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Generalized Napoleon and Torricelli Transformations and Their Iterations

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Abstract. For given triangles $T = (A, B, C)$ and $D = (X, Y, Z)$, the D-Napoleon and D-Torricelli triangles $\mathcal{N}_D(T)$ and $\mathcal{T}_D(T)$ of a triangle $T = (A, B, C)$ are the triangles $A'B'C'$ and $A^*B^*C^*$, where ABC', BCA', CAB', A^*BC, AB^*C, ABC^* are similar to D. In this paper it is shown that the iteration $\mathcal{N}_D^n(T)$ either terminates or converges (in shape) to an equilateral triangle, and that the iteration $\mathcal{T}_D^n(T)$ either terminates or converges to a triangle whose shape depends only on D. It is also shown that if A°, B°, C°, $A^\circledcirc$, $B^\circledcirc$, $C^\circledcirc$ are the centroids of the triangles ABC', BCA', CAB', A^*BC, AB^*C, ABC^*, respectively, then the shape of $A^\circ B^\circ C^\circ$ depends on both shapes of T and D, while the shape of $A^\circledcirc B^\circledcirc C^\circledcirc$ depends only on that of D and, unexpectedly, equals the limiting shape of the iteration $\mathcal{T}_D^n(T)$.

Keywords: centroids, (plane of) complex numbers, Fermat-Torricelli point, generalized Napoleon configuration, generalized Napoleon triangle, generalized Torricelli configuration, generalized Torricelli triangle, Möbius transformation, shape convergence, shape function, similar triangles, smoothing iteration

1. Introduction

The configuration that arises from erecting equilateral (similarly oriented) triangles (A_1, B, C), (A, B_1, C), (A, B, C_1) on the sides of a given triangle $T =$

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(A, B, C), as shown in Figure 1, is quite well known as it appears in at least the following three contexts:

(a-1) It is the configuration that Torricelli used in solving Fermat's problem regarding the point P that minimizes the sum $|PA|+|PB|+|PC|$ of distances from the vertices of triangle $T = (A, B, C)$. Torricelli proved that the lines AA_1, BB_1, CC_1 are concurrent and that the point P of concurrence (when no angle of T exceeds $120°$) is the point that solves the problem. The point P is called the *Fermat-Torricelli point* of T; see [3, pp. 21–22], [20], and [1, Chapter 2].

(a-2) It is the configuration that underlies the celebrated Napoleon's theorem stating that the centers of the erected triangles form an equilateral triangle; see for example [15] and [24].

(a-3) If we set $\mathcal{T}(A, B, C)=(A_1, B_1, C_1)$, and apply $\mathcal{T}$ to (A_1, B_1, C_1), and so on, then the sequence $\mathcal{T}_D^n(A, B, C)$ that we obtain converges in shape, as proved in [15], to an equilateral triangle.

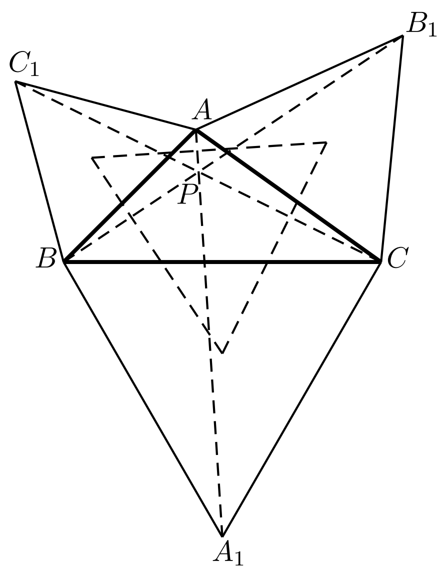

Figure 1. A Torricelli (or Napoleon) configuration in which ABC_1, BCA_1, CAB_1 are equilateral

A generalized configuration that has attracted a great deal of attention is shown in Figure 2, where the erected triangles (A^*, B, C), (A, B^*, C), (A, B, C^*) are (directly) similar to another given triangle $D = (X, Y, Z)$. Amazingly, the three results (1-a), (1-b), (1-c) generalize quite beautifully to this configuration. No matter what D is, we have the following:

(b-1) The lines AA^*, BB^*, CC^* are concurrent and the point P_D of concurrence is the point that minimizes the weighted sum $\alpha|PA| + \beta|PB| + \gamma|PC|$, where α, β, γ are the side lengths of D; see [19, §356, Theorem, pp. 222–223], [8], [31], [30], [2], [4], [9], [10], [11]. The point P_D is known as the *generalized*

Fermat-Torricelli point of T corresponding to the weights α, β, γ. For this to be valid, the weights α, β, γ must be realizable as the side lengths of a triangle (X, Y, Z) and, in addition, the angles X, Y, Z must satisfy the conditions $A + X,\ B + Y,\ C + Z < 180°$.

(b-2) The triangle formed by the circumcenters of the erected triangles is similar to D, and thus is equilateral when D is; see [27], [28], [29], [24].

(b-2') The triangle D_0 formed by the centroids of the erected triangles has a shape that depends only on D; see (24) of Theorem 6.2.

(b-3) The sequence $\mathcal{T}_D^n(A, B, C)$ either terminates or converges in shape to a triangle D_∞ whose shape depends only on D; see Theorem 6.3.

(b-4) It must come as a pleasant surprise to know that D_0 and D_∞ (of (b-2') and (b-3)) are similar. In fact, each is a triangle whose side lengths are proportional to $xu : yv : zw$, where x, y, z are the side lengths of D and u, v, w are the lengths of the medians of D; see Theorems 6.2, 6.3, and 3.3.

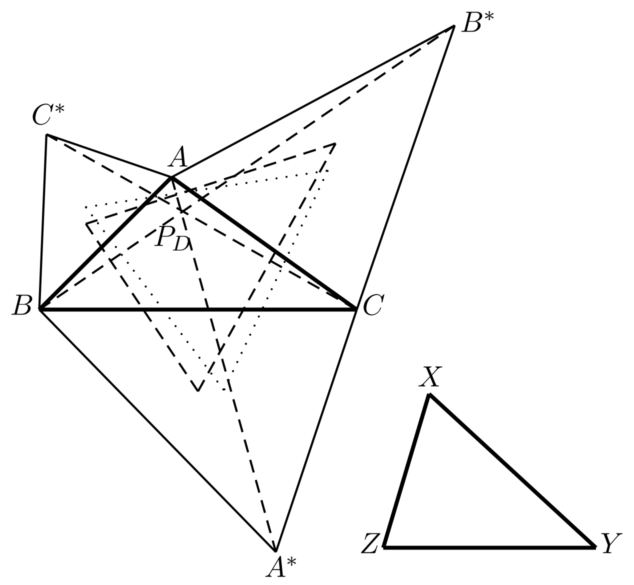

Figure 2. The Torricelli configuration, where A^*BC, AB^*C, ABC^* are similar to XYZ

In view of the above, the special role played by the circumcenter as manifested in (b-2) is now counterparted by a role of the centroid that, in view of (b-2'), (b-3), (b-4), is even more special. Also, the relation between the shapes of D_0 and D_∞ and that of D (as described in (b-4)) seems to be too intricate to be captured by geometric methods. Thus we do not expect the results above to have purely geometric proofs.

