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Hereditary Right Jacobson Radical of type-0(e) for Right Near-rings

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Abstract. Near-rings considered are right near-rings and R is a near-ring. The first two authors introduced right Jacobson radicals of type-0, 1 and 2 for right near-rings. Recently, the authors have shown that these right Jacobson radicals are Kurosh-Amitsur radicals (KA-radicals) in the class of all zero-symmetric near-rings but they are not ideal-hereditary in that class. In this paper right R-groups of type-0(e), right 0(e)-primitive ideals and right 0(e)-primitive near-rings are introduced. Using them the right Jacobson radical of type-0(e) is introduced for near-rings and is denoted by $\mathrm{J}^r_{0(e)}$. A right 0(e)-primitive ideal of R is an equiprime ideal of R. It is shown that $\mathrm{J}^r_{0(e)}$ is a KA-radical in the class of all near-rings and is an ideal-hereditary radical in the class of all zero-symmetric near-rings.

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1. Introduction

R denotes a right near-ring and all near-rings considered are right near-rings and not necessarily zero-symmetric.

The left Jacobson radical J_ν is not a Kurosh-Amitsur radical (KA-radical) in the class of all near-rings, $\nu \in \{0, 1, 2\}$. It is not known whether the left Jacobson radical J_3 is a KA-radical in the class of all near-rings. Veldsman [13] introduced the left Jacobson radicals $J_{2(0)}$ and $J_{3(0)}$ for near-rings. These two are the only known Jacobson-type radicals which are KA-radicals in the class of all near-rings. Moreover, these two radicals are ideal-hereditary in the class of all zero-symmetric near-rings. It is also known that there is no non-trivial ideal-hereditary radical in the class of all near-rings.

In [5] and [6] the first author studied the structure of near-rings in terms of right ideals and showed that as for rings, matrix units determined by right ideals identifies matrix near-rings. In order to show the importance of the right Jacobson radicals of near-rings in the extension of a form of the Wedderburn-Artin theorem of rings involving the matrix rings to near-rings, the right Jacobson radicals of type-ν were introduced and studied by the first and second author in [7], [8], [9] and [10], $\nu \in \{0, 1, 2, s\}$.

In [11] and [12] the authors have shown that the right Jacobson radicals of type-0, 1 and 2 introduced by the first two authors are KA-radicals in the class of all zero-symmetric near-rings but they are not ideal-hereditary in that class. In this paper right R-groups of type-0(e), right 0(e)-primitive ideals and right 0(e)-primitive near-rings are introduced. Using them the right Jacobson radical of type-0(e) is introduced for near-rings and is denoted by $J^r_{0(e)}$. A right 0(e)-primitive ideal of R is an equiprime ideal of R. It is shown that $J^r_{0(e)}$ is a KA-radical in the class of all near-rings and is an ideal-hereditary radical in the class of all zero-symmetric near-rings.

2. Preliminaries

Near-rings considered are right near-rings and not necessarily zero-symmetric. Unless otherwise specified R stands for a right near-ring. Near-ring notions not defined here can be found in Pilz [4].

R_0 and R_c denote the zero-symmetric part and constant part of R respectively.

Now we give here some definitions and results of [7] which will be used later.

An element a $\in$ R is called *right quasi-regular* if and only if the right ideal of R generated by the set $\{x - ax \mid x \in R\}$ is R. A right ideal (left ideal, ideal, subset) K of R is called a *right quasi-regular right ideal (left ideal, ideal, subset)* of R, if each element of K is right quasi-regular.

A right ideal K of R is called *right modular* if there is an element e $\in$ R such that x - ex $\in$ K for all x $\in$ R. In this case we say that K is *right modular by e.*

A maximal right modular right ideal of R is called a *right 0-modular right ideal* of R.

$J^r_{1/2}(R)$ is the intersection of all right 0-modular right ideals of R and if R has no right 0-modular right ideals, then $J^r_{1/2}(R)$ = R. The largest ideal of R contained in $J^r_{1/2}$(R) is denoted by $J^r_0(R)$ and is called the *right Jacobson radical of R of type-0*.

