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A Note on Ovals and their Evolutoides

Marco Hamann

Institute for Geometry, Dresden University of Technology
Zellescher Weg 12–14, Willersbau B107, D-1062 Dresden, Germany
e-mail: Marco.Hamann@tu-dresden.de

Abstract. In this paper we consider curves related to ovals, which can uniformly be generated by certain families of secants with respect to an arbitrary given oval. If e. g. the generating secants connect the points of contact of parallel tangents of the given oval then they envelop a curve, which contains information about the oval analogously to its evolute for example. Those curves are treated in different context in [2] and [6] too. In this article several properties can be treated into detail and extended respectively. In particular with help of this curve a global property to any evolutoide of the given oval can be shown, which is invariant under (regular) affine transformations. It generalizes a corresponding result about the evolute of the oval, see [2].

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1. Introduction

Curves related to ovals have been studied in many papers by mathematicians and engineers, too. Many questions were initiated by technical problems and have applications in engineering respectively. For example the study of isoptic curves is associated to cam mechanisms. As is generally known those curves are the loci, where the contour of the given oval is seen under fixed angle measure. Generalizations of isoptic curves are well-known: for example dually defined curves are generated by families of chords of equal length with respect to the given oval. They seem to be of some interest in current research too, see [4].

On the other hand in context to the generation of these curves by secants it seems to be natural to consider curves generated by a one-parameter family of certain secants with respect to an oval k. There exist numerous classical examples of it, in particular the evolute e associated to an oval k generated as envelope by the family of curve normals of k. As is generally known the definition of an evolute e can be generalized onto evolutoides e_α, if the normals of k are replaced by a one-parameter set of lines intersecting k isogonally. The constant angle (measure) of the intersection of the oriented line and the tangent in a point of k is denoted by α. For example those one-parameter sets of certain lines occur as ground view of horizontal tangents at points $P \in k$ of surfaces of constant slop, where k is itself a curve of constant slop.

In different context curves related to ovals are treated in [2]. The so-called *minimal curve* of a given oval k is defined as locus of certain centers Z at secants s, which connect the points of contact of parallel tangents of k. In a point Z the curvature of a certain convex curve k_Z related to k is stationary (with respect to $Z \in s$) and has an absolute minimum. It is shown in [2] that the set of points Z can be generated as envelope of the family $\{s\}$.

However the minimal curve might have further properties in context to the given oval k and its evolutoides e_α: in this article several properties can be treated into detail and extended respectively. In particular with help of this curve a global property to any evolutoide of the given oval can be shown, which is invariant under (regular) affine transformations. It generalizes a corresponding result about the evolute of the oval, see [2]. Finally when placing the emphasis on geometric optic minimal curves admit a generation by certain secants s with respect to an oval k.

2. Definition of a midenvelope

Let k be an oval in the Euclidean plane $\mathbb{E}^2$, that is a C^2-smooth strictly convex curve. Using a Cartesian coordinate system the curve k has non vanishing curvature κ. Without loss of generality the origin O of the coordinate system can be chosen as an inner point of the edge k. Then k is enveloped of all supporting (tangent) lines $x \cos t + y \sin t - p(t) = 0$ whereby the support function[1] $p : [0, 2\pi) \to \mathbb{R}^+$. Using complex numbers the oval k is parametrized by

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it},\ z(t) \in C^1 \tag{1}$$

whereby $t \in [0, 2\pi) \subset \mathbb{R}$. The dot in (1) stands for the derivation with respect to the real parameter t.

From the definition of k follows that for every $t \in [0, 2\pi)$ there exists a unique value $t+\pi$, so that the distinct supporting lines $l(t)$ and $l(t+\pi)$ with respect to k are parallel.[2] The points of contact are denoted by $z(t)$ and $z(t+\pi)$ respectively. Hence for $t \in [0, 2\pi)$ the oriented line $s(t)$ belonging to the ordered pair $(z(t), z(t+\pi))$ is well defined. In the following we investigate the family of secants $s(t)$

[1] Sometimes the support function is defined over the Gaussian image of a curve, see e. g. [1].

