-Pompeiu formula is important in the case of the unit disc $\mathbb{D} = \{z : |z| < 1\}$.

Theorem 2 *Any* $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ *is representable as*

$$w(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \mathrm{Re}\ w(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int_{|\zeta|=1} \mathrm{Im}\ w(\zeta)\frac{d\zeta}{\zeta}$$

$$-\frac{1}{\pi}\int_{|\zeta|<1} \Big(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta-z} + \frac{z\overline{w_{\overline{\zeta}}(\zeta)}}{1-z\overline{\zeta}}\Big)d\xi d\eta\ ,\ \ |z| < 1\ . \tag{11}$$

Corollary 1 *Any* $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ *can be represented as*

$$\begin{aligned} w(z) \ &= \ \frac{1}{2\pi i}\int_{|\zeta|=1} \mathrm{Re}\ w(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \\ &\quad -\frac{1}{2\pi}\int_{|\zeta|<1} \Big(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{w_{\overline{\zeta}}(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big)d\xi d\eta \\ &\quad +i\ \mathrm{Im}\ w(0), |z| < 1\ . \end{aligned} \tag{12}$$

Proof For fixed $z, |z| < 1$, formula (6) applied to $\mathbb{D}$ shows

$$\frac{1}{2\pi i}\int_{|\zeta|=1} w(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} - \frac{1}{\pi}\int_{|\zeta|<1} w_{\overline{\zeta}}(\zeta)\frac{\overline{z}}{1-\overline{z}\zeta}\ d\xi d\eta = 0\ .$$

Taking the complex conjugate and adding this to (7) in the case $D = \mathbb{D}$ gives for $|z| < 1$

$$w(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \Big(\frac{\zeta w(\zeta)}{\zeta-z} + \frac{z\overline{w(\zeta)}}{\zeta-z}\Big)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int_{|\zeta|<1} \Big(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta-z} + \frac{z\overline{w_{\overline{\zeta}}(\zeta)}}{1-z\overline{\zeta}}\Big)d\xi d\eta\ ,$$

where $\overline{\zeta}d\zeta = -\zeta d\overline{\zeta}$ for $|\zeta| = 1$ is used. This is (11). Subtracting i Im $w(0)$ from (11) proves (12).

Remark For analytic functions (12) is the Schwarz-Poisson formula

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \mathrm{Re}\ w(\zeta)\Big(\frac{2\zeta}{\zeta - z} - 1\Big)\frac{d\zeta}{\zeta} + i\ \mathrm{Im}\ w(0)\ . \tag{12'}$$

The kernel

$$\frac{\zeta + z}{\zeta - z} = \frac{2\zeta}{\zeta - z} - 1$$

is called the Schwarz kernel. Its real part

$$\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}$$

is the Poisson kernel. The Schwarz operator

$$S\varphi(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}$$

for $\varphi \in C(\partial\mathbb{D}; \mathbb{R})$ is known to provide an analytic function in $\mathbb{D}$ satisfying

$$\mathrm{Re}\ S\varphi = \varphi \text{ on } \partial\mathbb{D}$$

see [11] in the sense

$$\lim_{z\to\zeta} S\varphi(z) = \varphi(\zeta)\ ,\ \zeta \in \partial\mathbb{D}\ ,$$

for z in $\mathbb{D}$ tending to ζ. Poisson has proved the respective representation for harmonic functions, i.e. to solutions for the Laplace equation

$$\Delta u = \partial_x^2 u + \partial_y^2 u = 0$$

in $\mathbb{D}$. Re w for analytic w is harmonic.
The Schwarz operator can be defined for other simply and even multi-connected domains, see e.g. [1].

Formula (12) is called the Cauchy-Schwarz-Poisson-Pompeiu formula. Rewriting it according to

$$w_{\overline{z}} = f \text{ in } \mathbb{D}\ ,\ \mathrm{Re}\ w = \varphi \text{ on } \partial\mathbb{D}\ ,\ \mathrm{Im}\ w(0) = c\ ,$$

then

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} - \frac{1}{2\pi} \int\limits_{|\zeta|<1} \Big(\frac{f(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\Big) d\xi d\eta + ic \tag{12''}$$

is expressed by the given data. Applying the result of Schwarz it is easily seen that taking the real part on the right-hand side and letting z tend to a boundary point ζ this tends to $\varphi(\zeta)$.
Differentiating with respect to $\overline{z}$ as every term on the right-hand side is analytic besides the T-operator applied to f this gives $f(z)$. Also for $z = 0$ besides ic all other terms on the right-hand side are real.
Hence, (12″) is a solution to the so-called Dirichlet problem

$$w_{\overline{z}} = f \text{ in } \mathbb{D} \ , \ \text{Re } w = \varphi \text{ on } \partial\mathbb{D} \ , \ \text{Im } w(0) = c \ .$$

This shows how integral representation formulas serve to solve boundary value problems. The method is not restricted to the unit disc but in this case the solutions to the problems are given in an explicit way.