The largest ideal contained in a right 0-modular right ideal of R is called a *right 0-primitive ideal* of R. R is called a *right 0-primitive near-ring* if {0} is a right 0-primitive ideal of R.

A group (G, +) is called a *right R-group* if there is a mapping ((g, r) → gr) of G×R into G such that 1. (g + h)r = gr + hr, 2. g(rs) = (gr)s, for all g, h ∈ G and r, s ∈ R. A subgroup (normal subgroup) H of a right R-group G is called an *R-subgroup (ideal)* of G if hr ∈ H for all h ∈ H and r ∈ R.

Let G be a right R-group. An element g ∈ G is called a *generator* of G if gR = G and g(r + s) = gr + gs for all r, s ∈ R. G is said to be *monogenic* if G has a generator.

G is said to be *simple* if G ≠ {0}, and G, {0} are the only ideals of G.

A monogenic right R-group G is said to be a *right R-group of type-0* if G is simple.

The *annihilator* of G denoted by (0 : G) is defined as (0 : G) = {a ∈ R | Ga = {0}}.

Lemma 2.1. *The constant part of R is right quasi-regular.*

Lemma 2.2. *A nilpotent element of R is right quasi-regular.*

Theorem 2.3. *$J^r_{1/2}(R)$ is the largest right quasi-regular right ideal of R.*

Theorem 2.4. *$J^r_0(R)$ is the largest right quasi-regular ideal of R.*

Theorem 2.5. *$J^r_0(R)$ is the intersection of all right 0-primitive ideals of R.*

Theorem 2.6. *Let P be an ideal of R. P is a right 0-primitive ideal of R if and only if R/P is a right 0-primitive near-ring.*

Proposition 2.7. *Let G be a right R-group of type-0 and g_0 be a generator of G. Then (0 : g_0) := $\{r \in R \mid g_0 r = 0\}$ is a right 0-modular right ideal of R.*

Proposition 2.8. *Let G be a right R-group. G is a right R-group of type-0 if and only if there is a maximal right modular right ideal K of R such that G is R-isomorphic to R/K.*

Proposition 2.9. *Let P be an ideal of a zero-symmetric near-ring R. P is right 0-primitive if and only if P is the largest ideal of R contained in (0 : G) for some right R-group G of type-0.*

A near-ring R is called an *equiprime near-ring* if 0 ≠ a ∈ R, x, y ∈ R and arx = ary for all r ∈ R, implies x = y. An ideal I of R is called *equiprime* if R/I is an equiprime near-ring.

It is known that a near-ring R is equiprime if and only if

1. x, y ∈ R and xRy = {0} implies x = 0 or y = 0.

2. If $\{0\} \neq I$ is an invariant subnear-ring of R, x, y $\in$ R and ax = ay for all a $\in$ I implies x = y.

Moreover, an equiprime near-ring is zero-symmetric.

If I is an ideal of R, then we denote it by $I \lhd R$. A subset S of R is *left invariant* if $RS \subseteq S$. By a radical class we mean a radical class in the sense of Kurosh-Amitsur.