[2] More precisely it should be denoted $(t + \pi) \pmod{2\pi}$ instead of $t + \pi$.

through $z(t)$ and $z(t+\pi)$ for all t. Those secants can be geometrically interpreted as lines intersecting the (tangents of the) curve k in complementary angles $\alpha(t)$ and $\alpha(t+\pi)$. But notice the measure of $\alpha(t)$ changes generally when changing t.

Definition 1. *Let k be an oval parametrized by* (1). *Then the family of oriented lines $s(t)$ belonging to the ordered pairs $(z(t), z(t+\pi))$ of k for $t \in [0, 2\pi)$ determines a set of common points and a curve c respectively, which is the envelope of this family of certain secants of k. They should be named shortly diameters $s(t)$ and midenvelope c with respect to k.*[3]

Example 1. As is generally known the family of diameters $s(t)$ with respect to a curve k of constant width are the double normals of k. Hence the midenvelope c is the evolute of k. In particular if k is a circle then c coincides with the center of k, $s(t)$ are diameters of k.

To compute the equation for the midenvelope c and investigate respectively whether c contains common points of the diameters $s(t)$ we first have to compute the equation of $s(t)$ as the line connecting two distinct points $z(t)$ and $z(t+\pi)$ of k. Using the notation in (1) we obtain

$$0 = \big(a(t)\sin t + \dot{a}(t)\cos t\big)x + \big(-a(t)\cos t + \dot{a}(t)\sin t\big)y - b(t) \tag{2}$$

with $a(t) = p(t) + p(t+\pi)$ and $b(t) = p(t)\dot{p}(t+\pi) - \dot{p}(t)p(t+\pi)$. As a necessary condition that c is an envelope with respect to $\{s(t)\}$, we solve the linear equation system given by (2) and the subsequent equation (3), which is determined by derivation of left and right hand side of (2) with respect to t.

$$0 = \cos t\big(a(t) + \ddot{a}(t)\big)x + \sin t\big(a(t) + \ddot{a}(t)\big)y - \dot{b}(t). \tag{3}$$

Obviously (2) and (3) do not change if t is replaced by $t+\pi$, hence the solutions at t and $t+\pi$ coincide. If $b(t) = 0$ then the linear equation system is homogeneous. Since the radius of curvature of the oval k is greater than 0 for any t, furthermore evidently $a(t) \neq 0$, the unique solution of the homogeneous linear equation system is determined by $z_c = 0$.[4] It is easy to verify that $b(t) = 0$ if and only if the function $f : [0, \pi) \to \mathbb{R}^+$ with $t \mapsto p(t)/p(t+\pi)$ has a local extremum and a point of inflection with horizontal stationery tangent at t respectively or is constant. Therefore the condition $b(t) = 0$ for every $t \in [0, 2\pi)$ admits a geometric interpretation with respect to the given oval: k is then centrally symmetric. In this case the family of diameters $s(t)$ is a pencil of lines through $z_c = 0$.

Generaly the solution of the linear equation system given by (2) and (3) gives a parametrization of c. We excluded the case the solution z_c is fixed for every

[3] The pairs $(l(t), l(t+\pi))$ of parallel supporting lines, the family of diameters $s(t)$ and the midenvelope c are affine geometric properties. They are studied in different context in [2] too. With respect to the definition in [2] we will use different names for $s(t)$ and c.

[4] The radius r of curvature is determined by the support function, thus $r = p(t) + \ddot{p}(t) > 0$, see e. g. [5].

$t \in [0, 2\pi)$ and equivalent, k is a centrally symmetric oval respectively.[5] Using complex numbers we get

$$z_c(t) = \big(a(a+\ddot{a})\big)^{-1}\big(a\dot{b}e^{it} + (\dot{a}\dot{b} - b(a+\ddot{a}))ie^{it}\big)(t) \tag{4}$$

where $t \in [0, t) \pmod{\pi}$. Obviously the denominator in (4) cannot vanish in $[0, 2\pi)$.

