On the Directional Differentiability Properties of the max-min Function

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Abstract

In this paper, the directional lower and upper derivatives of the max-min function are investigated by using the directional lower and upper derivative sets of the max-min set valued map. Sufficient conditions ensuring the existence of the directional derivative of the max-min function are obtained.
Keywords: multivalued mapping, optimal control.

1 Introduction

It is well known that the max-min functions come into play in the control theory problems, the differential game theory problems and the parametric optimization problems (see, for example Danskin, 1966;1967). On the other hand the max-min functions are not usually differentiable. But in some problems it is necessary that the directional derivative or the directional lower and upper derivatives of the max-min functions should be calculated.

In this paper, by using the concepts of the directional upper and lower derivative sets of the max-min set valued map, the directional upper and lower derivatives of the max-min functions are given and sufficient conditions ensuring the existence of the directional derivative of the max-min function are obtained.

2 Derivative sets of the set valued map

Here and after, $cl(R^m)$ ($comp(R^m)$) denotes the set of all nonempty closed (compact) subsets in R^m. Let $a(\cdot) : R^n \to cl(R^m)$ be an upper semi-continuous set valued map. Let us consider the following sets. For $(x, y) \in R^n \times R^m$ and vector $f \in R^n$, we set

$$\begin{aligned} Da(x,y) \mid (f) &= \{d \in \mathbb{R}^m : \liminf_{\delta\to+0} \frac{1}{\delta} dist\ (y+\delta d, a(x+\delta f)) = 0\}, \\ D^*a(x,y) \mid (f) &= \{v \in \mathbb{R}^m : \limsup_{\delta\to+0} \frac{1}{\delta} dist\ (y+\delta d, a(x+\delta f)) = 0\}. \end{aligned}$$

Here for $x \in R^n$, $D \subset R^n$, $dist\,(x, D) = \inf_{d \in D} \|x - d\|$. $Da(x,y) \mid (f)$ $(D^*a(x,y) \mid (f))$ is called the upper (lower) derivative set of the set valued map $a(\cdot)$ at (x,y) in the direction f. Note that the directional upper (lower) derivative set of the set valued map $a(\cdot)$ is closed and there is a connection between the upper (lower) derivative set of the set valued map and the upper (lower) contingent cone which is used to investigate various problems in nonsmooth analysis (see, for example Aubin and Frankowska, 1990; Guseinov, et al., 1985; Clarke, et al., 1995). It is obvious that $D^*a(x,y) \mid (f) \subset Da(x,y) \mid (f)$.

$$A = gra(\cdot) = \{(x,y) \in R^n \times R^m : y \in a(x)\}$$

denotes the graph of the set valued map $a(\cdot)$. Since $a(\cdot)$ is upper semicontinuous, A is a closed set. It is possible to show that $Da(x,y) \mid (f) = D^*a(x,y) \mid (f) = \emptyset$ if $(x,y) \notin A$, $Da(x,y) \mid (f) = D^*a(x,y) \mid (f) = R^m$ if $(x,y) \in intA$ where $intA$ denotes the interior of A.

Suppose that the set valued map $a(\cdot)$ is given as

$$a(x) = \{y \in R^m : b(x,y) \leq 0\} \tag{2.1}$$

where $b(\cdot,\cdot) : R^n \times R^m \to R$ is a continuous function in $R^n \times R^m$ and locally Lipschitz in R^m. The lower and upper derivative of $b(\cdot,\cdot)$ at the point (x,y) in the direction (f,d) is denoted by $\dfrac{\partial^- b(x,y)}{\partial(f,d)}$ and $\dfrac{\partial^+ b(x,y)}{\partial(f,d)}$ respectively and defined by

$$\begin{aligned} \frac{\partial^- b(x,y)}{\partial(f,d)} &= \liminf_{\delta \to +0} \ [b(x + \delta f, y + \delta d) - b(x,y)]\, \delta^{-1}, \\ \frac{\partial^+ b(x,y)}{\partial(f,d)} &= \limsup_{\delta \to +0} \ [b(x + \delta f, y + \delta d) - b(x,y)]\, \delta^{-1} \end{aligned}$$

respectively. If

$$\frac{\partial b(x,y)}{\partial(f,d)} = \lim_{\delta \to +0} \ [b(x + \delta f, y + \delta d) - b(x,y)]\, \delta^{-1}$$

exists and is finite, then $b(\cdot,\cdot)$ is called differentiable at the point (x,y) in the direction (f,d) and $\dfrac{\partial b(x,y)}{\partial(f,d)}$ denotes the derivative of $b(\cdot,\cdot)$ at the point (x,y) in the direction (f,d).