Another popular configuration arises when the erected triangles (A, B, C'), (B, C, A'), (C, A, B') (and not (A, B, C'), (A', B, C), (A, B', C)) are required to be similar to D; see Figure 3 where the same T and D of Figure 2 are used. This configuration

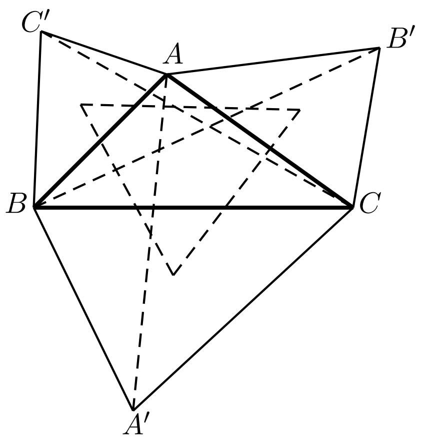

Figure 3. The Napoleon configuration, where ABC', BCA', CAB' are similar to XYZ

coincides with the previous one when D is equilateral. In this configuration, we have the following:

(c-1) The lines AA', BB', CC' are concurrent if and only if $D = (X, Y, Z)$ is isosceles at Z, i.e., has $ZX = ZY$; see [7, §D.11, p. 101] and [13]. It is worth mentioning that the point P_D of concurrence traces, as D ranges over all triangles (X, Y, Z) that are isosceles at Z, what is known as the Kiepert hyperbola; see [19, §357, Theorem, p. 223], [5], [6], and [24].

(c-2) The triangle formed by the centroids of the erected triangles has a shape that depends on both D and T; see (13) of Theorem 5.2.

(c-3) If we set $\mathcal{N}_D(A, B, C)=(A', B', C')$, and apply $\mathcal{N}_D$ to (A', B', C'), and so on, then the sequence $\mathcal{N}_D^n(A, B, C)$ that we obtain either terminates or converges in shape to an equilateral triangle; see Theorem 5.3.

(c-4) $\mathcal{N}_{D_1}$ and $\mathcal{N}_{D_2}$ commute for all permissible D_1 and D_2.

The results in (b-3) and (c-3) generalize the results in [15] (where $\mathcal{T}_D^n = \mathcal{N}_D^n$ is investigated for equilateral D), and [14] (where $\mathcal{N}_D^n$ is investigated for degenerate D) and they supplement the results in [33] (where $\mathcal{N}_D^n$ is investigated for general D). The result in (c-4) is proved in [32] in the special case when D_1 and D_2 are isosceles. As for (b-2), (b-2'), and (c-2), they raise the question of what happens if other centers are considered. This issue awaits further investigation.

Remark 1.1. The Torricelli and Napoleon configurations above do not have direct analogues in higher dimensions since it is not possible to erect similar d-simplices on the facets of an arbitrary d-simplex even when $d = 3$. However, some of the properties of these configurations that are listed above can be reformulated in such a way that they hold in higher dimensions. This is already done for some of these properties in [25] and [16], where interesting results are obtained. One hopes that the same can be done for the remaining properties.

2. Basic definitions

We adhere to the notation used in [17] and [15]. Thus we identify the Euclidean plane with the plane $\mathbb{C}$ of complex numbers, and we define a *triangle* to be any ordered triple (A, B, C) of points in $\mathbb{C}$. A triangle (A, B, C) is said to be *degenerate* if A, B, C are collinear, and is said to be *trivial* if $A = B = C$. We avoid denoting a triangle (A, B, C) by ABC and we reserve ABC to stand for the product of the complex numbers A, B, C. Similarly, we let (A, B) denote the line segment that joins A and B.

If a triangle (A, B, C) is non-degenerate, then we call it *negatively* (respectively, *positively*) oriented if the movement $A \mapsto B \mapsto C \mapsto A$ is clockwise (respectively, counter-clockwise). If (A, B, C) is degenerate, then we can simply declare that it does not have a well-defined orientation. However, it may be convenient to think of a degenerate triangle (A, B, C) as having a copy that is positively oriented and another copy that is negatively oriented. These two triangles have the same vertex sequence (A, B, C) and are referred to as the positively and negatively oriented triangles (A, B, C). If needed, one may denote them by $(A, B, C)^+$ and $(A, B, C)^-$, respectively.

Two non-trivial triangles (A, B, C) and (A', B', C') are said to be *directly similar* or simply *similar* if they have the same orientation and if

$$|A - B| : |B - C| : |C - A| = |A' - B'| : |B' - C'| : |C' - A'|.$$

Here $|A - B|$ denotes the length of the line segment (A, B).

3. The shape function ϕ

A *shape function* is any function Ψ from the set of non-trivial triangles to the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ having the property that

$$\Psi(T) = \Psi(T') \iff T \text{ and } T' \text{ are similar.}$$

June Lester's shape function S (introduced and investigated in [21], [22], [23]) is defined by

$$S(A, B, C) = \frac{A - B}{A - C},$$

where the right-hand side is ∞ if $C = A \neq B$. The shape function ϕ defined by

$$\phi(A, B, C) = \frac{A + \zeta B + \zeta^2 C}{A + \zeta^2 B + \zeta C}, \tag{1}$$

where $\zeta = e^{2\pi i/3}$, was introduced in [15], where it was seen to be more appropriate for handling the Napoleon transformation. These two shape functions are related by the linear fractional relations

$$\phi = \frac{1 + \zeta^2 S}{\zeta^2 + S}, \qquad S = \frac{\zeta^2 \phi - 1}{\zeta^2 - \phi},$$

and some of their properties and correspondences between them are given in Table 1 in [15]. This table is reproduced, after some minor corrections and additions, later in this paper. Specifically, corrections are made on rows 2, 4, 5, 6, 7, 9, and rows 11–16 are added. These are justified by Theorem 3.1 below and by the fact that

$$\phi(B, C, A) = \zeta\ \phi(A, B, C).$$

For simplicity, triangle (A, B, C) is referred to in the table as triangle ABC.

Remark 3.1. We shall freely use the obvious fact that if f is a shape function and g is a bijection on the extended complex plane $\mathbb{C}_\infty$, then the composition gf is also a shape function. This is true in particular if g is a Möbius transformation. For properties of Möbius transformations, one may consult [12, Chapter 9], [26, Chapter III.6], or any book on complex analysis.

