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# Generic Warped Product Submanifolds in Nearly Kaehler Manifolds

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**Abstract.** Warped product manifolds provide excellent setting to model space-time near black holes or bodies with large gravitational force (cf. [1], [2], [14]). Recently, results are published exploring the existence (or non-existence) of warped product submanifolds in Kaehlerian and contact settings (cf. [6], [17], [20]). To continue the sequel, we have considered warped product submanifolds of nearly Kaehler manifolds with one of the factors a holomorphic submanifold. Such submanifolds are generic submanifolds in the sense of B. Y. Chen [5] and provide a generalization of CR and semi-slant submanifolds. It is shown that nearly Kaehler manifolds do not admit non-trivial warped product generic submanifolds, thereby generalizing the results of Chen [6] and Sahin [20]. However, non-trivial generic warped products (obtained by reversing the two factors of warped product generic submanifolds) exist in nearly Kaehler manifolds (cf. [21]). Some interesting results on the geometry of these submanifolds are obtained in the paper.

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## 1. Introduction

In [2], R. L. Bishop and B. O'Neill introduced the notion of warped product manifolds by homothetically warping the product metric of a product manifold $B \times F$ on the fibers $p \times F$ for each $p \in B$. This generalized product metric appears in differential geometric studies in a natural way. For example, a surface of revolution is a warped product manifold. With regard to the physical applications of these manifolds, one may realize that the space time around a massive star or a black hole can be modeled on a warped product manifold for instance, the relativistic model of Schwarzschild.

The study of differential geometry of warped product manifolds are intensified after the impulse given by B. Y. Chen's work on warped product CR-submanifolds of Kaehler manifolds (cf., [6], [7]). B. Sahin [20], extending the study of Chen proved that there do not exist warped product semi-slant submanifolds in Kaehler manifolds. In view of the physical applications of these manifolds, the question of existence or non-existence of warped product submanifolds assumes significance in general. In the present paper, we have addressed the same problem in the setting of nearly Kaehler manifolds for a larger class of warped product submanifolds and thus generalize the results of Chen [6] and Sahin [20]. To be more precise, we investigated warped product submanifolds of nearly Kaehler manifolds with one of the factors a holomorphic submanifold. Besides the fact, that a nearly Kaehler structure on an almost Hermitian manifold provides an interesting geometric study, our study is also relevant due to the fact that one of the most important nearly Kaehler manifolds, namely $S^6$ fails to admit CR-product submanifolds. However, it does admit CR-warped product submanifolds (cf., [21]). Some important geometric properties of these warped product submanifolds follow from our study. In particular an inequality for the squared norm of the second fundamental form of generic warped product submanifolds in nearly Kaehler manifolds is obtained which generalizes the similar inequality for CR-warped product submanifolds in the Kaehler setting (cf. [6]).

## 2. Preliminaries

Let $(\bar{M}, J, g)$ be a nearly Kaehler manifold with an almost complex structure $J$ and Hermitian metric $g$ and a Levi-Civita connection $\bar{\nabla}$ such that

$$g(JU, JV) = g(U, V), \tag{2.1}$$

$$(\bar{\nabla}_U J)U = 0 \tag{2.2}$$

for all vector fields $U$ and $V$ on $\bar{M}$. Let $M$ be a submanifold of $\bar{M}$. Then the induced Riemannian metric on $M$ is denoted by the same symbol $g$ and the induced connection on $M$ is denoted by the symbol $\nabla$. If $T\bar{M}$ and $TM$ denote the tangent bundle on $\bar{M}$ and $M$ respectively and $T^{\perp}M$, the normal bundle on $M$, then the Gauss and Weingarten formulae are respectively given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.3}$$

$$\bar{\nabla}_U \xi = -A_\xi U + \nabla^{\perp}_U \xi, \tag{2.4}$$

for $U, V \in TM$ and $\xi \in T^{\perp}M$ where $\nabla^{\perp}$ denotes the connection on the normal bundle $T^{\perp}M$. $h$ and $A_{\xi}$ are the second fundamental forms and the shape operator of the immersions of $M$ into $\bar{M}$ corresponding to the normal vector field $\xi$. They are related as

$$g(A_{\xi}U, V) = g(h(U, V), \xi). \tag{2.5}$$

The mean curvature vector $H$ of $M$ is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where $n$ is the dimension of $M$ and $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame of vector fields on $M$. The squared norm of the second fundamental form is defined as

$$\|h\|^2 = \sum_{i,\ j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.6}$$

A submanifold $M$ of $\bar{M}$ is said to be a *totally geodesic submanifold* if $h(U, V) = 0$ for each $U, V \in TM$ and a submanifold is said to be *totally umbilical submanifold* if $h(U, V) = g(U, V)H$.

For any $U \in TM$ and $\xi \in T^{\perp}M$ we write

$$JU = PU + FU, \tag{2.7}$$

$$J\xi = t\xi + f\xi, \tag{2.8}$$

where $PU$ and $t\xi$ are the tangential components of $JU$ and $J\xi$ respectively and $FU$ and $f\xi$ are the normal components of $JU$ and $J\xi$ respectively.