We introduce the sets

$$\begin{aligned} H^{-}(x,y) \mid (f) &= \{d \in R^m : \frac{\partial^{-} b(x,y)}{\partial(f,d)} < 0\}, \\ H(x,y) \mid (f) &= \{d \in R^m : \frac{\partial^{-} b(x,y)}{\partial(f,d)} \leq 0\}, \\ E^{-}(x,y) \mid (f) &= \{d \in R^m : \frac{\partial^{+} b(x,y)}{\partial(f,d)} < 0\}, \\ E(x,y) \mid (f) &= \{d \in R^m : \frac{\partial^{+} b(x,y)}{\partial(f,d)} \leq 0\} \end{aligned}$$

(Guseinov, Kucuk and Ekici, 2001).

Proposition 1 *Let the set valued map* $a(\cdot)$ *be in the form* (2.1)*. Then for all* $(x,y) \in \partial A$ *and* $f \in R^n$*,*

$$\begin{aligned} clH^{-}(x,y) \mid (f) &\subset Da(x,y) \mid (f) \subset H(x,y) \mid (f), \\ clE^{-}(x,y) \mid (f) &\subset D^{*}a(x,y) \mid (f) \subset E(x,y) \mid (f) \end{aligned}$$

where ∂A *denotes the boundary of* A*,* clA *denotes the closure of* A*.*

By using the previous proposition, we obtain the following corollary.

Corollary 2 *Let* $(x,y) \in \partial A$*,* $b(\cdot,\cdot)$ *be differentiable at* (x,y) *and* $\dfrac{\partial b(x,y)}{\partial y} \neq 0$*. Then it is possible to show that*

$$\begin{aligned} Da(x,y) \mid (f) &= D^{*}a(x,y) \mid (f) \\ &= \{d \in R^m : \left\langle \frac{\partial b(x,y)}{\partial x}, f \right\rangle + \left\langle \frac{\partial b(x,y)}{\partial y}, d \right\rangle \leq 0\} \end{aligned}$$

where the symbol $\langle \cdot,\cdot \rangle$ *denotes the inner product.*

Remark 3 *Now suppose that the set valued map* $a(\cdot)$ *is given as*

$$a(x) = \{y \in R^m : \min_{i \in I} \max_{j \in J} b_{ij}(x,y) \leq 0\} \tag{2.2}$$

where I *and* J *are finite sets and* $b_{ij}(\cdot,\cdot)$ *is a continuous differentiable functions for all* $i \in I$ *and for all* $j \in J$*. Then (see Demyanov and Vasilyev, 1981)* $b(x,y) = \min\limits_{i \in I} \max\limits_{j \in J} b_{ij}(x,y)$ *is a directional derivable function and*

$$\frac{\partial b(x,y)}{\partial(f,d)} = \min_{i \in I_*(x,y)} \max_{j \in J_*(x,y)} [\left\langle \frac{\partial b_{ij}(x,y)}{\partial x}, f \right\rangle + \left\langle \frac{\partial b_{ij}(x,y)}{\partial y}, d \right\rangle]$$

where

$$J_*(x,y) = \{j_* \in J : \ b_{ij_*}(x,y) = \max_{j\in J} b_{ij}(x,y)\},$$
$$I_*(x,y) = \{i_* \in I : \ \min_{i\in I} \ \max_{j\in J} b_{ij}(x,y) = \max_{j\in J} b_{i_*j}(x,y)\}.$$