1	ABC and UVW are similar.	$S(ABC) = S(UVW)$.	$\phi(ABC) = \phi(UVW)$.
2	ABC and UVW are anti-similar.	$S(ABC) = \overline{S(UVW)}$.	$\phi(ABC)\ \overline{\phi(UVW)} = 1$.
3	ABC is degenerate.	$S(ABC)$ is real.	$\Vert\phi(ABC)\Vert = 1$.
4	ABC is non-degenerate and positively oriented.	$\mathrm{Im}(S) < 0$.	$\Vert\phi(ABC)\Vert < 1$.
5	ABC is non-degenerate and negatively oriented.	$\mathrm{Im}(S) > 0$.	$\Vert\phi(ABC)\Vert > 1$.
6 7 8	$A = C$. $A = B$. $B = C$.	$S(ABC) = \infty$. $S(ABC) = 0$. $S(ABC) = 1$.	$\phi(ABC) = \zeta^2$. $\phi(ABC) = \zeta$. $\phi(ABC) = 1$.
9	ABC is degenerate with $AB = AC$ and $B \neq C$.	$S(ABC) = -1$.	$\phi(ABC) = -1$.
10	ABC is equilateral.	$S(ABC) = -\zeta$ or $-\zeta^2$.	$\phi(ABC) = 0$ or ∞.
11 12 13	$AB = BC$ and $\angle ABC = 120°$. $AB = AC$ and $\angle BAC = 120°$. $CA = CB$ and $\angle BCA = 120°$.	$S(ABC) = (1-\zeta^2)/3$ or $(1-\zeta)/3$. $S(ABC) = \zeta$ or ζ^2. $S(ABC) = 3/(1-\zeta)$ or $3/(1-\zeta^2)$.	$\phi(ABC) = -2\zeta^2$ or $-\zeta^2/2$. $\phi(ABC) = -2$ or $-1/2$. $\phi(ABC) = -2\zeta$ or $-\zeta/2$.
14 15 16	$AB = AC$. $BC = BA$. $CA = CB$.	$\Vert S(ABC)\Vert = 1$. $\Vert S(ABC)\Vert = \Vert 1 - S(ABC)\Vert$. $\Vert 1 - S(ABC)\Vert = 1$.	$\phi(ABC)$ is real. $\phi(ABC)/\zeta^2$ is real. $\phi(ABC)/\zeta$ is real.

Table 1.

Theorem 3.1. *Let ϕ be the shape function defined in* (1)*, and let (A, B, C) be a non-trivial triangle. Then*

$$\phi(A, B, C) = -\zeta^2/2 \iff \textit{ABC is positively oriented and isosceles with}$$

$$\angle ABC = 120^\circ$$
$$\phi(A,B,C) \in \mathbb{R} \iff ABC \textit{ is isosceles with } AB = AC.$$

Proof. To prove the first statement, apply (1) to the triangle $(A,B,C) = (-3,3,-i\sqrt{3}) = (-3,3,\zeta^2-\zeta)$. This triangle is clearly isosceles with $B = 120^\circ$ and positively oriented.

For the second statement, we may assume $A = 0$. Then

$$\begin{aligned}\phi(A,B,C) \in \mathbb{R} &\iff \frac{\zeta B + \zeta^2 C}{\zeta^2 B + \zeta C} = \frac{\zeta^2 \overline{B} + \zeta \overline{C}}{\zeta \overline{B} + \zeta^2 \overline{C}} \\ &\iff (\zeta^2 - \zeta) B\overline{B} = (\zeta^2 - \zeta) C\overline{C} \iff \|B\| = \|C\|,\end{aligned}$$

as desired. □

Remark 3.2. For a triangle (A,B,C), we define K, L, M by

$$3K = A + B + C,\ 3L = A + \zeta B + \zeta^2 C,\ 3M = A + \zeta^2 B + \zeta C. \tag{2}$$

If (A,B,C) is not trivial, then its shape is given by

$$\phi(A,B,C) = \frac{L}{M}.$$

It is also easy to see that

$$(A,B,C) \text{ is trivial } \text{ (i.e., } A = B = C) \iff L = M = 0. \tag{3}$$

The next theorem shows how to calculate the shape $\phi(A,B,C)$ from certain linear dependence relations among A, B, C. It will be used in the next section.

Theorem 3.2. *Let (A,B,C) and (α,β,γ) be non-trivial triangles and suppose that $\alpha+\beta+\gamma = 0$. Then*

$$\begin{aligned}\phi(\alpha,\beta,\gamma) &= \frac{-\zeta^2(\alpha - \zeta^2\beta)}{\alpha - \zeta\beta} \\ \phi(A,B,C) = -\phi(\alpha,\beta,\gamma) &\iff \alpha A + \beta B + \gamma C = 0.\end{aligned}$$

Here $-\infty$ is understood to be ∞.

Proof.

$$\begin{aligned}\phi(\alpha,\beta,\gamma) &= \frac{\alpha + \zeta\beta + \zeta^2\gamma}{\alpha + \zeta^2\beta + \zeta\gamma} = \frac{\alpha + \zeta\beta + \zeta^2(-\alpha-\beta)}{\alpha + \zeta^2\beta + \zeta(-\alpha-\beta)} \\ &= \frac{(1-\zeta^2)\alpha + (\zeta - \zeta^2)\beta}{(1-\zeta)\alpha + (\zeta^2 - \zeta)\beta} = \frac{-\zeta^2\alpha + \zeta\beta}{\alpha - \zeta\beta} = \frac{-\zeta^2(\alpha - \zeta^2\beta)}{\alpha - \zeta\beta},\end{aligned}$$

as claimed in the first statement.

To prove the second statement, let K, L, M be as in (2). Then

$$A = K + L + M \ , \quad B = K + \zeta^2 L + \zeta M \ , \quad C = K + \zeta L + \zeta^2 M.$$

Plugging these in $\alpha A + \beta B + \gamma C$ and using $\alpha + \beta + \gamma = 0$, we obtain

$$\alpha A + \beta B + \gamma C \quad = \quad (\alpha + \zeta^2 \beta + \zeta\gamma)L + (\alpha + \zeta\beta + \zeta^2\gamma)M.$$

If $M = L = 0$, then we obtain the contradiction $A = B = C$. If $M = 0$ and $L \neq 0$, i.e., if $\phi(A, B, C) = \infty$, then

$$\begin{aligned}
\alpha A + \beta B + \gamma C = 0 &\iff (\alpha + \zeta^2\beta + \zeta\gamma)L + (\alpha + \zeta\beta + \zeta^2\gamma)M = 0 \\
&\iff \alpha + \zeta^2\beta + \zeta\gamma = 0 \\
&\iff \phi(\alpha, \beta, \gamma) = \infty.
\end{aligned}$$

Thus $\phi(A, B, C) = \infty \iff \phi(\alpha, \beta, \gamma) = \infty$. If $M \neq 0$, then

$$\begin{aligned}
\alpha A + \beta B + \gamma C = 0 &\iff (\alpha + \zeta^2\beta + \zeta\gamma)L + (\alpha + \zeta\beta + \zeta^2\gamma)M = 0 \\
&\iff \frac{L}{M} = \frac{-(\alpha + \zeta\beta + \zeta^2\gamma)}{\alpha + \zeta^2\beta + \zeta\gamma} \\
&\iff \phi(A, B, C) = -\phi(\alpha, \beta, \gamma).
\end{aligned}$$

This completes the proof. □

Given a triangle D with $\phi(D) = t$, the triangle D_∞ with $\phi(T_\infty) = t^{-2}$ will play a special role in Section 6. It is the triangle formed by the centroids of the erected triangles in Figure 2, and it is also the limit of $\mathcal{T}_D^n(T)$ for all T. In view of this, a geometric description of D_∞ in terms of D would be desirable. This is done in Theorem 3.3 below.