To obtain a proposition, when a midenvelope c is regular at t, we consider the tangent vector to c at t. Equivalent to the zero tangent vector the condition $|\dot{z}_c(t)| = 0$ is fulfilled. Provided that k is an oval we get

$$0 = \big((a+\ddot{a})(ab - \dot{a}\dot{b} + a\ddot{\dot{b}} + b\ddot{a}) - a\dot{b}(\dot{a} + \dot{\ddot{a}})\big)(t) \tag{5}$$

and replacing a and b and their derivations by the support function p

$$0 = \big(p(t+\pi) + \ddot{p}(t+\pi)\big)\big(\dot{p}(t) + \dot{\ddot{p}}(t)\big) - \big(\dot{p}(t+\pi) + \dot{\ddot{p}}(t+\pi)\big)\big(p(t) + \ddot{p}(t)\big). \tag{6}$$

Note the functions a, b and the support function p in (5) and (6) respectively are assumed to be of class C^3. Evidently the second condition is geometrically interpreted by the curvature radii of k at t and $t+\pi$: as noticed before the radius of curvature of k at t is given by $r := p(t) + \ddot{p}(t)$, hence the midenvelope c with respect to k is singular at t if and only if the function $g : [0, \pi) \to \mathbb{R}^+$ with $t \mapsto r(t)/r(t+\pi)$ has a local extremum and a point of inflection with horizontal stationery tangent at t respectively or is constant.

Proposition 1. *Let k be an arbitrary oval parametrized by* (1) *and c the midenvelope with respect to k parametrized by* (4). *Furthermore the support function p in* (1) *is of class C^3. Then c is singular at t* $\pmod{\pi}$ *if and only if* (6) *is fulfilled and the function $g : [0, \pi) \to \mathbb{R}^+$ with $t \mapsto r(t)/r(t+\pi)$ has a local extremum and a point of inflection with horizontal stationary tangent at t respectively or is constant.*

Example 2. Obviously for centrally symmetric ovals $r(t)/r(t+\pi) = 1$ for every $t \in [0, 2\pi)$. The midenvelope c is determined by the center of k and therefore singular at any $t \pmod{\pi}$.

Example 3. The midenvelope c belonging to a curve k of constant width is determined by its evolute e. Therefore $a(t) = p(t)+p(t+\pi) = r(t)+r(t+\pi) = const.$ It is easy to verify, if k is different from a circle, the function $g : [0, \pi) \to \mathbb{R}^+$ has local extrema exactly at those t, where $r(t)$ is extremal too. This is not surprising, because the evolute of a smooth curve with nonvanishing curvature is generally singular at those t, at which k has vertices.

As is generally known the curvature of a smooth plane curve, which is parametrized by $z : I \to \mathbb{C}$ with $z(I) \in C^2(I)$, can be computed by

$$\kappa(t) = \frac{\dot{\bar{z}}(t)\ddot{z}(t) - \dot{z}(t)\ddot{\bar{z}}(t)}{2i|\dot{z}(t)|^3}. \tag{7}$$

[5] In the case of a fixed solution we easily get $b(t) = 0$ for all $t \in [0, 2\pi)$ by changing the coordinate system.

Using (4) and (5) the curvature of the midenvelope c with respect to an oval k can be computed with reference to a, b and their derivations by

$$\kappa_c(t) = \frac{a^4(a+\ddot{a})^4 b}{\left(a^2+\dot{a}^2\right)^{\frac{3}{2}}\left((a+\ddot{a})(ab-\dot{a}\dot{b}+a\ddot{b}+b\ddot{a})-a\dot{b}(\dot{a}+\dddot{a})\right)^2}(t)\,. \tag{8}$$

It easily can be verified the expression $\left(a^2+\dot{a}^2\right)^{\frac{1}{2}}(t)$ in (8) admits an geometric interpretation as the length of the secant segment between $z(t)$ and $z(t+\pi)$. For a strictly convex smooth curve k it never vanishes.