The covariant differentiation of the tensors $P, F, t$ and $f$ are defined respectively as

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{2.9}$$

$$(\bar{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{2.10}$$

$$(\bar{\nabla}_U t)\xi = \nabla_U t\xi - t\nabla^{\perp}_U \xi, \tag{2.11}$$

$$(\bar{\nabla}_U f)\xi = \nabla^{\perp}_U f\xi - f\nabla^{\perp}_U \xi. \tag{2.12}$$

Furthermore, for any $U, V \in TM$, let us decompose $(\bar{\nabla}_U J)V$ into tangential and normal parts as

$$(\bar{\nabla}_U J)V = \mathcal{P}_U V + \mathcal{Q}_U V. \tag{2.13}$$

By making use of equations (2.3)–(2.10), we may obtain that

$$\mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - th(U, V), \tag{2.14}$$

$$\mathcal{Q}_U V = (\bar{\nabla}_U F)V + h(U, PV) - fh(U, V). \tag{2.15}$$

Similarly for $\xi \in T^{\perp}M$, denoting by $\mathcal{P}_U\xi$ and $\mathcal{Q}_U\xi$ respectively the tangential and normal parts of $(\bar{\nabla}_U J)\xi$, we find that

$$\mathcal{P}_U\xi = (\bar{\nabla}_U t)\xi + PA_{\xi}U - A_{f\xi}U, \tag{2.16}$$

$$\mathcal{Q}_U\xi = (\bar{\nabla}_U f)\xi + h(t\xi, U) + FA_\xi U. \tag{2.17}$$

The following properties of $\mathcal{P}$ and $\mathcal{Q}$ are used in our subsequent sections and can be verified through a straightforward computation

$$\begin{array}{llcl}
(\mathrm{p}_1) & \text{(i)} \quad \mathcal{P}_{U+V}W & = & \mathcal{P}_U W + \mathcal{P}_V W, \\
 & \text{(ii)} \quad \mathcal{Q}_{U+V}W & = & \mathcal{Q}_U W + \mathcal{Q}_V W, \\
(\mathrm{p}_2) & \text{(i)} \quad \mathcal{P}_U(V+W) & = & \mathcal{P}_U V + \mathcal{P}_U W, \\
 & \text{(ii)} \quad \mathcal{Q}_U(V+W) & = & \mathcal{Q}_U V + \mathcal{Q}_U W, \\
(\mathrm{p}_3) & \text{(i)} \quad g(\mathcal{P}_U V,\ W) & = & -\,g(V, \mathcal{P}_U W), \\
 & \text{(ii)} \quad g(\mathcal{Q}_U V,\ \xi) & = & -\,g(V, \mathcal{P}_U \xi), \\
(\mathrm{p}_4) & \quad \mathcal{P}_U JV + \mathcal{Q}_U JV & = & -\,J(\mathcal{P}_U V + \mathcal{Q}_U V).
\end{array}$$

On a submanifold $M$ of a nearly Kaehler manifold, by equations (2.2) and (2.13)

$$\text{(a)} \quad \mathcal{P}_U V + \mathcal{P}_V U = 0, \qquad \text{(b)} \quad \mathcal{Q}_U V + \mathcal{Q}_V U = 0 \tag{2.18}$$

for any $U,\ V \in TM$.

## 3. Some basic results

Let $\bar{M}$ be an almost Hermitian manifold with an almost complex structure $J$ and Hermitian metric $g$ and $M$, a submanifold of $\bar{M}$. For each $x \in M$, let $D_x = T_xM \cap JT_xM$ be the maximal holomorphic subspace of the tangent space $T_xM$ at $x \in M$. If $D : x \to D_x$ defines a smooth distribution on $M$, then $M$ is called a *generic submanifold* of $\bar{M}$ [5]. A generic submanifold $M$ of an almost Hermitian manifold is said to be a *CR-submanifold* if the orthogonal complementary distribution $D^0$ of $D$ in $TM$ is totally real i.e., $JD^0_x \subseteq T^\perp_x M$ for each $x \in M$. On a generic submanifold of an almost Hermitian manifold $\bar{M}$, the tangent bundle $TM$ and the normal bundle $T^\perp M$ are decomposed as

$$\text{(a)} \quad TM = D \oplus D^0, \quad \text{(b)} \quad T^\perp M = FD^0 \oplus \mu, \tag{3.1}$$

where $\mu$ is the orthogonal complementary distribution to $FD^0$ and is invariant under $J$. Moreover, following are some easy observations

$$\left.\begin{array}{ll}
\text{(a)}\ FD = \{0\}, & \text{(b)}\ PD = D, \\[2ex]
\text{(c)}\ PD^0 \subseteq D^0 \ \text{ and} & \text{(d)}\ t(T^\perp M) = D^0.
\end{array}\right\} \tag{3.2}$$

In terms of $P,\ F,\ t$ and $f$ we have

$$\left.\begin{array}{ll}
\text{(e)}\ P^2 + tF = -I, & f^2 + Ft = -I, \\[2ex]
\text{(f)}\ FP + fF = 0, & tf + Pt = 0.
\end{array}\right\} \tag{3.3}$$

An immersed submanifold $M$ of an almost Hermitian manifold $\bar{M}$ is said to be a *slant submanifold* if the Wirtinger angle $\theta(X) \in [0, \pi/2]$ between $JX$ and $T_xM$ has the same value $\theta$ for any $x \in M$ and $X \in T_xM$, $X \neq 0$ (cf. [3]). Holomorphic and totally real submanifolds are slant submanifolds with Wirtinger angle 0 and

$\pi/2$ respectively. A slant submanifold is called *proper slant* if it is neither holomorphic nor totally real. More generally, a distribution $D^\theta$ on $\bar{M}$ is called a *slant distribution* if the angle $\theta(X)$ between $JX$ and $D_x^\theta$ has the same value $\theta$ for each $x \in \bar{M}$ and $X \in D_x^\theta,\ X \neq 0$.