Theorem 3.3. *Let D and D_∞ be triangles with $\phi(D_\infty) = (\phi(D))^{-2}$. If x, y, z are the side lengths of D, and u, v, w are the lengths of the medians of D (in the standard order), then the side lengths of D_∞ are proportional to $xu : yv : zw$.*

Proof. Let $\phi(D) = t$, and assume that $D = (X, Y, Z)$. Then

$$\begin{aligned}
\phi(D_\infty) \quad = \quad t^{-2} \quad &= \quad \left(\frac{X + \zeta Y + \zeta^2 Z}{X + \zeta^2 Y + \zeta Z}\right)^{-2} \\
&= \quad \frac{(X^2 + 2YZ) + \zeta(Y^2 + 2XZ) + \zeta^2(Z^2 + 2XY)}{(X^2 + 2YZ) + \zeta^2(Y^2 + 2XZ) + \zeta(Z^2 + 2XY)} \\
&= \quad \phi(X^2 2YZ, Y^2 + 2ZX, Z^2 + 2XY).
\end{aligned}$$

The first side length of the triangle $(X^2 + 2YZ, Y^2 + 2ZX, Z^2 + 2XY)$ is given by

$$\|(Z^2 + 2XY) - (Y^2 + 2ZX)\| = 2\ \|Z - Y\| \left\|\frac{Z + Y}{2} - X\right\| = 2xu.$$

Similarly for the other sides. □

Corollary 3.1. *If x, y, z are the side lengths of a triangle D, and if u, v, w are the lengths of the medians of D (in the standard order), then there is a triangle whose side lengths are xu, yv, zw.*

In the context of Corollary 3.1, we recall that the existence and construction of a triangle whose side lengths are u, v, w is quite well known; see for example [19] and [18].

4. General linear transformations of triangles

Throughout this paper, we let $\zeta = e^{2i\pi/3}$. We also define $\mathcal{P}$ and $\mathcal{R}$ (and record $\mathcal{P}^{-1}$) as follows:

$$\mathcal{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{bmatrix}, \quad \mathcal{P}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{bmatrix}, \quad \mathcal{R} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{4}$$

Theorem 4.1. *Let A, B, C be complex numbers and let*

$$[A' \quad B' \quad C'] = [A \quad B \quad C]\,\mathcal{M}, \tag{5}$$

where $\mathcal{M}$ be a 3 by 3 matrix with complex entries. Let A°, B°, C°, respectively, be the centroids of triangles (A', B, C), (A, B', C), (A, B, C'). Let

$$\begin{bmatrix} K & L & M \\ K' & L' & M' \\ K^\circ & L^\circ & M^\circ \end{bmatrix} = \begin{bmatrix} A & B & C \\ A' & B' & C' \\ A^\circ & B^\circ & C^\circ \end{bmatrix} \mathcal{P}.$$

Then

$$[K' \quad L' \quad M'] = [K \quad L \quad M]\left(\mathcal{P}^{-1}\mathcal{M}\mathcal{P}\right), \tag{6}$$

$$3\,[K^\circ \quad L^\circ \quad M^\circ] = [K \quad L \quad M]\left(\mathcal{P}^{-1}\mathcal{M}\mathcal{P} + \mathcal{R}\right). \tag{7}$$

Proof. (6) follows from

$$\begin{aligned} [K' \quad L' \quad M'] &= [A' \quad B' \quad C']\,\mathcal{P} = [A \quad B \quad C]\,\mathcal{M}\mathcal{P} \\ &= [A \quad B \quad C]\,\mathcal{P}\left(\mathcal{P}^{-1}\mathcal{M}\mathcal{P}\right) = [K \quad L \quad M]\left(\mathcal{P}^{-1}\mathcal{M}\mathcal{P}\right). \end{aligned}$$

For (7), we have

$$\begin{aligned} 3\,[A^\circ \quad B^\circ \quad C^\circ] &= [A' + B + C \quad A + B' + C \quad A + B + C'] \\ &= [A \;\; B \;\; C]\,\mathcal{H} + [A' \;\; B' \;\; C'], \text{ where } \mathcal{H} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ &= [A \quad B \quad C]\,\mathcal{H} + [A \quad B \quad C]\,\mathcal{M} \\ &= [A \quad B \quad C]\,(\mathcal{H} + \mathcal{M}). \end{aligned}$$

Therefore

$$\begin{aligned} 3\,[K^\circ \quad L^\circ \quad M^\circ] &= 3\,[A^\circ \quad B^\circ \quad C^\circ]\,\mathcal{P} \\ &= [A \quad B \quad C]\,(\mathcal{H}+\mathcal{M})\mathcal{P} \\ &= [A \quad B \quad C]\,\mathcal{P}\mathcal{P}^{-1}(\mathcal{H}+\mathcal{M})\mathcal{P} \\ &= [A \quad B \quad C]\,\mathcal{P}\left(\mathcal{P}^{-1}\mathcal{H}\mathcal{P}+\mathcal{P}^{-1}\mathcal{M}\mathcal{P}\right) \\ &= [K \quad L \quad M]\left(\mathcal{R}+\mathcal{P}^{-1}\mathcal{M}\mathcal{P}\right), \end{aligned}$$

as desired. □

5. The generalized Napoleon transformation $\mathcal{N}_D$

Fix a non-trivial triangle $D = (X, Y, Z)$. For a triangle (A, B, C), we would like to define $\mathcal{N}_D(A, B, C)$ to be the triangle (A', B', C') for which the triangles (A, B, C'), (B, C, A'), (C, A, B') are similar to $D = (X, Y, Z)$; see Figure 3. Theorem 5.1 below shows that we have to restrict ourselves to triangles $D = (X, Y, Z)$ in which $X \neq Y$, and to triangles (A, B, C) in which no two vertices coincide. Note that [15] is concerned with the transformation $\mathcal{N}_D(A, B, C)$ and its iterations in the special case when D is equilateral.