3. Relation to the evolute of *k*

As shown in Section 2 the midenvelope c with respect to an oval k coincides with the evolute e of k, if and only if k is a curve of constant width. In general the points of c and e at t and $t+\pi$ respectively are different from each other. It seems natural to ask whether they have a certain position to each other, compare also [2].

Let k be an oval parametrized by (1). Using complex numbers the evolute e of k can be described by equation

$$z_e(t) = \dot{p}(t)ie^{it} - \ddot{p}(t)e^{it}. \tag{9}$$

We ask whether the points $z_e(t)$, $z_e(t+\pi)$ and $z_c(t)$ are collinear. Therefore we compute an equation of the (oriented) line g belonging to the (ordered) pair $(z_e(t), z_e(t+\pi))$ and verify a condition when g contains $z_c(t)$. Using the notation in (9) we obtain

$$z_e(t+\pi) = -\dot{p}(t+\pi)ie^{it} + \ddot{p}(t+\pi)e^{it} \tag{10}$$

and further an equation of $g(t)$

$$0 = \left(\dot{a}(t)\cos t - \ddot{a}(t)\sin t\right)x + \left(\dot{a}(t)\sin t + \ddot{a}(t)\cos t\right)y - d(t) \tag{11}$$

with $d(t) = \dot{p}(t)\ddot{p}(t+\pi) - \ddot{p}(t)\dot{p}(t+\pi)$. Obviously (11) does not change if t is replaced by $t+\pi$. Hence the solutions at t and $t+\pi$ coincide. Furthermore $g(t)$ is undefined at t if and only if $z_e(t)$ and $z_e(t+\pi)$ coincide. Using (11) we obtain $\dot{a}=0$ and $\ddot{a}=0$. In particular these conditions are fulfilled if the width of k (with respect to a certain direction) is stationary.

Verifying a condition, when $g(t)$ contains $z_c(t)$ using the representations (11) and (4), we obtain a global property of c with respect to an arbitrary oval k.

Proposition 2. *Let k be an oval and e its evolute with parameter representations* (1) *and* (9) *respectively. Furthermore the midenvelope c with respect to k is parametrized by* (4)*. If $z_e(t)$ and $z_e(t+\pi)$ at $t \in [0,\pi)$ are different from each other the connecting line $g(t)$ is well defined and contains the associated point $z_c(t)$, i.e. the points $z_e(t)$, $z_e(t+\pi)$ and $z_c(t)$ are collinear. Otherwise those three points coincide at $t \in [0,\pi)$.*[6]

[6]Proposition 2 is similarly elaborated in [2]. With regard to a consistent argumentation and an extension of Proposition 2 for example we have decided to note it down.

The coincidence of the points $z_e(t)$, $z_e(t+\pi)$ and $z_c(t)$ in Proposition 2 can easily verified: If $z_e(t) = z_e(t+\pi)$ or equivalent $\dot{a} = 0$ and $\ddot{a} = 0$, then the diameter $s(t)$ is a double normal of k at t and the width of k is stationary. Hence the point $z_c(t)$ coincide with $z_e(t) = z_e(t+\pi)$, compare Example 1.

Corollary 3. *If the points $z_e(t)$, $z_e(t+\pi)$ and $z_c(t)$ are collinear (and do not coincide) at $t \in [0,\pi)$ then $z_c(t)$ divides the line segments given by $z_e(t)$ and $z_e(t+\pi)$ as well as $z(t)$ and $z(t+\pi)$ at the same ratio $r(t)/r(t+\pi)$ of the curvature radii of k at t and $t+\pi$.*[7] *Otherwise if those three points coincide at $t \in [0,\pi)$ then the statement is evidently true for the line segment given by $z(t)$ and $z(t+\pi)$.*

Note Corollary 3 can be verified using the theorem on intersecting lines: Let k be an oval parametrized by (1). Then the normals $n(t)$ and $n(t+\pi)$ to k at t and $t+\pi$ are parallel to each other. If $z_e(t)$ and $z_e(t+\pi)$ do not coincide the line $g(t)$ connecting $z_e(t)$ and $z_e(t+\pi)$ contains $z_c(t)$ by Proposition 2. The points $z(t)$, $z(t+\pi)$ and $z_c(t)$ are collinear by Definition 1. Certainly $g(t)$ is different from $s(t)$.