If $M$ is a slant submanifold of an almost Hermitian manifold $\bar{M}$, then we have (cf. [3])

$$P^2 = -\cos^2(\theta) I, \tag{3.4}$$

where $\theta$ is the Wirtinger angle of $M$ in $\bar{M}$. Hence, we have

$$g(PU, PV) = \cos^2(\theta) g(U, V), \tag{3.5}$$

$$g(FU, FV) = \sin^2(\theta) g(U, V), \tag{3.6}$$

for $U$, $V$ tangent to $M$.

A natural generalization of CR-submanifolds in terms of slant distribution was given by N. Papaghiuc [19]. These submanifolds are known as semi-slant submanifolds. He defined these submanifolds as

**Definition 3.1.** *A submanifold $M$ of an almost Hermitian manifold is called a semi-slant submanifold if it is endowed with two orthogonal complementary distributions $D$ and $D^\theta$ such that $D$ is holomorphic and $D^\theta$ is slant.*

It is straight forward to see that CR-submanifolds and slant submanifolds are semi-slant submanifolds with $\theta = \pi/2$ and $D = \{0\}$ respectively, whereas a generic submanifold $M$ is a semi-slant submanifold if the complementary distribution $D^0$ of $D$ on $M$ is a slant distribution.

On using some of the formulas developed in the last section, we obtain the following criteria for the integrability of the distributions involved in the setting of generic submanifold of a nearly Kaehler manifold.

**Theorem 3.1.** *Let $M$ be a generic submanifold of a nearly Kaehler manifold $\bar{M}$. Then the holomorphic distribution $D$ on $M$ is integrable if and only if*

$$h(X, PY) = h(PX, Y) \quad \textit{and} \quad \mathcal{Q}_X Y = 0$$

*for any $X, Y \in D$.*

*Proof.* By equations (2.3), (2.15) and (2.18)(b), we have

$$g(F[X, Y], \xi) = g(h(X, PY) - h(PX, Y) + 2\mathcal{Q}_Y X, \xi)$$

for any $X, Y \in TM$ and $\xi \in T^\perp M$. Hence, $D$ is integrable if and only if

$$2\mathcal{Q}_X Y = h(X, PY) - h(PX, Y). \tag{3.7}$$

It is known that the Nijenhuis tensor $S$ of $J$ on $\bar{M}$ satisfies

$$S(U, V) = 4J(\bar{\nabla}_U J)V$$

for each $U, V \in T\bar{M}$. Moreover, as $(\bar{\nabla}_U J)JV = -J(\bar{\nabla}_U J)V$, we get

$$S^{\perp}(X,Y) = -4\mathcal{Q}_X JY \tag{3.8}$$

where $S^{\perp}(X,Y)$ denotes the normal part of $S(X,Y)$. Now as

$$S(U,V) = [JU, JV] - [U,V] - J[U,JV] - J[JU,V].$$

By equation (2.7), we have

$$S^{\perp}(X,Y) = F([X,PY] + [PX,Y])$$

and therefore by equation (3.5)

$$F([X,PY] + [PX,Y]) = -4\mathcal{Q}_X JY. \tag{3.9}$$

The assertion is proved by virtue of (3.7), (3.8) and (3.9).

A generic submanifold $M$ is called a *generic product* if the distributions $D$ and $D^0$ are parallel on $M$. In this case $M$ is foliated by the leaves of these distributions, forcing $M$ to be locally a Riemannian product of their leaves. In particular if $M$ is a CR-submanifold with parallel distributions then it is called a *CR-product.* B. Y. Chen [4] proved that a CR-submanifold of a Kaehler manifold is a CR-product if and only if $\bar{\nabla} P = 0$ or equivalently $A_{JD^0} D = 0$.

In general, if $M_1$ and $M_2$ are Riemannian manifolds with Riemannian metric $g_1$ and $g_2$ respectively then the product manifold $(M_1 \times M_2,\ g)$ is a Riemannian manifold with the Riemannian metric $g$ defined as

$$g(U,V) = g_1(d\pi_1 U, d\pi_1 V) + g_2(d\pi_2 U, d\pi_2 V)$$

where $\pi_i (i = 1,2)$ are the projection maps of $M$ onto $M_1$ and $M_2$ respectively and $d\pi_i (i = 1,2)$ are their differentials.

As a generalization of the product manifold and in particular of a generic product submanifold, one can consider warped product of manifolds which are defined as

**Definition 3.2.** *Let $(B,\ g_B)$ and $(F,\ g_F)$ be two Riemannian manifolds with Riemannian metrics $g_B$ and $g_F$ respectively and $f$, a positive differentiable function on $B$. The warped product of $B$ and $F$ is the Riemannian manifold*

$$B \times_f F = (B \times F,\ g),$$

*where*

$$g = g_B + f^2 g_F. \tag{3.10}$$

More explicitly if $U$ is tangent to $M = B \times_f F$ at $(p,q)$ then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2,$$

where the function $f$ is known as the warping function.