Theorem 5.1. *Let $D = (X, Y, Z)$, $T = (A, B, C)$ be non-trivial triangles, and let $t = \phi(D)$. Then there exist points A', B', C' such that the triangles (A, B, C'), (B, C, A'), (C, A, B') are similar to D if and only if*

(i) $X \neq Y$ (*i.e.*, $t \neq \zeta$), (8)

(ii) *no two vertices of (A, B, C) coincide (i.e., $\phi(T) \notin \{1, \zeta, \zeta^2\}$).* (9)

When these conditions hold, A', B', C' are unique and given by

$$[A' \ B' \ C'] = [A \ B \ C]\mathcal{M}, \ \mathcal{M} = \frac{1}{\zeta(t-\zeta)}\begin{bmatrix} 0 & \zeta(1-\zeta t) & 1-t \\ 1-t & 0 & \zeta(1-\zeta t) \\ \zeta(1-\zeta t) & 1-t & 0 \end{bmatrix}. \tag{10}$$

Proof. The case when D is equilateral is trivial and completely studied in [15], as mentioned earlier. Thus we assume that D is not equilateral, i.e., $t \notin \{0, \infty\}$.

Suppose that there exist triangles (A, B, C'), (B, C, A'), (C, A, B') that are similar to $D = (X, Y, Z)$. We are to prove (i) and (ii).

If $A = B$, then the assumption that (A, B, C') is similar to (B, C, A') would imply that $B = C$, leading to the contradiction $A = B = C$. Thus $A \neq B$. Similarly, $B \neq C$ and $C \neq A$. This proves (ii).

To prove (i), note that if (A, B, C') is non-trivial, then

$$\begin{aligned} (A, B, C') \text{ is similar to } D &\iff \phi(A, B, C') = t \\ &\iff \frac{A + \zeta B + \zeta^2 C'}{A + \zeta^2 B + \zeta C'} = t \\ &\iff \zeta(t-\zeta)C' = (1-t)A + \zeta(1-\zeta t)B. \end{aligned} \tag{11}$$

Thus if $X = Y$, then $t = \zeta$, by row 7 of Table 1, and $(1-\zeta)A + \zeta(1-\zeta^2)B = 0$. Therefore $A = B$, contradicting (ii). Thus $X \neq Y$. This proves (i).

Conversely, suppose that (i) and (ii) are satisfied. Then $t \neq \zeta$ and therefore there exists C' such that $\zeta(t-\zeta)C' = (1-t)A + \zeta(1-\zeta t)B$. For this C', it follows from (11) that either (A, B, C') is trivial or (A, B, C') is similar to (X, Y, Z). By (ii), (A, B, C') is non-trivial and therefore (A, B, C') is similar to (X, Y, Z). Similarly for the other two triangles.

The last statement follows from (11) and analogous statements for A' and B'. □

Definition 5.1. *Let $D = (X, Y, Z)$ be a triangle in which $X \neq Y$. For any triangle $T = (A, B, C)$ with pairwise distinct vertices, the D-Napoleon triangle $\mathcal{N}_D(T)$ of T is defined to be the triangle (A', B', C') where A', B', C' are as given in* (10)*. In other words, (A', B', C') is the triangle for which the triangles (A, B, C'), (B, C, A'), (C, A, B') are similar to D.*

Remark 5.1. Let $D = (X, Y, Z)$ be a triangle in which $X \neq Y$. Theorem 5.2 below expresses $\phi(\mathcal{N}_D(T))$ in terms of $\phi(T)$ for every triangle T with pairwise distinct vertices and thus makes the study of the sequence $\mathcal{N}_D^n(T)$ a simple matter. There are, however, some technicalities that must be taken care of first. First, since ϕ is not defined for a trivial triangle, we must answer the question of when $\mathcal{N}_D(T)$ is trivial. Secondly, we must answer the question of when two vertices of $\mathcal{N}_D(T)$ coincide since in that case $\mathcal{N}_D^2(T)$ will be undefined. Both questions are answered in Theorem 5.2.

Theorem 5.2. *Let $D = (X, Y, Z)$ be a triangle in which $X \neq Y$ and let $T = (A, B, C)$ be a triangle in which no two vertices coincide. Let $\mathcal{N}_D(T) = (A', B', C')$, and let A°, B°, C° be the centroids of triangles (B, C, A'), (C, A, B'), (A, B, C'), respectively. Let $t = \phi(D)$. Then*

$$\begin{aligned}
(A', B', C') \text{ is trivial} &\iff (A, B, C) \text{ and } (X, Y, Z) \text{ have the same orientation,} \\
&\ (A, B, C) \text{ is equilateral, and } (X, Y, Z) \text{ is isosceles} \\
&\ \text{with } \angle YZX = 120^\circ. \qquad (12) \\
(A^\circ, B^\circ, C^\circ) \text{ is trivial} &\iff (A, B, C) \text{ and } (X, Y, Z) \text{ are equilateral and} \\
&\ \text{have the same orientation.}
\end{aligned}$$

If (A', B', C') is not trivial, then

$$\phi(A', B', C') = \frac{-(t+2\zeta)}{2t+\zeta}\,\phi(A, B, C), \qquad (13)$$

and

$$\left\| \frac{-(t+2\zeta)}{2t+\zeta} \right\|^2 = 1 + \frac{3\,(1-\|t\|^2)}{\|2t+\zeta\|^2}. \qquad (14)$$

If $(A^\circ, B^\circ, C^\circ)$ *is not trivial, then*

$$\phi\,(A^\circ, B^\circ, C^\circ) \quad = \quad \frac{-\zeta}{t}\,\phi(A,B,C). \tag{15}$$

In particular,

$$\begin{aligned}
(A', B', C') \text{ is equilateral} &\iff (A,B,C) \text{ is equilateral} \\
&\qquad \text{or } (X,Y,Z) \text{ is isosceles with } \angle YZX = 120^\circ. \\
(A^\circ, B^\circ, C^\circ) \text{ is equilateral} &\iff (A,B,C) \text{ or } (X,Y,Z) \text{ is equilateral.}
\end{aligned}$$

Also, two vertices of (A', B', C') *coincide if and only if*

$$\phi(A,B,C) = \frac{-\zeta^i(2t+\zeta)}{t+2\zeta}$$

for some i*, where the values* $i = 0, 1, 2$ *correspond to* $B' = C'$*,* $A' = B'$*,* $C' = A'$*, respectively.*