4 Iteration of integral representation formulas

Integral representation formulas for solutions to first order equations can be used to get such formulas for higher order equations via iteration. The principle will be eluminated by iterating the main theorem of calculus in one real variable.
Main Theorem of Calculus *Let (a, b) be a segment of the real line $a < b$ and $f \in C^1((a, b); \mathbb{R}) \cap C([a, b]; \mathbb{R})$. Then for $x, x_0 \in (a, b)$*

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt \ . \tag{13}$$

Assuming now $f \in C^2((a, b); \mathbb{R}) \cap C^1([a, b]; \mathbb{R})$ then besides (13) also

$$f'(x) = f'(x_0) + \int_{x_0}^{x} f''(t)dt \ .$$

Inserting this into (13) and applying integration by parts gives

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^{x} (x - t)f''(x)dt \ .$$

Taylor Theorem *Let $f \in C^{n+1}((a, b); \mathbb{R}) \cap C^n([a, b]; \mathbb{R})$, then*

$$f(x) = \sum_{\nu=0}^{n} \frac{f^{(\nu)}(x_0)}{\nu!} \ (x - x_0)^\nu + \int_{x_0}^{x} \frac{(x - t)^n}{n!} \ f^{(n+1)}(t)dt \ . \tag{14}$$

Proof In case $n = 0$ formula (14) is just (13). Assuming (14) to hold for $n-1$ rather than for n and applying (13) to $f^{(n)}$ and inserting this in (14) provides (14) for n after partial integration.

Applying this iteration procedure to the representations (7) and (7′) leads to a hierarchy of kernel functions and higher order integral representations of Cauchy-Pompeiu type.

Theorem 3 *Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$, then*

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int\limits_{\partial D} w_{\overline{\zeta}}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta + \frac{1}{\pi}\int\limits_{D} w_{\overline{\zeta}\overline{\zeta}}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}\,d\xi d\eta \quad (15)$$

and

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{\partial D} w_{\overline{\zeta}}(\zeta)\log|\zeta - z|^2\,d\overline{\zeta}$$
$$+\frac{1}{\pi}\int\limits_{D} w_{\zeta\overline{\zeta}}(\zeta)\log|\zeta - z|^2\,d\xi d\eta\ . \quad (15')$$

Proof (1) For proving (15) formula (7) applied to $w_{\overline{z}}$ giving

$$w_{\overline{\zeta}}(\zeta) = \frac{1}{2\pi i}\int\limits_{\partial D} w_{\overline{\tilde{\zeta}}}(\tilde{\zeta})\frac{d\tilde{\zeta}}{\tilde{\zeta} - \zeta} - \frac{1}{\pi}\int\limits_{D} w_{\overline{\tilde{\zeta}}\overline{\tilde{\zeta}}}(\tilde{\zeta})\frac{d\tilde{\xi}d\tilde{\eta}}{\tilde{\zeta} - \zeta}$$

is inserted into (7) from what after having interchanged the order of integrations

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{\partial D} w_{\overline{\tilde{\zeta}}}(\tilde{\zeta})\psi(z,\tilde{\zeta})d\tilde{\zeta} - \frac{1}{\pi}\int\limits_{D} w_{\overline{\tilde{\zeta}}\overline{\tilde{\zeta}}}(\tilde{\zeta})\psi(z,\tilde{\zeta})d\tilde{\xi}d\tilde{\eta} \quad (16)$$

follows with

$$\psi(z,\zeta) = \frac{1}{\pi}\int\limits_{D}\frac{d\xi d\eta}{(\zeta - \tilde{\zeta})(\zeta - z)} = \frac{1}{\tilde{\zeta} - z}\frac{1}{\pi}\int\limits_{D}\Big(\frac{1}{\zeta - \tilde{\zeta}} - \frac{1}{\zeta - z}\Big)d\xi d\eta\ .$$