As a conclusion of Proposition 2 and Corollary 3 we get a global property on the locus of a midenvelope c with respect to an arbitrary oval k.

Corollary 4. *Let k be an oval parametrized by* (1)*, further c the midenvelope with respect to k parametrized by* (4)*. Since k is a closed strictly convex curve, furthermore the curvature $\kappa > 0$, the midenvelope c with respect to k lies in the interior of k.*

4. Evolutoides

Let k be an oval and c the midenvelope with respect to k. According to [8] the envelope e_α of a one-parameter set of oriented lines $l_\alpha(t)$, which isogonally intersect k, is called *evolutoide*. The constant angle measure $\measuredangle(l(t), l_\alpha(t))$ of intersection at any $t \in [0, 2\pi)$ is denoted by α. In the cases $\alpha = 0$ and $\alpha = \frac{\pi}{2}$ we speak of k and its evolute e respectively. Since for these curves the lines $s(t)$ and $g(t)$ connecting $z(t)$, $z(t+\pi)$ and $z_e(t)$, $z_e(t+\pi)$ both contain $z_c(t)$ at every $t \in [0,\pi)$, it seems natural to ask whether for $\alpha \in [0,\pi)$ ($\alpha \neq 0, \frac{\pi}{2}$) exist evolutoides e_α of k with the same property with respect to c.

First we compute a parametrization of the evolutoides e_α taking the same methods used in Section 2. Using the representation in (1) we obtain an equation for the line $l_\alpha(t)$

$$0 = x\cos(t-\alpha) + y\sin(t-\alpha) - \big(p(t)\cos\alpha - \dot{p}(t)\sin\alpha\big) \tag{12}$$

where $t \in [0, 2\pi)$ and α is fixed. As a necessary condition that e_α is an envelope with respect to $\{l_\alpha(t)\}$, we solve the linear equation system given by (12) and

[7] More precisely $z_c(t)$ divide the line segments at their interior.

the subsequent equation (13), which is determined by derivation of left and right hand side of (12) with respect to t.

$$0 = -x \sin(t-\alpha) + y \cos(t-\alpha) - \big(\dot{p}(t)\cos\alpha - \ddot{p}(t)\sin\alpha\big) \tag{13}$$

The solution of the equation system generally gives a parametrization of the evolutoides e_α. Using complex numbers we obtain

$$z_e(t,\alpha) = p(t)e^{i(t-\alpha)}\cos\alpha + \dot{p}(t)ie^{it} - \ddot{p}(t)ie^{i(t-\alpha)}\sin\alpha. \tag{14}$$

Obviously, the parametrization (14) describes the oval k and its evolute e respectively if and only if $\alpha = 0$ and $\alpha = \frac{\pi}{2}$ respectively, compare (1) and (9). Therefore the lines connecting $z_e(t,0)$, $z_e(t+\pi,0)$ and $z_e(t,\frac{\pi}{2})$, $z_e(t+\pi,\frac{\pi}{2})$ are determined by $s(t)$ and $g(t)$, further both contain $z_c(t)$.

Now we ask whether for $\alpha \in [0,\pi)$ ($\alpha \neq 0, \frac{\pi}{2}$) exist evolutoides e_α of k with the same property with respect to c. Therefore we compute an equation of the line $g(t,\alpha)$ connecting the points $z_e(t,\alpha)$ and $z_e(t+\pi,\alpha)$. $g(t,\alpha)$ is well defined as long as those points differ from each other. Using the notation as before we get

$$\begin{aligned} 0 = {} & \big(a(t)\sin(t-\alpha)\cos\alpha + \dot{a}(t)\cos t - \ddot{a}(t)\cos(t-\alpha)\sin\alpha\big)x \\ & - \big(a(t)\cos(t-\alpha)\cos\alpha - \dot{a}(t)\sin t + \ddot{a}(t)\sin(t-\alpha)\sin\alpha\big)y \\ & - \big(b(t)\cos^2\alpha - \dot{b}(t)\cos\alpha\sin\alpha + d(t)\sin^2\alpha\big). \end{aligned} \tag{15}$$