In that case, it follows from here that

$$\begin{aligned} &E^-(x,y) \mid (f) \\ &= H^-(x,y) \mid (f) \\ &= \{d \in R^m : \min_{i\in I_*(x,y)} \max_{j\in J_*(x,y)} [\left\langle \frac{\partial b_{ij}(x,y)}{\partial x}, f \right\rangle + \left\langle \frac{\partial b_{ij}(x,y)}{\partial y}, d \right\rangle] < 0\}, \end{aligned}$$

$$\begin{aligned} &E(x,y) \mid (f) \\ &= H(x,y) \mid (f) \\ &= \{d \in R^m : \min_{i\in I_*(x,y)} \max_{j\in J_*(x,y)} [\left\langle \frac{\partial b_{ij}(x,y)}{\partial x}, f \right\rangle + \left\langle \frac{\partial b_{ij}(x,y)}{\partial y}, d \right\rangle] \leq 0\}. \end{aligned}$$

Theorem 4 *Let the set valued map $a(\cdot)$ be in the form (2.2), $(x,y) \in \partial A$, $f \in R^n$ and $H^-(x,y) \mid (f) \neq \emptyset$. Then*

$$Da(x,y) \mid (f) = D^*a(x,y) \mid (f) = H(x,y) \mid (f).$$

Proof : It is obtained by using the previous proposition, the previous corollary and the previous remark. ■

Remark 5 *Above theorem is not true when $H^-(x,y) \mid (f) = \emptyset$ for $(x,y) \in \partial A$ and for $f \in R^n$.*

Example 6 *We take the set valued map $a(\cdot) : [0,1] \to cl(\mathbb{R}^2)$, $x \to a(x) = \{(y_1,y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 0\}$. We know that $a(x) = \{(0,0)\}$ for all $x \in [0,1]$ and $b(\cdot,\cdot,\cdot) : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $(x,y_1,y_2) \to b(x,y_1,y_2) = y_1^2 + y_2^2$ is a differentiable function. Then we obtain $H(x,0,0) \mid (1) = \mathbb{R}^2$, $H^-(x,0,0) \mid (1) = \emptyset$ and $Da(x,0,0) \mid (1) = \{(0,0)\}$ for $(x,0,0) \in \partial A$.*

3 Directional differentiability of the max-min function

Let $a(\cdot) : R^n \to comp(R^m)$, $b(\cdot) : R^n \to comp(R^k)$ be set valued maps and $\sigma(\cdot,\cdot,\cdot) : R^n \times R^m \times R^k \to R$ be a continuous function on $R^n \times R^m \times R^k$. The max-min function is denoted by $m(\cdot)$ and is defined by

$$m(x) = \max_{y\in a(x)} \min_{z\in b(x)} \sigma(x,y,z).$$

Here and after we will assume that $a(\cdot) : R^n \to comp(R^m)$, $b(\cdot) : R^n \to comp(R^k)$ are continuous set valued maps and $\sigma(\cdot,\cdot,\cdot) : R^n \times R^m \times R^k \to R$ is a continuous function on $R^n \times R^m \times R^k$ and locally Lipschitz on $R^m \times R^k$, i. e. for every bounded $D \subset R^n \times R^m \times R^k$, there exists $L(D) > 0$ such that

$$|\sigma(x, y_1, z_1) - \sigma(x, y_2, z_2)| \leq L(D). \|(y_1 - y_2, z_1 - z_2)\|$$

for any (x, y_1, z_1), $(x, y_2, z_2) \in D$. Under these conditions $m(\cdot)$ is a continuous function (see, for example Aubin and Frankowska, 1990). Let

$$Y_*(x) = \{(y_*, z_*) \in a(x) \times b(x) : m(x) = \max_{y \in a(x)} \min_{z \in b(x)} \sigma(x, y, z) = \sigma(x, y_*, z_*)\}.$$

$x \to Y_*(x)$ is an upper semicontinuous set valued map and it is called max-min set valued map. Now we give a characterization of the upper and lower directional derivatives of $m(\cdot)$.