Formula (7) applied to the function $\overline{z}$ shows

$$\frac{\overline{\tilde{\zeta} - z}}{\tilde{\zeta} - z} = \frac{1}{2\pi i}\int\limits_{\partial D}\frac{\overline{\zeta}d\zeta}{(\zeta - \tilde{\zeta})(\zeta - z)} - \frac{1}{\pi}\int\limits_{D}\frac{d\xi d\eta}{(\zeta - \tilde{\zeta})(\zeta - z)} = \tilde{\psi}(z,\tilde{\zeta}) - \psi(z,\tilde{\zeta}) \quad (17)$$

with a function $\tilde{\psi}$ analytic in both its variables. Hence by (6)

$$\frac{1}{2\pi i}\int\limits_{\partial D} w_{\overline{\tilde{\zeta}}}(\tilde{\zeta})\tilde{\psi}(z,\tilde{\zeta})d\tilde{\zeta} - \frac{1}{\pi}\int\limits_{D} w_{\overline{\tilde{\zeta}}\overline{\tilde{\zeta}}}(\tilde{\zeta})\tilde{\psi}(z,\tilde{\zeta})d\tilde{\xi}d\tilde{\eta} = 0\ .$$

Subtracting this from (16) and applying (17) gives (15).
(2) In order to show (15′) formula (7′) giving

$$w_{\overline{\zeta}}(\zeta) = -\frac{1}{2\pi i}\int_{\partial D} w_{\overline{\tilde{\zeta}}}(\tilde{\zeta})\frac{d\overline{\tilde{\zeta}}}{\overline{\tilde{\zeta}-\zeta}} - \frac{1}{\pi}\int_D w_{\tilde{\zeta}\overline{\tilde{\zeta}}}(\tilde{\zeta})\frac{d\tilde{\xi}d\tilde{\eta}}{\overline{\tilde{\zeta}-\zeta}}$$

is inserted in (7) so that after interchanging the order of integrations

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial D} w_{\overline{\tilde{\zeta}}}(\tilde{\zeta})\Psi(z,\zeta)d\overline{\tilde{\zeta}} - \frac{1}{\pi}\int_D w_{\tilde{\zeta}\overline{\tilde{\zeta}}}(\tilde{\zeta})\Psi(z,\zeta)d\tilde{\xi}d\tilde{\eta} \tag{18}$$

with

$$\Psi(z,\zeta) = \frac{1}{\pi}\int_D \frac{d\xi d\eta}{\overline{(\zeta-\tilde{\zeta})}(\zeta - z)}\ .$$

The function $\log|\tilde{\zeta} - z|^2$ is a C^1-function in $D\setminus\{\tilde{\zeta}\}$ for fixed $\tilde{\zeta}\in D$. Hence formula (7) may be applied in $D_\varepsilon = D\setminus\{z : |z-\tilde{\zeta}|\leq\varepsilon\}$ for small enough positive ε giving for $z\in D_\varepsilon$

$$\log|\tilde{\zeta} - z|^2 = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \log|\tilde{\zeta}-\zeta|^2\,\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D_\varepsilon}\frac{1}{\overline{\zeta-\tilde{\zeta}}}\frac{d\xi d\eta}{\zeta - z}\ .$$

As for $\varepsilon < |z-\tilde{\zeta}|$

$$\frac{1}{\pi}\int_{|\zeta-\tilde{\zeta}|<\varepsilon}\frac{1}{\overline{\zeta-\tilde{\zeta}}}\frac{d\xi d\eta}{\zeta - z} = \frac{1}{\pi}\int_0^\varepsilon\int_0^{2\pi}\frac{e^{i\varphi}}{\tilde{\zeta}-z+te^{i\varphi}}\,d\varphi dt$$

exists and tends to zero with ε tending to zero and because for $\varepsilon < |z-\tilde{\zeta}|$

$$\frac{1}{2\pi i}\int_{|\zeta-\tilde{\zeta}|=\varepsilon}\log|\tilde{\zeta}-\zeta|^2\,\frac{d\zeta}{\zeta - z} = \frac{2\log\varepsilon}{2\pi i}\int_{|\zeta-\tilde{\zeta}|=\varepsilon}\frac{d\zeta}{\zeta - z} = 0$$

this relation results in

$$\log|\tilde{\zeta} - z|^2 = \frac{1}{2\pi i}\int_{\partial D}\log|\tilde{\zeta}-\zeta|^2\,\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D\frac{1}{\overline{\zeta-\tilde{\zeta}}}\frac{d\xi d\eta}{\zeta - z} = \tilde{\Psi}(z,\tilde{\zeta}) - \Psi(z,\tilde{\zeta}) \tag{19}$$