Let $\mathcal{E}$ a class of near-rings. $\mathcal{E}$ is called *regular*, if $\{0\} \neq I \lhd R \in \mathcal{E}$ implies that $0 \neq I/K \in \mathcal{E}$ for some $K \lhd I$. It is known that, if $\mathcal{E}$ is a regular class, then $\mathcal{U}\mathcal{E} = \{R \mid R$ has no non-zero homomorphic image in $\mathcal{E}\}$ is a radical class, called the *upper radical* determined by $\mathcal{E}$. The *subdirect closure* of a class of near-rings $\mathcal{E}$ is the class $\overline{\mathcal{E}} = \{R \mid R$ is a subdirect sum of near-rings from $\mathcal{E}\}$. A class $\mathcal{E}$ is called hereditary if $I \lhd R \in \mathcal{E}$ implies $I \in \mathcal{E}$. $\mathcal{E}$ is called *c-hereditary* if I is a left invariant ideal of $R \in \mathcal{E}$ implies $I \in \mathcal{E}$. It is clear that a hereditary class is a regular class. If $I \lhd R$ and for every non zero ideal J of R, $J \cap I \neq \{0\}$, then I is called an *essential ideal* of R and is denoted by $I \lhd \cdot R$. A class of near-rings $\mathcal{E}$ is called closed *under essential extensions (essential left invariant extensions)* if $I \in \mathcal{E}$, $I \lhd \cdot R$ (I is an essential ideal of R which is left invariant) implies $R \in \mathcal{E}$. A class of near-rings $\mathcal{E}$ is said to satisfy condition (F_l) if $K \lhd I \lhd R$, and I is left invariant in R and $I/K \in \mathcal{E}$, then $K \lhd R$.

In [2], G. L. Booth and N. J. Groenewald defined special radicals for near-rings. A class $\mathcal{E}$ consisting of equiprime near-rings is called a *special class* if it is hereditary and closed under left invariant essential extensions. If $\mathcal{R}$ is the upper radical in the class of all near-rings determined by a special class of near-rings, then $\mathcal{R}$ is called a *special radical.* If $\mathcal{R}$ is a radical class, then the class $\mathcal{S}\mathcal{R} = \{R \mid \mathcal{R}(R) = \{0\}\}$ is called the *semisimple class* of $\mathcal{R}$.

We also need the following theorem:

Theorem 2.10. (Theorem 2.4 of [13]) *Let $\mathcal{E}$ be a class of zero-symmetric near-rings. If $\mathcal{E}$ is regular, closed under essential left invariant extensions and satisfies condition* (F_l)*, then $\mathcal{R} := \mathcal{U}\mathcal{E}$ is c-hereditary radical class in the variety of all near-rings, $\mathcal{S}\mathcal{R} = \overline{\mathcal{E}}$ and $\mathcal{S}\mathcal{R}$ is hereditary. So, $\mathcal{R}(R) = \cap \{I \lhd R \mid R/I \in \mathcal{E}\}$ for any near-ring R.*

Remark 2.11. Since all ideals in a zero-symmetric near-ring are left invariant, under the hypothesis of Theorem 2.10, in the variety of zero-symmetric near-rings both $\mathcal{R}$ and $\mathcal{S}\mathcal{R}$ are hereditary and hence the radical is ideal-hereditary, that is, if $I \lhd R$, then $\mathcal{R}(I) = I \cap \mathcal{R}(R)$.

Proposition 2.12. (Proposition 3.3 of [1]) *The class of all equiprime near-rings is closed under essential left invariant extensions.*

Proposition 2.13. (Corollary 2.4 of [1]) *The class of all equiprime near-rings satisfies condition* (F_l).

3. Right Jacobson radical of type-0(e)

Throughout this section R stands for a right near-ring. R_0 and R_c denote the zero-symmetric part and the constant part of R respectively.

Note that if G is a right R-group, then H := {g ∈ G | gR = {0}} is an ideal of G. This means, if G is a right R-group of type-0, then gR = {0} implies g = 0.

Proposition 3.1. *Let G be a right R-group. Then $GR_c = \{0\}$ if and only if G0 = {0}.*

Proof. If $GR_c = \{0\}$, then clearly, G0 = {0}. Suppose that G0 = {0}. Let g ∈ G. Now $0 = (gr_c)0 = g(r_c0) = gr_c$ for all $r_c \in R_c$. Therefore, $GR_c = \{0\}$. □

Proposition 3.2. *Let G be a right R-group of type-0. If R_c is contained in a right quasi-regular right ideal of R, then $GR_c = \{0\}$.*