Bishop and O'Neill [2] obtained the following basic results for warped product manifolds

**Theorem 3.2.** [2] *Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in TB$ and $V, W \in TF$ then*

(i) $\nabla_X Y \in TB$,

(ii) $\nabla_X V = \nabla_V X = (\frac{Xf}{f})V$,

(iii) $nor(\nabla_V W) = \frac{-g(V,W)}{f}\nabla f$

*where $nor(\nabla_V W)$ is the component of $\nabla_V W$ in $TB$ and $\nabla f$ is the gradient of $f$, defined by $g(\nabla f, U) = Uf$, for all $U \in TM$.*

**Corollary 3.1.** *On a warped product manifold $M = M_1 \times_f M_2$*

(i) *$M_1$ is totally geodesic in $M$,*

(ii) *$M_2$ is totally umbilical in $M$.*

Throughout, we denote by $N_T$ and $N_\perp$ a holomorphic and a totally real submanifold respectively of an almost Hermitian manifold $\bar{M}$.

B. Y. Chen [5] studied CR-submanifolds of a Kaehler manifold which are warped product of the form $N_\perp \times_f N_T$ and $N_T \times_f N_\perp$, known as warped product CR-submanifolds and CR-warped product submanifolds respectively. He proved that warped product CR-submanifolds are simply CR-products whereas CR-warped product submanifolds in a Kaehler manifold are non-trivial. He further worked out a characterization for a CR-submanifold of a Kaehler manifold to be locally a CR-warped product submanifold.

A warped product manifold is said to be *trivial* if its warping function $f$ is constant. More generally, a trivial warped product manifold $M = N_1 \times_f N_2$ is a Riemannian product $N_1 \times N_2^f$ where $N_2^f$ is the manifold with Riemannian metric $f^2 g_2$ which is homothetic to the original metric $g_2$ of $N_2$. For example, a trivial CR-warped product is a CR-product.

B. Sahin [20] extended the study of warped product CR-submanifolds and CR-warped product submanifolds of Kaehler manifolds by introducing warped product submanifolds $N_\theta \times_f N_T$ and $N_T \times_f N_\theta$ where $N_\theta$ denotes a slant submanifold with Wirtinger angle $\theta$. He obtained the following theorems.

**Theorem 3.3.** [20] *Let $\bar{M}$ be a Kaehler manifold. Then there do not exist warped product submanifolds $M = N_\theta \times_f N_T$ in $\bar{M}$ such that $N_\theta$ is proper slant submanifold and $N_T$ is a holomorphic submanifold of $\bar{M}$.*

**Theorem 3.4.** [20] *Let $\bar{M}$ be a Kaehler manifold. Then there do not exist warped product submanifolds $M = N_T \times_f N_\theta$ in $\bar{M}$ such that $N_T$ is a holomorphic submanifold and $N_\theta$, a proper slant submanifold of $\bar{M}$.*

**Note.** Theorem 3.3 is valid for all $\theta \in [0, \pi/2]$ whereas there are many examples of CR-warped product submanifolds $N_T \times_f N_\perp$ in Kaehler manifolds which are not CR-products strengthening the fact that Theorem 3.4 is valid for the case of proper semi-slant warped product submanifolds(cf. [6]).

As a step forward, we study warped product submanifolds of nearly Kaehler manifolds with one of the factors a holomorphic submanifold, namely $N\times_f N_T$ and $N_T\times_f N$ in a nearly Kaehler manifold $\bar{M}$ where $N$ is an arbitrary submanifold and $N_T$ a holomorphic submanifold of $\bar{M}$. In the sequel we call these warped products as warped product generic submanifolds and generic warped product submanifolds respectively. The trivial case of these warped products is a generic product.

## 4. Warped product generic submanifolds of nearly Kaehler manifolds

Let $\bar{M}$ be a nearly Kaehler manifold and $M = N\times_f N_T$ be a warped product generic submanifold of $\bar{M}$. Then by Theorem 3.2,

$$\nabla_X Z = \nabla_Z X = (Z\ln f)X \tag{4.1}$$

for any $X\in TN_T$ and $Z\in TN$. Thus, for any $Y\in TN_T$

$$\begin{aligned} Z\ln f g(X,Y) &= g(\nabla_{JX}Z, JY)\\ &= -g(Z,\bar{\nabla}_{JX}JY)\\ &= -g(Z,\mathcal{P}_{JX}Y) - g(Z, J\bar{\nabla}_{JX}Y). \end{aligned}$$

On making use of the equations (2.18) (a), ($p_4$), (2.3), (2.7), (4.1) and the Theorem 3.1, the right hand side of the above equation takes the form

$$g(PZ,\mathcal{P}_Y X) - (PZ\ln f)g(JX,Y) + g(h(JX,Y),FZ).$$

Thus, we have

$$g(PZ,\mathcal{P}_Y X) = (Z\ln f)g(X,Y) + (PZ\ln f)g(JX,Y) - g(h(JX,Y),FZ).$$

Taking account of skew symmetry of $\mathcal{P}$ in $X$ and $Y$ and applying Theorem 3.1, the above equation yields

$$g(h(JX,Y),FZ) = (Z\ln f)g(X,Y). \tag{4.2}$$

Now by equations (2.13), (2.18) and property ($p_4$),

$$\begin{aligned} g(\bar{\nabla}_{JX}JX, JZ) &= g(\bar{\nabla}_{JX}X, Z)\\ &= g(\nabla_{JX}X, Z). \end{aligned}$$

Clearly the right hand side is zero by virtue of formula (4.1). Thus,

$$g(\bar{\nabla}_{JX}JX, JZ) = 0. \tag{4.3}$$

On the other hand by Gauss formula, we may write

$$g(h(JX,JX),FZ) = g(\bar{\nabla}_{JX}JX,FZ),$$
$$= g(\bar{\nabla}_{JX}JX,JZ) - g(\nabla_{JX}JX,PZ).$$