where the function $\tilde{\Psi}$ is analytic in z but anti-analytic in $\tilde{\zeta}$. Therefore from (6′)

$$\frac{1}{2\pi i}\int_{\partial D} w_{\overline{\tilde{\zeta}}}(\tilde{\zeta})\tilde{\Psi}(z,\tilde{\zeta})d\overline{\tilde{\zeta}} + \frac{1}{\pi}\int_D w_{\tilde{\zeta}\overline{\tilde{\zeta}}}(\tilde{\zeta})\tilde{\Psi}(z,\tilde{\zeta})d\tilde{\xi}d\tilde{\eta} = 0\ .$$

Adding this to (18) and observing (19) proves (15′).

Remark There are dual formulas to (15) and (15′) resulting from interchanging the roles of (7) and (7′) in the preceding procedure. They arise also from complex conjugation of (15) and (15′) after replacing w by $\overline{w}$. They are

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\overline{\zeta - z}}} + \frac{1}{2\pi i}\int_{\partial D} w_\zeta(\zeta)\frac{\zeta - z}{\overline{\overline{\zeta - z}}}\, d\overline{\zeta} + \frac{1}{\pi}\int_D w_{\zeta\zeta}(\zeta)\frac{\zeta - z}{\overline{\overline{\zeta - z}}}\, d\xi d\eta \tag{15''}$$

and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\overline{\zeta - z}}} - \frac{1}{2\pi i}\int_{\partial D} w_\zeta(\zeta)\log|\zeta - z|^2\, d\zeta$$

$$+\frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)\log|\zeta - z|^2\, d\xi d\eta\ . \tag{15'''}$$

The kernel functions $(\overline{\zeta - z})/(\zeta - z)$, $\log|\zeta - z|^2$, $(\zeta - z)/(\overline{\zeta - z})$ of the second order differential operators $\partial_{\overline{z}}^2, \partial_z\partial_{\overline{z}}, \partial_z^2$ respectively are thus obtained from those Cauchy and anti-Cauchy kernels $1/(\zeta - z)$ and $1/(\overline{\zeta - z})$ for the Cauchy-Riemann operator $\partial_{\overline{z}}$ and its complex conjugate ∂_z.

Continuing in this way in [4, 5], see also [1], a hierarchy of kernel functions and related integral operators are constructed and general higher order Cauchy-Pompeiu representation formulas are developed.

Definition 3 *For $m, n \in \mathbb{Z}$ satisfying $0 \le m + n$ and $0 < m^2 + n^2$ let*

$$K_{m,n}(z,\zeta) = \begin{cases} \dfrac{(-1)^n(-m)!}{(n-1)!\pi}\ (\zeta - z)^{m-1}(\overline{\zeta - z})^{n-1} \text{ if } m \le 0\ , \\ \dfrac{(-1)^m(-n)!}{(m-1)!\pi}\ (\zeta - z)^{m-1}(\overline{\zeta - z})^{n-1} \text{ if } n \le 0\ , \\ \dfrac{(\zeta - z)^{m-1}(\overline{\zeta - z})^{n-1}}{(m-1)!(n-1)!\pi}\ \Big[\log|\zeta - z|^2 \\ \qquad -\sum\limits_{\mu=1}^{m-1}\frac{1}{\mu} - \sum\limits_{\nu=1}^{n-1}\frac{1}{\nu}\Big] \qquad \text{if } 1 \le m, n\ , \end{cases} \tag{20}$$

and for $f \in L_1(D;\mathbb{C}), D \subset \mathbb{C}$ a domain,

$$\begin{aligned} T_{0,0}f(z) &= f(z) \text{ for } (m,n) = (0,0)\ , \\ T_{m,n}f(z) &= \int_D K_{m,n}(z,\zeta)f(\zeta)d\xi d\eta \text{ for } (m,n) \neq (0,0)\ . \end{aligned} \tag{21}$$