Proof. Let g_0 be a generator of G. Suppose that R_c is contained in a right quasi-regular right ideal K of R. Since $(0 : g_0) = \{r \in R \mid g_0r = 0\}$ contains the largest right quasi-regular right ideal of R, $K \subseteq (0 : g_0)$. So $g_0K = \{0\}$ and hence $g_0R_c = \{0\}$. Let g ∈ G. Now $g = g_0s$ for some s ∈ R. So, $gr_c = (g_0s)r_c = g_0(sr_c) = 0$ for all $r_c \in R_c$ as $sr_c \in R_c$. Therefore, $GR_c = \{0\}$. □

Corollary 3.3. *Let G be a right R-group of type-0. If the normal subgroup of (R, +) generated by R_c is right quasi-regular, then $GR_c = \{0\}$.*

Proof. Suppose that $< R_c >_n$ is the normal subgroup of (R, +) generated by R_c. Let $x \in < R_c >_n$. Now $x = (r_1 + y_1 - r_1) + (r_2 + y_2 - r_2) + ... + (r_k + y_k - r_k)$, where $r_i \in R$, $y_i \in R_c$. Now $xr = ((r_1 + y_1 - r_1) + (r_2 + y_2 - r_2) + ... + (r_k + y_k - r_k))r = (r_1r + y_1r - r_1r) + (r_2r + y_2r - r_2r) + ... + (r_kr + y_kr - r_kr) \in < R_c >_n$ as $y_ir \in R_c$. So, $< R_c >_n$ is a right ideal of R. Since $< R_c >_n$ is a right quasi-regular right ideal of R containing R_c, by Proposition 3.2, $GR_c = \{0\}$. □

Corollary 3.4. *Let G be a right R-group of type-0. If R_c is a normal subgroup of (R, +), then $GR_c = \{0\}$.*

Corollary 3.5. *Let G be a right R-group of type-0. If (R, +) is an abelian group, then $GR_c = \{0\}$.*

Corollary 3.6. *Let G be a right R-group of type-0. If R is zero-symmetric, then $GR_c = G0 = \{0\}$.*

Proposition 3.7. *Let G be a right R-group of type-0 and G0 = {0}. Then there is a largest ideal of R contained in (0 : G) = {r ∈ R | Gr = {0}}.*

Proof. Since G0 = {0}, the zero ideal of R is contained in (0 : G). Let I and J be ideals of R contained in (0 : G). We show now that I + J is contained in (0 : G). Let g_0 be a generator of the right R-group G. Let i ∈ I, j ∈ J and g ∈ G. We get r ∈ R such that $g = g_0r$. Then $g(i + j) = (g_0r)(i + j) = g_0(r(i + j)) = g_0(r(i$

+ j) - ri + ri) = g_0(r(i + j) - ri) + g_0(ri) = $g_0 j'$ + (g_0r)i = 0 + 0 = 0, where $j' \in$ J. So i + j ∈ (0 : G) and hence I + J ⊆ (0 : G). From this we get that for any collection of ideals of R contained in (0 : G) their sum is an ideal of R contained in (0 : G). Therefore, the sum K of all ideals T of R such that T ⊆ (0 : G) is the largest ideal of R contained in (0 : G). □

Definition 3.8. *Let G be a right R-group of type-0 and G0 = {0}. Suppose that P is the largest ideal of R contained in (0 : G) = {r ∈ R | Gr = {0}}. Then G is said to be a right R-group of type-0(e) if 0 ≠ g ∈ G,* r_1, $r_2 \in R$ *and* $gxr_1 = gxr_2$ *for all* $x \in R$ *implies* $r_1 - r_2 \in P$.

Remark 3.9. Let G be a right R-group of type-0(e) and P be the largest ideal of R contained (0 : G). Let g_0 be a generator of G. Since g_0R = G, if r_1, r_2 ∈ R and $gr_1 = gr_2$ for all g ∈ G, then $r_1 - r_2$ ∈ P.

Let G be a finite additive group and let N be a maximal normal subgroup of G. Let K := (N : G) = {f ∈ M_0(G) | f(G) ⊆ N}. We show in the following example that M_0(G)/K is a right M_0(G)-group of type-0(e).