Proof. Following the notation of Theorem 4.1, we let

$$\begin{bmatrix} K & L & M \\ K' & L' & M' \\ K^\circ & L^\circ & M^\circ \end{bmatrix} \quad = \quad \begin{bmatrix} A & B & C \\ A' & B' & C' \\ A^\circ & B^\circ & C^\circ \end{bmatrix} \mathcal{P},$$

where $\mathcal{P}$ is as given in (4). Then using (6), (7), (10), we see that

$$[K' \quad L' \quad M'] \;=\; [K \quad L \quad M] \;\; \text{Diag}\left[1 \quad \frac{t+2\zeta}{t-\zeta} \quad \frac{-(2t+\zeta)}{t-\zeta}\right], \tag{16}$$

$$[K^\circ \quad L^\circ \quad M^\circ] \;=\; [K \quad L \quad M] \;\; \text{Diag}\left[1 \quad \frac{\zeta}{t-\zeta} \quad \frac{-t}{t-\zeta}\right], \tag{17}$$

where Diag$[a \quad b \quad c]$ stands for the diagonal matrix with a, b, c on its main diagonal. Thus

$$\begin{aligned}
& (A', B', C') \text{ is trivial} \\
\iff\; & L' = M' = 0 \;\; (\text{by } (3)) \\
\iff\; & \frac{(t+2\zeta)L}{t-\zeta} = \frac{(2t+\zeta)M}{t-\zeta} = 0 \\
\iff\; & (t = -2\zeta \text{ or } L = 0) \;\text{ and }\; (t = -\zeta/2 \text{ or } M = 0) \\
\iff\; & (L = 0 \text{ and } t = -\zeta/2) \;\text{ or }\; (M = 0 \text{ and } t = -2\zeta) \\
\iff\; & (\phi(T) = 0 \text{ and } \phi(D) = -\zeta/2) \;\text{ or }\; (\phi(T) = \infty \text{ and } \phi(D) = -2\zeta)\,,
\end{aligned}$$

as claimed.

Similarly,

$$\begin{aligned}
(A^\circ, B^\circ, C^\circ) \text{ is trivial} \;\iff\; & L^\circ = M^\circ = 0 \\
\iff\; & \frac{\zeta L}{t-\zeta} = \frac{-tM}{t-\zeta} = 0 \\
\iff\; & (L = 0 \text{ or } t = \infty) \;\text{ and }\; (M = 0 \text{ or } t = 0) \\
\iff\; & (L = 0 \text{ and } t = 0) \;\text{ or }\; (M = 0 \text{ and } t = \infty) \\
\iff\; & (\phi(T) = \phi(D) = 0) \;\text{ or }\; (\phi(T) = \phi(D) = \infty)\,,
\end{aligned}$$

as claimed.

The statement (13) follows from $\phi(A', B', C') = L'/M'$ and (16); (15) follows from $\phi(A^\circ, B^\circ, C^\circ) = L^\circ/M^\circ$ and (17); and (14) follows from

$$\left\| \frac{-(t+2\zeta)}{2t+\zeta} \right\|^2 - 1 \;=\; \frac{(t+2\zeta)(\bar{t}+2\zeta^2) - (2t+\zeta)(2\bar{t}+\zeta^2)}{\|2t+\zeta\|^2} \;=\; \frac{3\,(1-\|t\|^2)}{\|2t+\zeta\|^2}.$$

The last statement follows from (13) and the facts that $B' = C' \iff \phi(A', B', C') = 1$, etc. □

Remark 5.2. Let us now study separately the case when $D = (X, Y, Z)$ is degenerate. Since $X \neq Y$, we may assume that $(X, Y, Z) = (0, 1, s)$ for some real s. The relation between s and $t = \phi(D)$ is given by

$$t \;=\; \phi(D) \;=\; \frac{\zeta + s\zeta^2}{\zeta^2 + s\zeta} \;=\; \frac{s\zeta + 1}{s + \zeta},$$

or equivalently

$$s \;=\; \frac{1 - t\zeta}{t - \zeta}. \tag{18}$$

Letting $\mathcal{N}_D(A, B, C) = (A', B', C')$ be the D-Napoleon triangle of (A, B, C), we see that A', B', C' are the points that divide the sides (B, C), (C, A), (A, B) in the ratio $s : 1 - s$. Thus (A', B', C') is what is referred to in [14] as the s-medial triangle of (A, B, C). In other words, $\mathcal{N}_D = \mathcal{M}_s$, where $t = \phi(D)$ and s are related by (18). The sequence $\mathcal{M}_s^n(T)$ of s-medial triangles is studied in detail in [14]. In particular, we mention the fact that the sequence $\mathcal{M}_s^n(T)$ converges in shape if and only if T is equilateral or $s = 1/2$. To see this once more, we use (13) and (14) and the fact that $\|t\| = 1$ (because D is degenerate) to conclude that $\phi\left(\mathcal{M}_s^n(A, B, C)\right) = c^n\ \phi(A, B, C)$, where

$$c \;=\; \frac{-(t + 2\zeta)}{2t + \zeta}.$$

Since $\|c\| = 1$, it follows that $\phi\left(\mathcal{M}_s^n(A, B, C)\right)$ converges if and only if $\phi(A, B, C) = 0$, $\phi(A, B, C) = \infty$, or $c = 1$. Using (18), we see that $c = 1 \iff t = -\zeta \iff s = 1/2$, as claimed.

In view of the remark above, we can assume that D is non-degenerate, i.e., $\|t\| \neq 1$. We can also exclude the exceptional cases mentioned in (12) in Theorem 5.2. We also mention that when D is equilateral, then we obtain the ordinary (positive and negative) Napoleon triangles of (A, B, C) studied in [15].

Theorem 5.3. *Let $D = (X, Y, Z)$ be a non-degenerate triangle, and let $T = (A, B, C)$ be a triangle with pairwise distinct vertices.*

(i) *If D and T have the same orientation with T equilateral and D isosceles with $\angle YZX = 120^\circ$, then $\mathcal{N}_D(T)$ is trivial.*

(ii) *If there exist positive integers j and k such that*

$$\phi(T) = \zeta^j \left(\frac{2t+\zeta}{-(t+2\zeta)} \right)^k , \tag{19}$$

then the sequence $\mathcal{N}_D^n(T)$ terminates at the stage when n is the smallest such k.

(iii) *Otherwise, the sequence $\mathcal{N}_D^n(T)$ converges in shape to an equilateral triangle whose orientation is opposite to that of D.*

Proof. To prove (ii), let k be the smallest positive integer satisfying (19). Using (13), we see that k is the smallest positive integer such that $\phi\left(\mathcal{N}_D^k(T)\right) \in \{1, \zeta, \zeta^2\}$, and $\mathcal{N}_D^n(T)$ terminates at $n = k$.
For (iii), suppose that there is no k such that $\phi\left(\mathcal{N}_D^k(T)\right) \in \{1, \zeta, \zeta^2\}$. Thus the sequence does not terminate. Using (14), we see that the limit of $\phi\left(\mathcal{N}_D^n(T)\right)$, as $n \to \infty$, is ∞ if $\|t\| < 1$ and 0 if $\|t\| > 1$. Thus $\mathcal{N}_D^n(T)$ converges to an equilateral triangle whose orientation is opposite to that of D. □

We conclude this section by proving that $\mathcal{N}_{D_1}$ and $\mathcal{N}_{D_2}$ commute for all feasible D_1 and D_2. This was proved in [32, Remark 1, p. 128] in the special case when D_1 and D_2 are isosceles.