Examples

$$T_{0,1}f(z) = -\frac{1}{\pi}\int_D \frac{f(\zeta)}{\zeta - z}\, d\xi d\eta\ ,\ T_{1,0}f(z) = -\frac{1}{\pi}\int_D \frac{f(\zeta)}{\overline{\zeta - z}}\, d\xi d\eta\ ,$$

$$T_{0,2}f(z) = \frac{1}{\pi}\int_D f(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}\, d\xi d\eta\ ,\ T_{2,0}f(z) = \frac{1}{\pi}\int_D f(\zeta)\frac{\zeta - z}{\overline{\zeta - z}}\, d\xi d\eta,$$

$$T_{1,1}f(z) = \frac{1}{\pi}\int_D f(\zeta)\log|\zeta - z|^2\, d\xi d\eta\ ,$$

$$T_{-1,1}f(z) = -\frac{1}{\pi}\int_D \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta\ ,\ T_{1,-1}f(z) = -\frac{1}{\pi}\int_D \frac{f(\zeta)}{(\overline{\zeta - z})^2} d\xi d\eta.$$

The kernel functions are weakly singular as long as $0 < m + n$. But for $m + n = 0, 0 < m^2 + n^2$ they are strongly singular and the related integral operators are strongly singular of Calderon-Zygmund type to be understood as Cauchy principle value integrals. They are useful to solve higher order partial differential equations. $K_{m,n}$ turns out to be the fundamental solution to $\partial_{\overline{z}}^m \partial_z^n$ for $0 \le m, n$. As special cases to the general Cauchy-Pompeiu representation deduced in [4] two particular situations are considered.

Theorem 4 *Let $w \in C^n(D;\mathbb{C}) \cap C^{n-1}(\overline{D};\mathbb{C})$ for some $n \ge 1$. Then*

$$w(z) = \sum_{\nu=0}^{n-1} \frac{1}{2\pi i}\int_{\partial D} \partial_{\overline{\zeta}}^{\nu} w(\zeta)\frac{(\overline{z - \zeta})^{\nu}}{\nu!(\zeta - z)}\, d\zeta - \frac{1}{\pi}\int_D \partial_{\overline{\zeta}}^{n} w(\zeta)\frac{(\overline{z - \zeta})^{n-1}}{(n-1)!(\zeta - z)}\, d\xi d\eta \tag{22}$$

This formula obviously is a generalization to (15) and can be proved inductively in the same way as (15).

Theorem 5 *Let $w \in C^{2n}(D;\mathbb{C}) \cap C^{2n-1}(\overline{D};\mathbb{C})$ for some $n \ge 1$. Then*

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial D} \frac{w(\zeta)}{\zeta - z}\, d\zeta + \sum_{\nu=1}^{n-1} \frac{1}{2\pi i}\int_{\partial D} \frac{(\zeta - z)^{\nu-1}(\overline{\zeta - z})^{\nu}}{(\nu-1)!\nu!} \\ &\quad \Big[\log|\zeta - z|^2 - \sum_{\rho=1}^{\nu-1}\frac{1}{\rho} - \sum_{\sigma=1}^{\nu}\frac{1}{\sigma}\Big](\partial_\zeta \partial_{\overline{\zeta}})^{\nu} w(\zeta) d\zeta \\ &+ \sum_{\nu=1}^{n} \frac{1}{2\pi i}\int_{\partial D} \frac{|\zeta - z|^{2(\nu-1)}}{(\nu-1)!^2}\Big[\log|\zeta - z|^2 - 2\sum_{\rho=1}^{\nu-1}\frac{1}{\rho}\Big]\partial_\zeta^{\nu-1}\partial_{\overline{\zeta}}^{\nu} w(\zeta) d\overline{\zeta} \\ &+ \frac{1}{\pi}\int_D \frac{|\zeta - z|^{2(n-1)}}{(n-1)!^2}\Big[\log|\zeta - z|^2 - 2\sum_{\rho=1}^{n-1}\frac{1}{\rho}\Big](\partial_\zeta \partial_{\overline{\zeta}})^{n} w(\zeta) d\xi d\eta\ . \end{aligned} \tag{23}$$

This representation contains (15′) as a particular case for $n = 1$. The proof also follows by induction on the basis of (7) and (7′).

For the general case related to the differential operator $\partial_z^m \partial_{\overline{z}}^n$ some particular notations are needed which are not introduced here, see [5].

5 Basic boundary value problems

As was pointed out in connection with the Schwarz-Poisson formula in the case of the unit disc boundary value problems can be solved explicitly. For this reason this particular domain is considered. This will give necessary information about the nature of the problems considered. The simplest and therefore fundamental cases occur with respect to analytic functions.