Example 3.10. Let G be a non-zero finite additive group and let N be a maximal normal subgroup of G. Let K := (N : G) = {f ∈ M_0(G) | f(G) ⊆ N}. Since N is a maximal normal subgroup of G, K is a maximal right ideal of M_0(G). Define (f + K)h := fh + K, f, h ∈ M_0(G). Now M_0(G)/K is a right M_0(G)-group of type-0 as K is maximal and 1 + K is a generator, where 1 is the identity element in M_0(G). Since M_0(G) is a simple near-ring, {0} is the largest ideal of M_0(G) contained in (0 : M_0(G)/K). Suppose that 0 ≠ s + K ∈ M_0(G)/K, f, h ∈ M_0(G) and (s + K)tf = (s + K)th for all t ∈ M_0(G). So, stf - sth ∈ K. Assume that s(g_0) ∉ N and f(g) ≠ h(g) for some g_0, g ∈ G. Let h(g) ≠ 0. We get t ∈ M_0(G) such that t(f(g)) = 0 and t(h(g)) = g_0. So, stf - sth ∉ K, a contradiction. Therefore f = h, that is, f - h ∈ {0}. Hence, M_0(G)/K is a right M_0(G)-group of type-0(e).

From the above example it follows that if (G, +) is a finite simple group, then M_0(G) is a right M_0(G)-group of type-0(e).

Now we give an example of a right R-group of type-0 which is not of type-0(e).

Example 3.11. Consider G := Z_8, the group of integers under addition modulo 8. Now T : G → G defined by T(g) = 5g, for all g ∈ G is an automorphism of G. T fixes 0, 2, 4, 6 and maps 1 to 5, 5 to 1, 7 to 3 and 3 to 7. A := {I, T} is an automorphism group of G. {0}, {2}, {4}, {6}, {1, 5} and {3, 7} are the orbits. Let R be the centralizer near-ring M_A(G), the near-ring of all self maps of G which fix 0 and commute with T. An element of R is completely determined by its action on {1, 2, 3, 4, 6}. Note that for f ∈ R, we have f(2), f(4), f(6) are arbitrary in 2G and f(1), f(3) are arbitrary in G. This example was considered in [3] where it was shown that J := (0 : 2G) = {f ∈ R | f(h) = 0 for all h ∈ 2G} is the only non-trivial ideal of R. Let K := (2G : G) = {t ∈ R | t(G) ⊆ 2G} ≠ R. Let t_0 be the identity element in R. Now t_0 + K is a generator of the right R-group R/K. Let h ∈ R - K. We show now that (h + K)R = R/K. Since h ∉ K, there is an a ∈ G - 2G such that b := h(a) ∉ 2G. We construct an element s ∈ R such that s = 0 on 2G and s(1) = s(3) = a, so that s(5) = s(7) = a + 4. Since s maps G - 2G to G - 2G, we get that t_0 - hs ∈ K and hence (h + K)s = t_0 + K. So (h + K)R = R/K. Therefore, R/K is a right R-group of type-0. Moreover, (R/K)J ≠ {K}. Therefore, {0} is the largest ideal of R contained in (K : R) = {f ∈ R | Rf ⊆ K} and hence $J_0^r(R) = \{0\}$. Consider s_1, s_2 ∈ R, where $s_1(1) = 1$ and 0 on G - {1, 5} and $s_2(1) = 5$ and 0 on G - {1, 5}. Clearly, (h + K)s_1 = (h + K)s_2 for all h ∈ R as h(1) - h(5) ∈ 2G for all h ∈ R. But $s_1 - s_2 \notin \{0\}$. Therefore, by Remark 3.9, R/K is not a right R-group of type-0(e).