On applying equations (4.1) and (4.3), the above equation yields

$$g(h(JX,JX),FZ) = (PZ\ln f)\|X\|^2. \tag{4.4}$$

It follows from equation (4.2) and (4.4) that

$$PZ\ln f = 0.$$

Hence, we have proved

**Theorem 4.1.** *Let $\bar{M}$ be a nearly Kaehler manifold and $M = N \times_f N_T$ a warped product submanifold of $\bar{M}$ with $N$ and $N_T$ a Riemannian and a holomorphic submanifolds respectively of $\bar{M}$. Then $M$ is trivial i.e., a generic product submanifold of $\bar{M}$.*

As a consequence of the above theorem, it follows that there does not exist a warped product semi-slant submanifold of nearly Kaehler manifolds. In other words, Theorem 4.1 generalizes Theorem 3.3.

## 5. Generic warped product submanifolds of a nearly Kaehler manifold

To complete the study, we now investigate generic warped product submanifolds $N_T \times_f N$ and in particular semi-slant warped product submanifolds $N_T \times_f N_\theta$ of a nearly Kaehler manifold $\bar{M}$.

Let $M = N_T \times_f N$ be a generic warped product submanifold of a nearly Kaehler manifold $\bar{M}$. Then by Theorem 3.2,

$$\nabla_X Z = \nabla_Z X = (X\ln f)Z \tag{5.1}$$

whereas by formula (2.9) and Theorem 3.2,

$$(\bar{\nabla}_Z P)W = g(Z,W)P(\nabla \ln f) - g(Z,PW)\nabla \ln f \tag{5.2}$$

for each $X \in TN_T$ and $Z,\ W \in TN$. We begin the section exhibiting some important relations of the second fundamental form $h(U,V)$ and $\mathcal{P}_U V$.

**Lemma 5.1.** *On a generic warped product submanifold $M = N_T \times_f N$ of a nearly Kaehler manifold $\bar{M}$, we have*

(i) $g(\mathcal{P}_X Y, Z) = g(h(X,Y),\ FZ) = 0$

(ii) $g(\mathcal{P}_X Z, PZ) = g(h(X,Z), FPZ) - g(h(X,PZ),\ FZ)$

(iii) $g(h(PX,Z),FZ) = (X\ln f)\|Z\|^2$

*for each* $X,\ Y \in TN_T$ *and* $Z \in TN$.

*Proof.* As $N_T$ is totally geodesic in $M$, $(\bar{\nabla}_X P)Y \in TN_T$ and therefore by formula (2.14)

$$\begin{aligned} g(\mathcal{P}_X Y, Z) &= -g(th(X,Y),\ Z) \\ &= g(h(X,Y),\ FZ). \end{aligned}$$

The right hand side of the above equation is symmetric in $X$ and $Y$ whereas the left hand side is skew symmetric in $X$ and $Y$. Hence (i) holds.

By formulae (2.14) and (5.1),

$$\mathcal{P}_X PZ = -A_{FPZ}X - th(X, PZ).$$

Taking product with $Z$, the above equation on making use of equation (2.5) and property ($p_3$) yields,

$$g(\mathcal{P}_X Z, PZ) = g(h(X,Z),\ FPZ) - g(h(X,PZ),\ FZ).$$

This verifies statement (ii).

Now, by formula (2.14) and equation (2.18)(a),

$$0 = (\bar{\nabla}_X P)Z + (\bar{\nabla}_Z P)X - 2th(X,Z) - A_{FZ}X. \tag{5.3}$$

As by formulae (2.9) and (5.1), $(\bar{\nabla}_X P)Z = 0$, equation (5.3), on taking account of this fact and formula (5.1), yields

$$(\bar{\nabla}_Z P)X = (PX \ln f)Z - (X \ln f)PZ = 2th(X,Z) + A_{FZ}X. \tag{5.4}$$

Taking product with $Z$ in (5.4) and replacing $X$ by $PX$, statement (iii) follows completing the proof of the lemma.

**Theorem 5.1.** *A generic submanifold* $M$ *with involutive distributions* $D$ *and* $D^0$ *in a nearly Kaehler manifold* $\overline{M}$ *is locally a generic warped product if and only if* $\nabla_Z PZ \in D^0$ *and there exists a* $C^\infty$*-function* $\alpha$ *on* $M$ *with* $Z\alpha = 0$ *for all* $Z \in D^0$ *such that*

$$A_{FZ}X = -[(JX\alpha)Z + \frac{1}{3}(X\alpha)PZ] \tag{5.5}$$

*for each* $X$ *in* $D$ *and* $Z$ *in* $D^0$.

*Proof.* If $M$ is a generic warped product submanifold $N_T \times_f N^0$ of a nearly Kaehler manifold $\overline{M}$, then by (5.1), $\nabla_Z PZ \in TN^0$. Further, by (2.9) and (5.1) $(\overline{\nabla}_X P)Z = 0$ for each $X \in TN_T$ and $Z \in TN^0$. Thus, (2.14) gives

$$\mathcal{P}_X Z = -A_{FZ}X - th(X,Z). \tag{5.6}$$

On the other hand by (2.9), (2.14) and (5.1),

$$\mathcal{P}_Z X = (PX \ln f)Z - (X \ln f)PZ - th(X,Z). \tag{5.7}$$

As $\overline{M}$ is nearly Kaehler, adding (5.6) and (5.7) gives

$$A_{FZ}X = (PX \ln f)Z - (X \ln f)PZ - 2th(X, Z).$$

Therefore, for any $W \in TN^0$,

$$-3g(A_{FZ}X, W) = 3(PX \ln f)\ g(Z, W) + (X \ln f)\ g(PZ, W),$$

or,

$$g(A_{FZ}X, W) = -[(PX \ln f)\ g(Z, W) + \frac{X \ln F}{3} g(PZ, W)]. \tag{5.8}$$

(5.5) is proved in view of (5.8) and Lemma (5.1)(i).