Schwarz boundary value problem *Find an analytic function w in the unit disc, i.e. a solution to $w_{\overline{z}} = 0$ in $\mathbb{D}$, satisfying*

$$\mathrm{Re}\ w = \gamma \text{ on } \partial\mathbb{D}\ ,\ \mathrm{Im}\ w(0) = c$$

for $\gamma \in C(\partial\mathbb{D}; \mathbb{R}), c \in \mathbb{R}$ given.

Theorem 6 *This Schwarz problem is uniquely solvable. The solution is given by the Schwarz formula*

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + ic\ . \tag{24}$$

The proof follows from the Schwarz-Poisson formula (12′) together with a detailed study of the boundary behaviour, see [11].

Dirichlet boundary value problem *Find an analytic function w in the unit disc, i.e. a solution to $w_{\overline{z}} = 0$ in $\mathbb{D}$, satisfying for given $\gamma \in C(\partial\mathbb{D}; \mathbb{C})$*

$$w = \gamma \textit{ on } \partial\mathbb{D}\ .$$

Theorem 7 *This Dirichlet problem is solvable if and only if for $|z| < 1$*

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\overline{z} d\zeta}{1 - \overline{z}\zeta} = 0\ . \tag{25}$$

The solution is then uniquely given by the Cauchy integral

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{\zeta - z}\ . \tag{26}$$

Remark This result is a consequence of the Plemelj-Sokhotzki formula, see e.g. [9, 8, 1]. The Cauchy integral (26) obviously provides an analytic function in $\mathbb{D}$ and one in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}, \hat{\mathbb{C}}$ the Riemann sphere. The Plemelj-Sokhotzki formula states that for $|\zeta|= 1$

$$\lim_{z\to\zeta,|z|<1} w(z) - \lim_{z\to\zeta,1<|z|} w(z) = \gamma(\zeta) \ .$$

In order that for any $|\zeta|= 1$

$$\lim_{z\to\zeta,|z|<1} w(z) = \gamma(\zeta)$$

the condition

$$\lim_{z\to\zeta,1<|z|} w(z) = 0$$

is necessary and sufficient. However, the Plemelj-Sokhotzki formula in its classical formulation holds if γ is Hölder continuous. Nevertheless, for the unit disc Hölder continuity is not needed, see [9].

Proof 1. (25) is shown to be necessary. Let w be a solution to the Dirichlet problem. Then w is analytic in $\mathbb{D}$ having continuous boundary values

$$\lim_{z\to\zeta} w(z) = \gamma(\zeta) \tag{27}$$

for all $|\zeta|= 1$.

Consider for $1 <|z|$ the function

$$w\Big(\frac{1}{\overline{z}}\Big) = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{\overline{z}}{\overline{\zeta - z}}\frac{d\zeta}{\zeta} \ .$$

As with $z, 1 <|z|$, tending to $\zeta, |\zeta|= 1, 1/\overline{z}$ tends to ζ too, $\lim_{z\to\zeta} w(1/\overline{z})$ exists, i.e. $\lim_{z\to\zeta} w(z)$ exists for $1 <|z|$. From

$$w(z) - w\Big(\frac{1}{\overline{z}}\Big) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta}$$

and the properties of the Poisson kernel for $|\zeta|= 1$

$$\lim_{z\to\zeta,|z|<1} w(\zeta) - \lim_{z\to\zeta,1<|z|} w(z) = \gamma(\zeta) \tag{28}$$

follows. Comparison with (27) shows $\lim_{z\to\zeta} w(z) = 0$ for $1 <|z|$. As $w(\infty) = 0$ then the maximum principle for analytic functions tells that $w(z) \equiv 0$ in $1 <|z|$. This is condition (25).

2. The sufficiency of (25) follows at once from adding (25) to (26) leading to

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{z}}{\overline{\zeta - z}}\Big)\frac{d\zeta}{\zeta} \\ &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta} \ . \end{aligned}$$

Thus for $|\zeta|=1$

$$\lim_{z\to\zeta,|z|<1} w(z)=\gamma(\zeta)$$

follows again from the properties of the Poisson kernel.