Proposition 7 *For all $x \in R^n$ and $f \in R^n$*

$$\frac{\partial^- m(x)}{\partial f} \leq \inf_{(y,z) \in Y_*(x)} \inf_{(d,n) \in DY_*(x,y,z)|(f)} \frac{\partial^+ \sigma(x, y, z)}{\partial(f, d, n)}, \tag{3.1}$$

$$\frac{\partial^+ m(x)}{\partial f} \leq \inf_{(y,z) \in Y_*(x)} \inf_{(d,n) \in D^* Y_*(x,y,z)|(f)} \frac{\partial^+ \sigma(x, y, z)}{\partial(f, d, n)}. \tag{3.2}$$

Proof : Let $(y, z) \in Y_*(x)$. Let $DY_*(x, y, z) \mid (f) = \emptyset$. Then

$$\inf_{(d,n) \in DY_*(x,y,z)|(f)} \frac{\partial^+ \sigma(x, y, z)}{\partial(f, d, n)} = +\infty$$

and the inequality (3.1) holds. Now let $(y, z) \in Y_*(x)$, $DY_*(x, y, z) \mid (f) \neq \emptyset$. Choose arbitrary $(d, n) \in DY_*(x, y, z) \mid (f)$. Then from the definition of $DY_*(x, y, z) \mid (f)$, there exists a sequence $(y_k, z_k) \in Y_*(x + \delta_k f)$, where $\delta_k \to +0$ as $k \to \infty$, such that

$$(y_k, z_k) = (y, z) + \delta_k(d, n) + (o_1(\delta_k), o_2(\delta_k))$$

where $\|(o_1(\delta_k), o_2(\delta_k))\| / \delta_k \to 0$ as $k \to \infty$. Since $(y, z) \in Y_*(x)$, it follows that $m(x) = \sigma(x, y, z)$ and $(y_k, z_k) \in Y_*(x + \delta_k f)$ $(k = 1, 2, ...)$ then it follows that $m(x + \delta_k f) = \sigma(x + \delta_k f, y_k, z_k)$. Consequently

$$\begin{aligned}
&\frac{\partial^- m(x)}{\partial f} \\
&= \liminf_{\delta \to +0} \; [m(x + \delta f) - m(x)] \, \delta^{-1} \\
&\leq \liminf_{k \to \infty} \; [\sigma(x + \delta_k f, y_k, z_k) - \sigma(x, y, z)] \, \delta_k^{-1} \\
&= \liminf_{k \to \infty} \; [\sigma(x + \delta_k f, y + \delta_k d + o_1(\delta_k), z + \delta_k n + o_2(\delta_k)) - \sigma(x, y, z)] \, \delta_k^{-1} \\
&\leq \liminf_{\delta \to +0} \; [\sigma(x + \delta_k f, y + \delta_k d, z + \delta_k n) - \sigma(x, y, z)] \, \delta_k^{-1} \\
&\leq \limsup_{\delta \to +0} \; [\sigma(x + \delta f, y + \delta d, z + \delta n) - \sigma(x, y, z)] \, \delta^{-1} \\
&= \frac{\partial^+ \sigma(x, y, z)}{\partial(f, d, n)}.
\end{aligned}$$

So we have $\dfrac{\partial^- m(x)}{\partial f} \le \dfrac{\partial^+ \sigma(x,y,z)}{\partial(f,d,n)}$ for any $(d,n) \in DY_*(x,y,z) \mid (f)$ and consequently we obtain the inequality (3.1).

Let us prove (3.2). Let $(y,z) \in Y_*(x)$. Let $D^*Y_*(x,y,z) \mid (f) = \emptyset$. Then

$$\inf_{(d,n)\in D^*Y_*(x,y,z)|(f)} \frac{\partial^+ \sigma(x,y,z)}{\partial(f,d,n)} = +\infty$$

and the inequality (3.2) holds.

Now let $(y,z) \in Y_*(x)$, $D^*Y_*(x,y,z) \mid (f) \neq \emptyset$. Choose arbitrary $(d,n) \in D^*Y_*(x,y,z) \mid (f)$. From the definition of $D^*Y_*(x,y,z) \mid (f)$, there exists a $\delta_* > 0$ such that for all $\delta \in [0,\delta_*]$

$$(y(\delta), z(\delta)) = (y,z) + \delta(d,n) + (o_1(\delta), o_2(\delta)) \in Y_*(x+\delta f)$$