With help of the previous equation we can specify a condition, whether the points $z_e(t,\alpha)$, $z_e(t+\pi,\alpha)$ and $z_c(t)$ are collinear: For a chosen $\alpha \in [0,\pi)$ the line $g(t,\alpha)$ connecting $z_e(t,\alpha)$ and $z_e(t+\pi,\alpha)$ contains $z_c(t)$ if and only if the parametrization (4) of c fulfills (15) at t. Based on this condition we obtain an generalization of Proposition 2.

Proposition 5. *Let k be an oval and c its midenvelope parametrized by* (1) *and* (4) *respectively. Furthermore the family of evolutoides e_α with respect to k is denoted by $\{e_\alpha | \alpha \in [0,\pi)\}$. Any e_α admits a parametrization by* (14)*. If $z_e(t,\alpha)$ and $z_e(t+\pi,\alpha)$ at $t \in [0,\pi)$ are different from each other, the connecting line $g(t,\alpha)$ is well defined and contains the associated point $z_c(t)$, i.e. the points $z_e(t,\alpha)$, $z_e(t+\pi,\alpha)$ and $z_c(t)$ are collinear.*

Note the points $z_e(t,\alpha)$ and $z_e(t+\pi,\alpha)$ coincide if and only if $|z_e(t,\alpha) - z_e(t+\pi,\alpha)| = 0$. Using (14) we obtain

$$0 = a^2(t)\cos^2\alpha + \dot{a}^2(t) + \ddot{a}^2(t)\sin^2\alpha - 2a(t)\dot{a}(t)\sin\alpha - 2\dot{a}(t)\ddot{a}(t)\cos\alpha \tag{16}$$

which is evidently fulfilled if the width of k is stationary and $\alpha = \frac{\pi}{2}$, compare Examples 1 and 3.

In the case the points $z_e(t,\alpha)$, $z_e(t+\pi,\alpha)$ and $z_c(t)$ are collinear (and do not coincide) we obtain a generalization of Corollary 3, which analogously can be verified.

Corollary 6. *If the points $z_e(t,\alpha)$, $z_e(t+\pi,\alpha)$ and $z_c(t)$ are collinear (and do not coincide) at $t \in [0,\pi)$ then $z_c(t)$ divides the line segments given by $z_e(t,\alpha)$ and $z_e(t+\pi,\alpha)$ as well as $z(t)$ and $z(t+\pi)$ at the same ratio $r(t)/r(t+\pi)$ of the curvature radii of k at t and $t+\pi$.*

Verifying Corollary 6 the case $g(t,\alpha) = g(t+\pi,\alpha) = s(t)$ has to be considered for its own: It is a matter of common knowledge for a continuous motion of a moving 2-frame along a plane curve $k \in C^2$ that the instantaneous center of this motion coincides with the center of curvature with respect to k, see [7]. Furthermore the normal of a kinematically generated curve at t contains the instantaneous center at t. Referring to the current corollary the locus of any points $z_e(t,\alpha)$ at t with $\alpha \in [0,\pi)$ is a circle with diameter $[z(t), z_e(t)]$. Hence the statement in Corollary 6 evidently continues to be valid if $g(t,\alpha) = g(t+\pi,\alpha) = s(t)$, because the functions $\alpha \to |z_e(t,\alpha) - z_c(t)|$ and $\alpha \to |z_e(t+\pi,\alpha) - z_c(t)|$ are continuous.