Theorem 5.4. *Let $D_1 = (X_1, Y_1, Z_1)$ and $D_2 = (X_2, Y_2, Z_2)$ be non-trivial triangles in which $X_1 \neq Y_1$ and $X_2 \neq Y_2$. Then $\mathcal{N}_{D_1}(\mathcal{N}_{D_2}) = \mathcal{N}_{D_2}(\mathcal{N}_{D_1})$.*

Proof. Let $t_1 = \phi(D_1)$, $t_2 = \phi(D_2)$, and let $\mathcal{M}$ be the matrix given in (10). Let $\mathcal{M}_1$, $\mathcal{M}_2$ be obtained from $\mathcal{M}$ by replacing t by t_1, t_2, respectively. Then $\mathcal{N}_{D_2}(\mathcal{N}_{D_1}(A,B,C)) = [A \quad B \quad C]\mathcal{M}_1\mathcal{M}_2$. Direct calculations show that $\zeta^2(t_1 - \zeta)(t_2 - \zeta)\mathcal{M}_1\mathcal{M}_2$ is the circulant matrix whose first row is $[\zeta(1-\zeta t_1)(1-t_2) + \zeta(1-\zeta t_2)(1-t_1) \quad (1-t_1)(1-t_2) \quad \zeta^2(1-\zeta t_1)(1-\zeta t_2)]$. Thus $\mathcal{M}_1\mathcal{M}_2 = \mathcal{M}_2\mathcal{M}_1$ and therefore $\mathcal{N}_{D_2}(\mathcal{N}_{D_1}) = \mathcal{N}_{D_1}(\mathcal{N}_{D_2})$, as desired. □

6. The generalized Torricelli transformation $\mathcal{T}_D$

We now turn to the Torricelli transformation $\mathcal{T}_D(T)$ shown in Figure 2, and we study the sequence of iterations $\mathcal{T}_D^n(T)$. We see that $\mathcal{T}_D^n(T)$ either terminates or converges, in shape. Unexpectedly, the limiting shape is not that of an equilateral triangle, but that of a triangle whose shape depends only on that of D.

Our first theorem shows that for $\mathcal{T}_D^n(T)$ to be defined, we must restrict ourselves to triangles D and T in which no two vertices coincide.

Theorem 6.1. *Let $D = (X, Y, Z)$ and $T = (A, B, C)$ be non-trivial triangles and let $t = \phi(D)$.*

If no two vertices of D coincide (i.e., $t \notin \{1, \zeta, \zeta^2\}$), then there exist points A^*, B^*, C^* such that the triangles (A, B, C^*), (A, B^*, C), (A^*, B, C) are similar *to D if and only if no two vertices of T coincide. In this case, A^*, B^*, C^* are unique and given by*

$$[A^* \quad B^* \quad C^*] = [A \quad B \quad C]\, M, \quad M = \begin{bmatrix} 0 & \frac{-\zeta(t-1)}{t-\zeta^2} & \frac{-\zeta^2(t-1)}{t-\zeta} \\ \frac{-\zeta^2(t-\zeta^2)}{t-1} & 0 & \frac{-\zeta(t-\zeta^2)}{t-\zeta} \\ \frac{-\zeta(t-\zeta)}{t-1} & \frac{-\zeta^2(t-\zeta)}{t-\zeta^2} & 0 \end{bmatrix}. \quad (20)$$

If $Y = Z$, then such points A^*, B^*, C^* exist if and only if $B = C$, in which case $B^* = C^* = B = C$ and A^* is any point different from $B = C$. Similar statements hold for the cases $Z = X$ and $X = Y$.

Proof. Suppose that the vertices of D are distinct, i.e., $t \notin \{1, \zeta, \zeta^2\}$.

If there exist points A^*, B^*, C^* such that the triangles (A, B, C^*), (A, B^*, C), (A^*, B, C) are similar to D, then clearly A, B, C must be distinct, and (20) follows directly from the equations

$$\phi(A, B, C^*) = \phi(A, B^*, C) = \phi(A^*, B, C) = t$$

and the definition of ϕ.

Conversely, if A, B, C are distinct, we define A^*, B^*, C^* by (20), and we verify that (A, B, C^*), (A, B^*, C), (A^*, B, C) are similar to D.

The last statement about $Y = Z$ etc. is immediate. □

In view of the last statement in Theorem 6.1, we restrict our attention in the next definition to the cases when each of D and T has distinct vertices.

Definition 6.1. *Let $D = (X, Y, Z)$ and $T = (A, B, C)$ be triangles in which no two vertices coincide. Then the D-Torricelli triangle $\mathcal{T}_D(T)$ of T is defined to be the triangle (A^*, B^*, C^*) whose vertices A^*, B^*, C^* are as given in* (20)*. In other words, (A^*, B^*, C^*) is the triangle for which the triangles (A, B, C^*), (A, B^*, C), (A^*, B, C) are similar to D.*

Theorem 6.2. *Let $D = (X, Y, Z)$ and $T = (A, B, C)$ be triangles in which no two vertices coincide. Let $\mathcal{T}_D(T) = (A^*, B^*, C^*)$ be the D-Torricelli triangle of T, and let $A^{\circledcirc}$, $B^{\circledcirc}$, $C^{\circledcirc}$ be the centroids of the triangles (A^*, B, C), (A, B^*, C), (A, B, C^*), respectively. Let $T^* = (A^*, B^*, C^*)$ and $T^{\circledcirc} = (A^{\circledcirc}, B^{\circledcirc}, C^{\circledcirc})$. Let $t = \phi(D)$, and let ψ be the shape function defined by*

$$\psi = \frac{-t^2\phi + 1}{\phi - t}; \tag{21}$$

see Remark 3.1*. Then (A^*, B^*, C^*) is never trivial and*

$$\phi(A^*, B^*, C^*) = \frac{-(t^3 + 2)\phi(A, B, C) + 3t}{(2t^3 + 1) - 3t^2\phi(A, B, C)}, \tag{22}$$

$$\psi(A^*, B^*, C^*) = \frac{-1}{2}\psi(A, B, C). \tag{23}$$

Also, $(A^{\circledcirc}, B^{\circledcirc}, C^{\circledcirc})$ *is trivial if and only if* (A, B, C) *and* (X, Y, Z) *are similar, in which case* $(A^*, B^*, C^*) = (A, B, C)$*. Otherwise,*

$$\phi\left(A^{\circledcirc}, B^{\circledcirc}, C^{\circledcirc}\right) = \frac{1}{t^2}. \tag{24}$$

In particular,

$$\begin{aligned} (A^*, B^*, C^*) \text{ is equilateral} &\iff \phi(A, B, C) = \frac{3t}{t^3+2} \text{ or } \frac{2t^3+1}{3t^2}, \\ \left(A^{\circledcirc}, B^{\circledcirc}, C^{\circledcirc}\right) \text{ is equilateral} &\iff (X, Y, Z) \text{ is equilateral.} \end{aligned}$$