where $\|(o_1(\delta), o_2(\delta))\| / \delta \to 0$ as $\delta \to +0$. Since $(y,z) \in Y_*(x)$ then it follows that $m(x) = \sigma(x,y,z)$ and $(y(\delta), z(\delta)) \in Y_*(x+\delta f)$ then it follows that $m(x+\delta f) = \sigma(x+\delta f, y(\delta), z(\delta))$ for any $\delta \in [0,\delta_*]$. Then

$$\begin{aligned}
&\frac{\partial^+ m(x)}{\partial f} = \limsup_{\delta\to+0}\ [m(x+\delta f) - m(x)]\,\delta^{-1}\\
&= \limsup_{\delta\to+0}\ [\sigma(x+\delta f, y(\delta), z(\delta)) - \sigma(x,y,z)]\,\delta^{-1}\\
&= \limsup_{\delta\to+0}\ [\sigma(x+\delta f, y+\delta d + o_1(\delta), z+\delta n + o_2(\delta)) - \sigma(x,y,z)]\,\delta^{-1}\\
&\le \limsup_{\delta\to+0}\ [\sigma(x+\delta f, y+\delta d, z+\delta n) - \sigma(x,y,z)]\,\delta^{-1} = \frac{\partial^+ \sigma(x,y,z)}{\partial(f,d,n)}.
\end{aligned}$$

Hence $\dfrac{\partial^+ m(x)}{\partial f} \le \dfrac{\partial^+ \sigma(x,y,z)}{\partial(f,d,n)}$ for any $(d,n) \in D^*Y_*(x,y,z) \mid (f)$, we obtain the inequality (3.2). ■

Proposition 8 *Let $x \in R^n$, $f \in R^n$ and there exists $(y_*, z_*) \in Y_*(x)$ such that $DY_*(x,y_*,z_*) \mid (f) \neq \emptyset$. Then*

$$\frac{\partial^+ m(x)}{\partial f} \ge \inf_{(y,z)\in Y_*(x)}\ \inf_{(d,n)\in DY_*(x,y,z)|(f)} \frac{\partial^- \sigma(x,y,z)}{\partial(f,d,n)} \tag{3.3}$$

Moreover if there exists $(y^*, z^*) \in Y_*(x)$ such that $D^*Y_*(x,y^*,z^*) \mid (f) \neq \emptyset$ then

$$\frac{\partial^- m(x)}{\partial f} \ge \inf_{(y,z)\in Y_*(x)}\ \inf_{(d,n)\in D^*Y_*(x,y,z)|(f)} \frac{\partial^- \sigma(x,y,z)}{\partial(f,d,n)} \tag{3.4}$$

Proof : Take any $(d,n) \in DY_*(x,y_*,z_*) \mid (f)$. From the definition of $DY_*(x,y_*,z_*) \mid (f)$, there exists a sequence $(y_k, z_k) \in Y_*(x+\delta_k f)$, where $\delta_k \to +0$ as $k \to \infty$, such that

$$(y_k, z_k) = (y_*, z_*) + \delta_k(d,n) + (o_1(\delta_k), o_2(\delta_k))$$

where $\|(o_1(\delta_k), o_2(\delta_k))\| / \delta_k \to 0$ as $k \to \infty$. Since $(y_*, z_*) \in Y_*(x)$, it follows that $m(x) = \sigma(x, y_*, z_*)$ and $(y_k, z_k) \in Y_*(x + \delta_k f)$ $(k = 1, 2, ...)$ then it follows that $m(x + \delta_k f) = \sigma(x + \delta_k f, y_k, z_k)$. Consequently

$$\begin{aligned}
&\frac{\partial^+ m(x)}{\partial f} \\
&= \limsup_{\delta \to +0} \, [m(x + \delta f) - m(x)] \, \delta^{-1} \\
&\geq \limsup_{k \to \infty} \, [\sigma(x + \delta_k f, y_k, z_k) - \sigma(x, y_*, z_*)] \, \delta_k^{-1} \\
&= \limsup_{k \to \infty} \, [\sigma(x + \delta_k f, y_* + \delta_k d + o_1(\delta_k), z_* + \delta_k n + o_2(\delta_k)) - \sigma(x, y_*, z_*)] \, \delta_k^{-1} \\
&\geq \liminf_{\delta \to +0} \, [\sigma(x + \delta f, y_* + \delta d, z_* + \delta n) - \sigma(x, y_*, z_*)] \, \delta_k^{-1} \\
&= \frac{\partial^- \sigma(x, y_*, z_*)}{\partial (f, d, n)}.
\end{aligned}$$

So we have $\dfrac{\partial^+ m(x)}{\partial f} \geq \dfrac{\partial^- \sigma(x, y_*, z_*)}{\partial (f, d, n)}$ for any $(d, n) \in DY_*(x, y_*, z_*) \mid (f)$ and consequently we obtain the inequality (3.3).