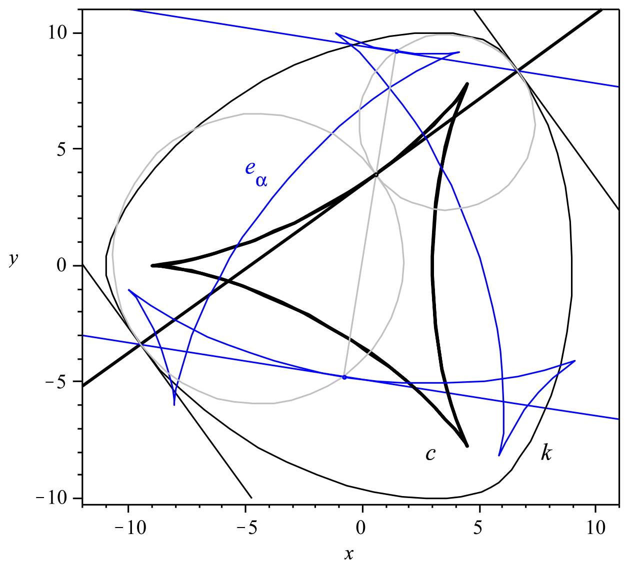

Figure 1. Midenvelope c and evolutoide e_α with respect to an oval k

Example 4. In Figure 1 a given oval k and its midenvelope c are depicted. The support function p of k is given by $p(t) := \cos(nt) + m$ with $n = 2l+1$, $l \in \mathbb{Z}^+$, further $m \in \mathbb{R}^+$ and $m > n^2 - 1$, see [3]. Obviously $a(t) := p(t) + p(t+\pi) = 2m$ at any $t \in [0, 2\pi)$ and therefore k is a curve of constant width: c and the evolute e of k coincide. Further on the evolutoide $e(\alpha_0)$ and the loci of points $\{z_e(t_0,\alpha) | \alpha \in [0,\pi)\}$ and $\{z_e(t_0+\pi,\alpha) | \alpha \in [0,\pi)\}$ are depicted too. The figure visualizes the statements in Proposition 5 and Corollary 6.

5. Conclusion

As conclusions of Proposition 5 and Corollary 6 we can give geometric interpretations on the points of the midenvelope c with respect to corresponding points of any evolutoide e_α of an arbitrary oval k.

Since the connecting lines of $z_c(t)$ and $z_e(t,\alpha)$ at t are contained in a pencil with carrier $z_c(t)$, furthermore the induced ratios equal $r(t)/r(t+\pi)$ at any $\alpha \in [0,\pi)$, we can treat all corresponding points $z_e(t,\alpha)$, $z_e(t+\pi,\alpha)$ as preimage and image respectively of a dilatation $\sigma(t)$ given by the center $z_c(t)$ and the fixed ratio $r(t)/r(t+\pi)$. In particular the point $z(t)$, its tangent $l(t)$ and center of curvature $z_e(t)$ of k at t are mapped by $\sigma(t)$ onto $z(t+\pi)$, $l(t+\pi)$ and $z_e(t+\pi)$ respectively. Hence $k^{\sigma(t)}$ is in 2^{nd} order contact to k in $z(t+\pi)$. Furthermore any pair $\big(z_e(t,\alpha), l_\alpha(t)\big)$ with $\alpha \in [0,\pi)$ is mapped onto $\big(z_e(t+\pi,\alpha), l_\alpha(t+\pi)\big)$ by $\sigma(t)$.

When t passes through $[0,2\pi)$ the midenvelope c can therefore be treated as the locus of all centers of dilatations $\sigma(t)$ as described before. Note there are evidently alternative ways to define contact with respect to k at arbitrary $(t_1,t_2) \in [0,2\pi) \times [0,2\pi)$ by similarities. Otherwise the pairs $(t_1,t_2) = (t, t+\pi)$ with $t \in [0,\pi)$ and therefore c are geometrically determined by k. In particular for central symmetric ovals the midenvelope c is given by the center of symmetry, compare Example 2. In addition in case of an ellipse k the evolutoides e_α are affine parastroides, hence central symmetric (with centre c), see [8].

With help of the midenvelope c we can finally simplify the construction of evolutoides e_α with respect to an oval k.

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