The third basic boundary value problem is based on the outward normal derivative at the boundary of a regular domain. This directional derivative on a circle $|z-a|=r$ is in the direction of the radius vector, i.e. the outward normal vector is $\nu=(z-a)/r$, and the normal derivative in this direction ν given by

$$\partial_\nu=\partial_r=\frac{z}{r}\,\partial_z+\frac{\overline{z}}{r}\,\partial_{\overline{z}}\ .$$

In particular for the unit disc $\mathbb{D}$

$$\partial_r=z\partial_z+\overline{z}\partial_{\overline{z}}\ .$$

Neumann boundary value problem *Find an analytic function w in the unit disc, i.e. a solution to $w_{\overline{z}}=0$ in $\mathbb{D}$, satisfying for some $\gamma\in C(\partial\mathbb{D};\mathbb{C})$ and $c\in\mathbb{C}$*

$$\partial_\nu w=\gamma \text{ on } \partial\mathbb{D}\ ,\ w(0)=c\ .$$

Theorem 8 *This Neumann problem is solvable if and only if for $|z|<1$*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta}=0 \tag{29}$$

is satisfied. The solution then is

$$w(z)=c-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta}\ . \tag{30}$$

Proof The boundary condition reduced to the Dirichlet condition

$$zw'(z)=\gamma(z) \text{ for } |z|=1$$

because of the analyticity of w. Hence from the preceding result

$$zw'(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta-z}$$

if and only if for $|z|<1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta}=0\ . \tag{31}$$

But as $zw'(z)$ vanished at the origin this imposes the additional condition

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta}=0 \tag{32}$$

on γ. Then

$$w'(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{(\zeta-z)\zeta}\ .$$

Integrating shows

$$w(z)=c-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\log\frac{\zeta-z}{\zeta}\frac{d\zeta}{\zeta}$$

which is (30). Adding (31) and (32) leads to

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{1}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta} &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{\zeta}}{\overline{\zeta-z}}\frac{d\zeta}{\zeta} \\ &= -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta-z}}=0\ ,\end{aligned}$$

i.e. to (29). By integration this gives

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\log(1-\overline{z}\zeta)d\overline{\zeta}=0\ .$$

Next these boundary value problems will be studied for the inhomogeneous Cauchy-Riemann equation. Using the T-operator the problems will be reduced to the ones for analytic functions. Here in the case of the Neumann problem it will make a difference if the normal derivative on the boundary or only the effect of $z\partial_z$ on the function is prescribed.

Theorem 9 *The Schwarz problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\overline{z}}=f \textit{ in } \mathbb{D}\ ,\ \mathrm{Re}\, w=\gamma \textit{ on } \partial\mathbb{D}\ ,\ \mathrm{Im}\, w(0)=c$$

for $f\in L_1(\mathbb{D};\mathbb{C}),\gamma\in C(\partial\mathbb{D};\mathbb{R}),c\in\mathbb{R}$ *is uniquely solvable by the Cauchy-Schwarz-Pompeiu formula*

$$w(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}+ic-\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\overline{\zeta}}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\Big]d\xi d\eta \tag{33}$$

This representation (33) follows just from (12) assuming that the solution w exists. But (33) can easily be justified to be a solution. That this solution is unique follows from Theorem 6.

Theorem 10 *The Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\overline{z}} = f \text{ in } \mathbb{D} \ , \ w = \gamma \text{ on } \partial\mathbb{D}$$

for $f \in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma \in C(\partial\mathbb{D};\mathbb{C})$ *is solvable if and only if for* $|z|< 1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} = \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{\overline{z}d\xi d\eta}{1-\overline{z}\zeta} \ . \tag{34}$$

The solution then is uniquely given by

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{\zeta - z} \ . \tag{35}$$

Representation (35) follows from (7) if the problem is solvable. The unique solvability is a consequence of Theorem 7. That (35) actually is a solution under (34) follows by observing the properties of the T-operator on one hand and from

$$\begin{aligned} w(z) \ &= \ \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\Big(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\Big)\frac{d\zeta}{\zeta} \\ &- \ \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\Big(\frac{1}{\zeta - z} + \frac{\overline{z}}{1-\overline{z}\zeta}\Big)d\xi d\eta \ = \ \gamma(z) \end{aligned}$$

for $|z|= 1$ on the other.

That (34) is also necessary follows from Theorem 7. Applying condition (25) to the boundary value of the analytic function $w - Tf$ in $\mathbb{D}$, i.e. to $\gamma - Tf$ on $\partial\mathbb{D}$ gives (34) because of

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{d\tilde{\xi}d\tilde{\eta}}{\tilde{\zeta} - \zeta}\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} \quad =$$

$$-\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \frac{\overline{z}}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta - \tilde{\zeta}}\, d\tilde{\xi}d\tilde{\eta} \quad = \quad -\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|=1} f(\tilde{\zeta})\frac{\overline{z}}{1-\overline{z}\tilde{\zeta}}\, d\tilde{\xi}d\tilde{\eta}$$

as is seen from the Cauchy formula.