Let us prove (3.4). Take any $(d, n) \in D^* Y_*(x, y^*, z^*) \mid (f)$. From the definition of $D^* Y_*(x, y^*, z^*) \mid (f)$, there exists a $\delta_* > 0$ such that for all $\delta \in [0, \delta_*]$

$$(y(\delta), z(\delta)) = (y^*, z^*) + \delta(d, n) + (o_1(\delta), o_2(\delta)) \in Y_*(x + \delta f)$$

where $\|(o_1(\delta), o_2(\delta))\| / \delta \to 0$ as $\delta \to +0$. Since $(y^*, z^*) \in Y_*(x)$ then it follows that $m(x) = \sigma(x, y^*, z^*)$ and $(y(\delta), z(\delta)) \in Y_*(x + \delta f)$ then it follows that $m(x + \delta f) = \sigma(x + \delta f, y(\delta), z(\delta))$ for any $\delta \in [0, \delta_*]$. Then

$$\begin{aligned}
&\frac{\partial^- m(x)}{\partial f} \\
&= \liminf_{\delta \to +0} \, [m(x + \delta f) - m(x)] \, \delta^{-1} \\
&= \liminf_{\delta \to +0} \, [\sigma(x + \delta f, y(\delta), z(\delta)) - \sigma(x, y^*, z^*)] \, \delta^{-1} \\
&= \liminf_{\delta \to +0} \, [\sigma(x + \delta f, y^* + \delta d + o_1(\delta), z^* + \delta n + o_2(\delta)) - \sigma(x, y^*, z^*)] \, \delta^{-1} \\
&\geq \liminf_{\delta \to +0} \, [\sigma(x + \delta f, y^* + \delta d, z^* + \delta n) - \sigma(x, y^*, z^*)] \, \delta^{-1} \\
&= \frac{\partial^- \sigma(x, y^*, z^*)}{\partial (f, d, n)}.
\end{aligned}$$

Hence $\dfrac{\partial^- m(x)}{\partial f} \geq \dfrac{\partial^- \sigma(x, y^*, z^*)}{\partial (f, d, n)}$ for any $(d, n) \in D^* Y_*(x, y^*, z^*) \mid (f)$, we obtain the inequality (3.4). ■

From Proposition 2 and Proposition 3 we have the following statement.

Theorem 9 *Suppose that* $x \in R^n$, $f \in R^n$ *and there exists* $(y_*, z_*) \in Y_*(x)$ *such that* $D^* Y_*(x, y_*, z_*) \mid (f) \neq \emptyset$. *Let* $\sigma(\cdot, \cdot, \cdot) : R^n \times R^m \times R^k \to R$ *is a* *differentiable function at* (x, y, z) *in the direction* (f, d, n) *for any* $(y, z) \in Y_*(x)$, $d \in R^m$ *and* $n \in R^k$. *Then* $m(\cdot) : R^n \to R$ *is differentiable at* x *in the direction* f *and*

$$\frac{\partial m(x)}{\partial f} = \inf_{(y,z)\in Y_*(x)} \inf_{(d,n)\in DY_*(x,y,z)|(f)} \frac{\partial \sigma(x, y, z)}{\partial (f, d, n)}$$

4 Conclusions

By using the concepts of the directional lower and upper derivative sets of the max-min set valued map, the directional lower and upper derivatives of the max-min function are investigated. The results of this paper can be employed to calculate the directional lower and upper derivatives of the max-min functions in the control theory problems, the differential game problems and the parametric optimization problems. Sufficient conditions ensuring the existence of the directional derivative of the max-min function are obtained.

5 References

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