Proposition 3.12. *Let G be right R-group of type-0(e). Then (0 : G) is an ideal of R.*

Proof. Let P be the largest ideal of R contained in (0 : G). Let r ∈ (0 : G). We have gr = 0 = g0, for all g ∈ G. Since G is a right R-group of type-0(e), by Remark 3.9, r = r - 0 ∈ P. Therefore, (0 : G) ⊆ P and hence (0 : G) = P. □

Definition 3.13. *A right modular right ideal K of R is called right 0(e)-modular if R/K is a right R-group of type-0(e).*

Definition 3.14. *Let G be a right R-group of type-0(e). Then (0 : G) is called a right 0(e)-primitive ideal of R.*

Definition 3.15. *A near-ring R is called right 0(e)-primitive if {0} is a right 0(e)-primitive ideal of R.*

Definition 3.16. *The intersection of all right 0(e)-primitive ideals of R is called the right Jacobson radical of R of type-0(e) and is denoted by $J_{0(e)}^r(R)$. If R has no right 0(e)-primitive ideals, then $J_{0(e)}^r(R)$ is defined to be R.*

Note that if R is a ring, then $J_{0(e)}^r(R) = J(R)$, where J is the Jacobson radical of rings.

Proposition 3.17. *Let G be a monogenic right R-group. If g_0 is generator of G, then K := (0 : g_0) = $\{r \in R \mid g_0 r = 0\}$ is a right modular right ideal of R and G ≃ R/K as right R-groups. Hence, if G is a right R-group of type-0(e), then K is a right 0(e)-modular right ideal of R.*

Remark 3.18. Let K be a right ideal of R. Then the ideal {0} of R is contained in K. Since K is a subgroup of (R, +), if I and J are ideals of R contained in K, then I + J ⊆ K. So, there is a largest ideal of R contained in K.

Proposition 3.19. *Let G be right R-group of type-0(e) and P := (0 : G) = {r ∈ R | Gr = {0}}. Then P is the largest ideal of R contained in (0 : g_0), g_0 is a generator of the right R-group G.*

Proof. Let g_0 be a generator of the right R-group G. Since GP = $\{0\}$, we have $g_0P = \{0\}$. So P $\subseteq$ (0 : g_0). Let Q be the largest ideal of R contained in (0 : g_0). So, we have P $\subseteq$ Q. Since $R_c \subseteq$ P, $R_c \subseteq$ Q and hence RQ $\subseteq$ Q. Let g $\in$ G. Now g = g_0r for some r $\in$ R. So, gQ = (g_0r)Q = g_0(rQ) $\subseteq$ g_0Q = $\{0\}$. Therefore, Q $\subseteq$ (0 : G) = P and hence Q = P. □

Corollary 3.20. *Let P be an ideal of R. P is a right 0(e)-primitive ideal of R if and only if P is the largest ideal of R contained in a right 0(e)-modular right ideal of R.*

Proposition 3.21. *Let P be an ideal of R. P is a right 0(e)-primitive ideal of R if and only if R/P is a right 0(e)-primitive near-ring.*

Proof. Let P be a right 0(e)-primitive ideal of R. So, we get a right 0(e)-modular right ideal M of R such that P is the largest ideal of R contained in M. Now M/P is a right 0(e)-modular right ideal of R/P. Since P is the largest ideal of R contained in M, the zero ideal of R/P is the largest ideal of R/P contained in M/P. Therefore, R/P is a right 0(e)-primitive near-ring. Suppose now that R/P is a right 0(e)-primitive near-ring. So, we get a right 0(e)-modular right ideal M/P of R/P such that the zero ideal of R/P is the largest ideal of R/P contained in M/P. Clearly, M is a right 0(e)-modular right ideal of R. Since the zero ideal of R/P is the largest ideal of R/P contained in M/P, P is the largest ideal of R contained in M. Therefore, P is a right 0(e)-primitive ideal of R. □

From Proposition 3.21, we have the following:

Proposition 3.22. $\mathcal{J}^r_{0(e)}$ *is the Hoehnke radical corresponding to the class of all right 0(e)-primitive near-rings.*

Definition 3.23. *Let G be a right R-group of type-0(e). Then G is called faithful if (0 : G) = $\{0\}$.*