Conversely, let $M$ be a generic submanifold of $\overline{M}$ satisfying the hypothesis of the Theorem. Then for any $X, Y \in D$ and $Z \in D^0$,

$$g(h(X, Y), FZ) = g(A_{FZ}X, Y) = 0.$$

That means $h(X, Y) \in \mu$. Since $D$ is involutive it follows from (2.15) that

$$h(X, PY) = F\nabla_X Y + fh(X, Y).$$

Now, as $F(TM)$ and $\mu$ are orthogonal complementary sub bundles of the normal bundle $T^{\perp}(M)$, the last equation yields that $F\nabla_X Y = 0$ which means $\nabla_X Y \in D$. Hence each leaf of $D$ is totally geodesic in $M$. Further, let $N^0$ be a leaf of $D^0$ and $h^0$be the second fundamental form of the immersion of $N^0$ in $M$ , then for any $X, Y \in D$ and $Z, W \in D^0$,

$$g(h^0(Z, W), JX) = g(\nabla_Z W, JX) = g(\mathcal{P}_Z W - \nabla_Z PW + A_{FW}Z, X). \tag{5.9}$$

Interchanging $Z$ and $W$, the above relation takes the form

$$g(h^0(Z, W), JX) = g(P_W Z - \nabla_W PZ + A_{FZ}W, X). \tag{5.10}$$

As $\nabla_Z PZ \in D^0$ and $\overline{M}$ is nearly Kaehler, on adding (5.9) and (5.10) we obtain that

$$2g(h^0(Z, W), JX) = g(A_{FW}X, Z) + g(A_{FZ}X, W).$$

On using (5.5), the above equation yields,

$$g(h^0(Z, W), X) = -(JX\alpha)\ g(Z, W)$$

i.e.,

$$h^0(Z, W) = -g(Z, W)\nabla\alpha.$$

That shows that the leaves of $D^0$ are totally umbilical in $M$ with mean curvature vector $\nabla\alpha$. Moreover, the condition $Z\alpha = 0$ for all $Z \in D^0$, implies that the mean curvature vector is parallel. That is, the leaves of $D^0$ are extrinsic spheres in $M$. Hence, by virtue of a result in [13] which states that "If the tangent bundle of a Riemannian manifold $M$ splits into an orthogonal sum $TM = E_1 \oplus E_0$ of a non-trivial vector sub bundles such that $E_1$ is auto-parallel and its orthogonal complement $E_0$ is spherical, then the manifold $M$ is locally isometric to the warped product $N_1 \times_f N_0$", we conclude that $M$ is locally a generic warped product submanifold of $\overline{M}$ with warping function $e^{\alpha}$. This proves the theorem.

Let $N = N_\theta$ be a slant submanifold of a nearly Kaehler manifold $\bar{M}$ with Wirtinger angle $\theta$. Then the generic warped product submanifold $M = N_T \times_f N$ reduces to a semi-slant warped product submanifold. In what follows we study these submanifolds in $\bar{M}$ with the assumption that the Wirtinger angle $\theta \neq \pi/2$. When $\theta = \pi/2$, the submanifold $M$ is a CR-warped product submanifold which are studied in the setting of Kaehler manifolds by B. Y. Chen [6] and in the setting of nearly Kaehler manifolds by V. A. Khan et al. [15].

**Lemma 5.2.** *On a proper semi-slant warped product submanifold $M = N_T \times_f N_\theta$ of a nearly Kaehler manifold $\bar{M}$,*

$$g(h(X, PZ), FZ) = -g(h(X, Z), FPZ) = -\frac{1}{3}\cos^2(\theta)(X \ln f)\|Z\|^2 \tag{5.11}$$

*Proof.* Taking product with $PZ$ in equation (5.4) and making use of formula (3.5), we obtain

$$2g(h(X, Z), FPZ) - g(h(X, PZ), FZ) = \cos^2(\theta)(X \ln f)\|Z\|^2. \tag{5.12}$$

As $\theta \neq \pi/2$, replacing $Z$ by $PZ$ in the above equation and taking account of equation (3.4), we deduce that

$$2g(h(X, PZ), FZ) - g(h(X, Z), FPZ) = -\cos^2(\theta)(X \ln f)\|Z\|^2. \tag{5.13}$$

Adding equations (5.12) and (5.13), we get

$$g(h(X, Z), FPZ) = -g(h(X, PZ), FZ). \tag{5.14}$$

Using the relation (5.14) in (5.12), we obtain (5.11). That completes the proof.

**Corollary 5.1.** *Let $M$ be a proper semi-slant warped product submanifold of a nearly Kaehler manifold $\bar{M}$. Then*

$$g(\mathcal{P}_X Z, PZ) = \frac{2}{3}\cos^2(\theta)(X \ln f)\|Z\|^2. \tag{5.15}$$

*Proof.* The assertion follows on using (5.11) in relation (ii) of Lemma 5.1.