Theorem 11 *The Neumann problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\overline{z}} = f \text{ in } \mathbb{D} \ , \ \partial_\nu w = \gamma \text{ on } \partial\mathbb{D} \ , \ w(0) = c \ ,$$

for $f \in C^\alpha(\overline{\mathbb{D}};\mathbb{C}), 0 < \alpha < 1, \gamma \in C(\partial\mathbb{D};\mathbb{C}), c \in \mathbb{C}$ *is solvable if and only if for* $|z| < 1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} f(\zeta)\frac{d\overline{\zeta}}{1-\overline{z}\zeta} + \frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{\overline{z}f(\zeta)}{(1-\overline{z}\zeta)^2}\,d\xi d\eta = 0\ . \tag{36}$$

The unique solution then is

$$w(z) = c - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma(\zeta)-\overline{\zeta}f(\zeta))\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{zf(\zeta)}{\zeta(\zeta-z)}\,d\xi d\eta\ . \tag{37}$$

Proof The function $\varphi = w - Tf$ satisfies

$$\varphi_{\overline{z}} = 0 \text{ in } \mathbb{D}\ ,\ \partial_\nu\varphi = \gamma - z\Pi f - \overline{z}f \text{ on } \partial\mathbb{D}\ ,\ \varphi(0) = c - Tf(0)\ .$$

As the property of the Π-operator, see [12], Chapter 1, §8 and §9, guarantee $\Pi f \in C^\alpha(\overline{\mathbb{D}};\mathbb{C})$ for $f \in C^\alpha(\overline{\mathbb{D}};\mathbb{C})$ Theorem 8 shows

$$\varphi(z) = c - Tf(0) - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma(\zeta) - \zeta\Pi f(\zeta) - \overline{\zeta}f(\zeta))\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta}$$

if and only if

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma(\zeta) - \zeta\Pi f(\zeta) - \overline{\zeta}f(\zeta))\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} = 0\ .$$

From

$$\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\zeta\Pi f(\zeta)\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta} = \\
&-\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-z\overline{\zeta})}{(\zeta-\tilde{\zeta})^2}\,d\zeta d\tilde{\xi} d\tilde{\eta} = \\
&\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-z\overline{\zeta})}{(1-\tilde{\zeta}\overline{\zeta})^2}\,d\overline{\zeta} d\tilde{\xi} d\tilde{\eta} = 0\ ,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Pi f(\zeta)\frac{d\zeta}{1-\overline{z}\zeta} &= -\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\frac{1}{2\pi}\int\limits_{|\zeta|=1}\frac{1}{(\zeta-\tilde{\zeta})^2}\frac{d\zeta}{1-\overline{z}\zeta}\,d\tilde{\xi} d\tilde{\eta} \\
&= -\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} f(\tilde{\zeta})\partial_\zeta\frac{1}{1-\overline{z}\zeta}\Big|_{\zeta=\tilde{\zeta}}\,d\tilde{\xi} d\tilde{\eta} = -\frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{\overline{z}}{(1-\overline{z}\zeta)^2}\,d\xi d\eta
\end{aligned}$$

the result follows.
Theorem 12 *The problem*

$$w_{\overline{z}} = f \text{ in } \mathbb{D} \ , \ zw_z = \gamma \text{ on } \partial\mathbb{D} \ , \ w(0) = c$$

is solvable for $f \in C^\alpha(\overline{\mathbb{D}};\mathbb{C}), 0 < \alpha < 1, \gamma \in C(\partial\mathbb{D};\mathbb{C}), c \in \mathbb{C}$, *if and only if*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} + \frac{\overline{z}}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2} = 0 \ . \tag{38}$$

The solution is then uniquely given as

$$w(z) = c - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\log(1-z\overline{\zeta})\frac{d\zeta}{\zeta} - \frac{z}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{\zeta(\zeta - z)} \ . \tag{39}$$

Proof The function $\varphi = w - Tf$ satisfies

$$\varphi_{\overline{z}} = 0 \text{ in } \mathbb{D} \ , \ z\varphi'(z) = \gamma - z\Pi f \text{ on } \partial\mathbb{D} \ , \ \varphi(0) = c - Tf(0) \ .$$

Comparing this with the problem in the preceding proof leads to the result.

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