Proof. We adhere to the notation of Section 3 with the understanding that (i) A', B', C', K', L', M' are to be replaced by A^*, B^*, C^*, K^*, L^*, M^*, (ii) A°, B°, C°, K°, L°, M° are to be replaced by $A^{\circledcirc}$, $B^{\circledcirc}$, $C^{\circledcirc}$, $K^{\circledcirc}$, $L^{\circledcirc}$, $M^{\circledcirc}$, (iii) $\mathcal{M}$ is as given in (20). Then it follows from (6) and (7) by direct calculations of $\mathcal{P}^{-1}\mathcal{M}\mathcal{P}$ and $\mathcal{P}^{-1}\mathcal{M}\mathcal{P} + \mathcal{R}$ that

$$[K^* \quad L^* \quad M^*](t^3-1) = [K \quad L \quad M]\begin{bmatrix} t^3-1 & 0 & 0 \\ 3t & t^3+2 & 3t^2 \\ -3t^2 & -3t & -(2t^3+1) \end{bmatrix} \tag{25}$$

$$\left[K^{\circledcirc} \quad L^{\circledcirc} \quad M^{\circledcirc}\right](t^3-1) = [K \quad L \quad M]\begin{bmatrix} t^3-1 & 0 & 0 \\ t & 1 & t^2 \\ -t^2 & -t & -t^3 \end{bmatrix}. \tag{26}$$

Hence

$$\begin{aligned} (A^*, B^*, C^*) \text{ is trivial} &\iff L^* = M^* = 0 \\ &\iff -(t^3+2)L + 3tM = (2t^3+1)M - 3t^2L = 0 \\ &\iff \frac{L}{M} = \frac{3t}{t^3+2} = \frac{2t^3+1}{3t^2} \\ &\implies 3t(3t^2) = (t^3+2)(2t^3+1) \\ &\iff t^6 - 2t^3 + 1 = 0 \\ &\iff t^3 = 1 \\ &\iff t \in \{1, \zeta, \zeta^2\}, \end{aligned}$$

contradicting the assumption that no two vertices of D coincide. Therefore (A^*, B^*, C^*) is never trivial.

(22) follows immediately from $\phi(A^*, B^*, C^*) = L^*/M^8$ and (25). Also (23) follows directly from (22) and (21), Also,

$$\begin{aligned} (A^{\circledcirc}, B^{\circledcirc}, C^{\circledcirc}) \text{ is trivial} &\iff L^{\circledcirc} = M^{\circledcirc} = 0 \\ &\iff L - tM = t^2(L - tM) = 0 \\ &\iff L - tM = 0 \\ &\iff \frac{L}{M} = t \\ &\iff \phi(T) = t. \end{aligned}$$

Therefore $(A^{\circledcirc}, B^{\circledcirc}, C^{\circledcirc})$ is trivial if and only if T and D are similar. In this case, it is clear that $(A^*, B^*, C^*) = (A, B, C)$ and $(A^{\circledcirc}, B^{\circledcirc}, C^{\circledcirc}) = (G, G, G)$, where G is the centroid of (A, B, C). Otherwise, (24) follows from $\phi(A^{\circledcirc}, B^{\circledcirc}, C^{\circledcirc}) = L^{\circledcirc}/M^{\circledcirc}$ and (26).

The last statements follow from (22) and (24). □

Remark 6.1. It is worth explaining how ψ in (21) came about. Following the procedure used in solving a system of two linear difference equations in two variables, we form the matrix $\mathbb{M} = \begin{bmatrix} t^3+2 & -3t \\ 3t^2 & -(2t^3+1) \end{bmatrix}$ of (22). It is routine to see that the zeros of the characteristic equation of $\mathbb{M}$ are $t^3 - 1$ and $-2(t^3 - 1)$. The corresponding eigenvectors are $(-t^2, 1)$ and $(1, -t)$. Thus we introduce the new shape function ψ defined by (21) and we see that

$$\psi(A^*, B^*, C^*) = \frac{t^3 - 1}{-2(t^3 - 1)}\psi(A, B, C) = \frac{-1}{2}\psi(A, B, C),$$

as claimed.

The shape function ψ introduced in Theorem 6.2 is useful in deciding whether the sequence $\mathcal{T}_D^n(T)$ converges in shape. Here we note that (23) is valid provided that $\phi(A, B, C) \notin \{1, \zeta, \zeta^2\}$, i.e., $\psi(A, B, C) \notin \Omega = \{t + 1, \zeta(t + \zeta), \zeta^2(t + \zeta^2)\}$. Thus for $\mathcal{T}_D(T)$ to be defined, $\psi(T)$ should not belong to Ω. Similarly, for $\mathcal{T}_D^2(T)$ to be defined, $\psi(T^*) \notin \Omega$, i.e., $\psi(T) \notin -2\Omega$. Thus, for the sequence $\mathcal{T}_D^n(T)$ to be defined for all n, $\psi(T)$ must not belong to the set

$$\bigcup_{n=0}^{\infty} \left\{(-2)^n(1 + t), (-2)^n\zeta(t + \zeta), (-2)^n\zeta^2(t + \zeta^2)\right\}.$$

Equivalently, $\phi(T)$ must not belong to the set

$$\bigcup_0^{\infty} \left\{ \frac{(-2)^n\omega(t + \omega) + 1}{t^2 + (-2)^n\omega(t + \omega)} \ : \ \omega \in \Omega \right\}.$$

Thus we have proved:

Theorem 6.3. *Let $D = (X, Y, Z)$ and $T = (A, B, C)$ be triangles in which no two vertices coincide. Let $t = \phi(D)$, and let ψ be as defined in* (21)*.*

(i) *If (A, B, C) is similar to D, then $\mathcal{T}_D^n(T)$ is identical with (A, B, C) for all n.*

(ii) *If there exists a non-negative integer k and an $\omega \in \{1, \zeta, \zeta^2\}$ such that*

$$\phi(A, B, C) = \frac{(-2)^k\omega(t + \omega) + 1}{t^2 + (-2)^n\omega(t + \omega)},$$

then the sequence $\mathcal{T}_D^n(T)$ terminates when n equals the smallest such k.

(iii) *Otherwise, the sequence* $\mathcal{T}_D^n(T)$ *converges in shape to a triangle* D_∞ *with*

$$\phi\,(D_\infty) = \frac{1}{t^2}.$$

With an eye on Theorem 5.4, we conclude this section by mentioning that $\mathcal{T}_{D_1}$ and $\mathcal{T}_{D_2}$ do not commute, even shape-wise, when D_1 and D_2 have different shapes. This follows by using (22) to calculate $\phi(\mathcal{T}_{D_1}(\mathcal{T}_{D_2}(T)))$ and $\phi(\mathcal{T}_{D_2}(\mathcal{T}_{D_1}(T)))$ and compare the two results.

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