**Theorem 5.2.** *A proper semi-slant warped product submanifold $M = N_T \times_f N_\theta$ of a nearly Kaehler manifold $\bar{M}$ is trivial if and only if the following equivalent conditions hold*

(i) $g(h(X, Z), FPZ) = g(h(X, PZ), FZ)$

(ii) $g(\mathcal{P}_X Z, PZ) = 0$

*for each* $X \in TN_T$ *and* $Z \in TN_\theta$.

*Proof.* The two conditions are equivalent due to relation (ii) of Lemma 5.1. Moreover as $\theta \neq \pi/2$, the assertion follows from formula (5.15).

**Corollary 5.2.** *There exist no semi-slant warped product submanifolds* $N_T \times_f N_\theta$ *of a nearly Kaehler manifold such that* $h(X, Z) \in \mu$ *for* $X \in TN_T$ *and* $Z \in TN_\theta$.

**Corollary 5.3.** *There exist no mixed totally geodesic semi-slant warped product submanifolds in a nearly Kaehler manifold.*

Let us denote by $D$ and $D^\theta$ the tangent bundles on $N_T$ and $N_\theta$ respectively and let $\{X_1, \dots, X_p, X_{p+1} = JX_1, \dots, X_{2p} = JX_p\}$ and $\{Z_1, \dots, Z_q\}$ be local orthonormal frames of vector fields on $N_T$ and $N_\theta$ respectively with $2p$ and $q$ being their real dimensions. Then by (2.6),

$$\|h\|^2 = \sum_{i,\ j=1}^{2p} g(h(X_i, X_j), h(X_i, X_j)) + \sum_{i,\ j=1}^{2p} \sum_{r=1}^{q} g(h(X_i, Z_r), h(X_i, Z_r)) + \sum_{r,\ s=1}^{q} g(h(Z_r, Z_s), h(Z_r, Z_s)). \tag{5.16}$$

Now, on a semi-slant warped product submanifold of a nearly Kaehler manifold, we prove

**Theorem 5.3.** *Let* $M = N_T \times_f N_\theta$ *be a semi-slant warped product submanifold of a nearly Kaehler manifold* $\bar{M}$ *with* $N_T$ *and* $N_\theta$ *holomorphic and slant submanifolds respectively of* $\bar{M}$. *Then the squared norm of the second fundamental form* $h$ *satisfies*

$$\|h\|^2 \geq 2q \csc^2(\theta)\{1 + \frac{\cos^4(\theta)}{9}\}\|\nabla \ln f\|^2 \tag{5.17}$$

*where* $\nabla \ln f$ *is the gradient of* $\ln f$ *and* $q$ *is the dimension of* $N_\theta$.

*Proof.* In view of the decomposition (3.1), we may write

$$h(U, V) = h_{FD^\theta}(U, V) + h_\mu(U, V) \tag{5.18}$$

for each $U, V \in TM$, where $h_{FD^\theta}(U, V) \in FD^\theta$ and $h_\mu(U, V) \in \mu$ with

$$h_{FD^\theta}(U, V) = \sum_{r=1}^{q} h^r(U, V) FZ_r \tag{5.19}$$

where,

$$h^r(U, V) = \csc^2(\theta) g(h(U, V), FZ_r) \tag{5.20}$$

for each $U,\ V \in TM$.

Making use of formula (iii) of Lemma 5.1, formulae (5.19) and (3.6), we obtain

$$g(h_{FD^\theta}(PX_i, Z_r), h_{FD^\theta}(PX_i, Z_r)) = h^r(PX_i, Z_r)(X_i \ln f) + \sin^2(\theta) \sum_{s \neq r} (h^s(PX_i, Z_r))^2$$

for each $i = 1, \ldots, 2p$ and $r = 1, \ldots, q$. The above equation, on applying the formula (5.20) and formula (iii) of Lemma 5.1 takes the form

$$g(h_{FD^\theta}(PX_i, Z_r), h_{FD^\theta}(PX_i, Z_r)) = \csc^2(\theta)(X_i \ln f)^2 + \sin^2(\theta) \sum_{s \neq r} (h^s(PX_i, Z_r))^2.$$

Summing over $i = 1, \ldots, 2p$ and $r = 1, \ldots, q$, the last equation gives

$$\sum_{i=1}^{2p} \sum_{r=1}^{q} g(h_{FD^\theta}(PX_i, Z_r), h_{FD^\theta}(PX_i, Z_r)) = 2q \csc^2(\theta) \|\nabla \ln f\|^2 + \sin^2(\theta) \sum_{i=1}^{2p} \sum_{r,s=1,\ r \neq s}^{q} (h^s(PX_i, Z_r))^2. \tag{5.21}$$

Let us choose the orthonormal frame of vector fields on $D^\theta$ as $\{Z_1, \ldots, Z_{q/2}, PZ_1, PZ_2, \ldots, PZ_{q/2} = Z_q\}$. Then the second term in the right hand side of the last equation, on using (5.20) is written as

$$\csc^2(\theta) \sum_{i=1}^{2p} \Big[\sum_{r=1}^{q/2} \{(g(h(PX_i, Z_r), FPZ_r))^2 + (g(h(PX_i, PZ_r), FZ_r))^2\} + \sum_{r=1}^{q/2} \sum_{s=1,\ s \neq r}^{q/2} \{(g(h(PX_i, Z_r), FZ_s))^2 + (g(h(PX_i, PZ_r), FPZ_s))^2\}\Big].$$

On applying formula (5.11), the first two terms in the above expression reduce to

$$\csc^2(\theta) \sum_{i=1}^{2p} \Big[\sum_{r=1}^{q/2} \frac{4}{9} \cos^4(\theta)(PX_i \ln f)^2 \|Z_r\|^2\Big] = \frac{2q}{9} \cos^4(\theta) \csc^2(\theta) \|\nabla \ln f\|^2.$$

Taking account of the above into (5.21), we obtain

$$\sum_{i=1}^{2p} \sum_{r=1}^{q} g(h_{FD^\theta}(PX_i, Z_r), h_{FD^\theta}(PX_i, Z_r)) \geq 2q \csc^2(\theta)\{1 + \frac{1}{9} \cos^4(\theta)\} \|\nabla \ln f\|^2. \tag{5.22}$$

The inequality (5.17) follows from (5.16) and (5.22).

The equality in (5.17) holds if (i) $h(D, D) = 0$, (ii) $h(D^\theta, D^\theta) = 0$ and (iii) $h(X_i, PZ_r)$ is normal to $FPZ_s$ and $h(X_i, Z_r)$ is normal to $FZ_s$ for each $i = 1, \ldots, 2p$ and $r = 1, \ldots, q/2$, where $s \neq r$.

The inequality (5.17) provides a generalization to the inequality obtained in the setting of CR-warped products in Kaehler manifolds (cf. [6, Theorem 5.1]).

## References

[1] Beem, J. K.; Ehrlich, P. E.; Easley, K.: *Global Lorentzian geometry*. Marcel Dekker, New York 1996. Zbl 0846.53001

[2] Bishop, R. L.; O'Neill, B.: *Manifolds of Negative curvature*. Trans. Am. Math. Soc. **145** (1969), 1–49. Zbl 0191.52002

[3] Chen, B.-Y.: *Geometry of slant submanifolds*. Katholieke Universiteit Leuven, Leuven 1990. Zbl 0716.53006

[4] Chen, B.-Y.: *CR-submanifolds of a Kaehler Manifold. I.* J. Differ. Geom. **16** (1981), 305–322. Zbl 0431.53048

[5] Chen, B.-Y.: *Differential Geometry of Real Submanifolds in a Kaehler Manifold*. Monatsh. Math. **91** (1981), 257–275. Zbl 0451.53041

[6] Chen, B.-Y.: *Geometry of warped product CR-submanifolds in Kaehler Manifolds*. Monatsh. Math. **133** (2001), 177–195. Zbl 0996.53044

[7] Chen, B.-Y.: *Geometry of warped product CR-submanifolds in Kaehler Manifolds. II*. Monatsh. Math. **134** (2001), 103–119. Zbl 0996.53045

[8] Chen, B. Y.: *CR-warped products in complex projectives space with compact holomorphic factor*. Monatsh. Math. **141** (2004), 177–186. Zbl 1063.53056

[9] Chen, B.-Y.: *Geometry of warped products as Riemannian submanifolds and related problems*. Soochow J. Math. **28**(2) (2002), 125–156. Zbl 1012.53051

[10] Gray, A.: *Nearly Kaehler manifolds*. J. Differ. Geom. **4** (1970), 283–309. Zbl 0201.54401

[11] Gray, A.: *Almost Complex submanifolds of Six Sphere*. Proc. Am. Math. Soc. **20** (1969), 277–279. Zbl 0165.55803

[12] Gray, A.: *Some examples of almost Hermitian manifolds*. Ill. J. Math. **10** (1966), 353–366. Zbl 0183.50803

[13] Hiepko, S.: *Eine Innere Kennzeichnung verzerrter Produkte*. Math. Ann. **241** (1979), 209–215. Zbl 0387.53014

[14] Hong, S. T.: *Warped products and black holes*. Nuovo Cim. J. B **120** (2005), 1227–1234.

[15] Khan, K. A.; Khan, V. A.: *Totally umbilical CR-manifolds of a nearly Kaehler manifold*. Geom. Dedicata **50** (1994), 47–51. Zbl 0805.53053

[16] Khan, V. A.; Khan, K. A.; Siraj-Uddin: *Warped product CR-submanifolds in nearly Kaehler manifolds*. SUT J. Math. **43**(2) (2007), 201–213. Zbl 1147.53006

[17] Munteanu, M. I.: *A note on doubly warped product contact CR-submanifolds in trans-Sasakian manifolds*. Acta Math. Hung. **116**(1–2) (2007), 121–126. Zbl 1135.53319

[18] Mustafa, M. T.: *A non existence result for compact Einstein warped products*. J. Phys. A, Math. Gen. **38**(47) (2005), L791–L793. Zbl 1082.83007

[19] Papaghiue, N.: *Semi-slant submanifolds og Kaehlerian manifold*. Ann. St. Univ. Iasi, tom XL, S. I. 9($f_1$) (1994), 55–61.

[20] Sahin, B.: *Non existence of warped product semi-slant submanifolds of Kaehler manifolds*. Geom. Dedicata **117** (2006), 195–202. Zbl 1093.53059

[21] Sekigawa, K.: *Some CR-submanifolds in a 6-dimensional sphere*. Tensor (New Ser.) **41** (1984), 13–20. Zbl 0547.53027

[22] Seongtag, K.: *Warped products and Einstein metrics*. J. Phys. A, Math. Gen. **39** (2006), L329–L333. Zbl 1101.83006

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