Theorem 3.24. *Let R be a right 0(e)-primitive near-ring. Then R is an equiprime near-ring.*

Proof. Since $\{0\}$ is a right 0(e)-primitive ideal of R, by Proposition 3.12, $\{0\}$ = (0 : G) for a right R-group G of type-0(e). Let 0 $\neq$ a $\in$ R, r_1, r_2 $\in$ R and a$x$$r_1$ = a$x$$r_2$ for all x $\in$ R. Since (0 : G) = $\{0\}$, there is a g $\in$ G such that ga $\neq$ 0. Let h := ga. Now h$x$$r_1$ = h$x$$r_2$ for all x $\in$ R. Since G is a right R-group of type-0(e), by Proposition 3.12, r_1 - r_2 $\in$ (0 : G) = $\{0\}$. Therefore, r_1 = r_2 and hence R is an equiprime near-ring. □

Corollary 3.25. *A right 0(e)-primitive ideal of R is an equiprime ideal of R.*

Corollary 3.26. *A right 0(e)-primitive near-ring is a zero-symmetric near-ring.*

Theorem 3.27. *Let G be a right R-group of type-0. Suppose that S is an invariant subnear-ring and a right ideal of R. If GS $\neq$ $\{0\}$, then G is also a right S-group of type-0.*

Proof. Suppose that GS $\neq \{0\}$. Clearly, G is a right S-group. Let g $\in$ G and gS := {gs | s $\in$ S} $\subseteq$ G. Consider the normal subgroup $< gS >_n$ of (G, +) generated by gS. Let r $\in$ R, h $\in < gS >_n$. Now h = (x_1 + δ_1(gs_1) - x_1) + (x_2 + δ_2(gs_2) - x_2) + ... + (x_k + δ_k(gs_k) - x_k), $s_i \in$ S, $x_i \in$ G, $\delta_i \in \{1, -1\}$. Since SR $\subseteq$ S, hr = (x_1r + δ_1(g(s_1r)) - x_1r) + (x_2r + δ_2(g(s_2r)) - x_2) + ... + (x_kr + δ_k(g(s_kr)) - x_kr) $\in < gS >_n$. So, $< gS >_n$ is an ideal of the right R-group G and hence it is also an ideal of the right S-group G. Let 0 $\neq$ h $\in$ G. Suppose that hS = {0}. Since hR $\neq$ {0}, $< hR >_n$ is a non-zero ideal of the right R-group G. Since G is a simple right R-group, $< hR >_n$= G. So, GS = $< hR >_n$S $\subseteq < hS >_n$ = {0}, a contradiction to GS $\neq$ {0}. Therefore, hS $\neq$ {0}. Let g_0 be a generator of the right R-group G. So g_0 is a distributive element of the right R-group G and g_0R = G. Clearly, g_0 is a distributive element of the right S-group G and hence g_0S is a subgroup of (G, +). We have (g_0S)R = g_0(SR) $\subseteq$ g_0S. So g_0S is an R-subgroup of G. Let g $\in$ G and s $\in$ S. Since g_0R = G, g = g_0r for some r $\in$ R. So g + g_0s - g = g_0r + g_0s - g_0r = g_0(r + s - r) $\in$ g_0S, as S is a normal subgroup of (R, +). Therefore, g_0S is an ideal of the right R-group G and hence g_0S = G. So g_0 is also a generator of the right S-group G. Let K be a non-zero ideal of the right S-group G. Let 0 $\neq$ y $\in$ K. As seen above $< yS >_n$ is a non-zero ideal of the right R-group G and hence $< yS >_n$ = G. Since G = $< yS >_n$ $\subseteq$ K, G = K. Therefore, {0} and G are the only ideals of the right S-group G and hence G is a right S-group of type-0. □

Theorem 3.28. *Let G be a right R-group of type-0(e). Suppose that S is an invariant subnear-ring and a right ideal of R. If GS $\neq$ {0}, then G is also a right S-group of type-0(e).*

Proof. Suppose that GS $\neq$ {0}. By Theorem 3.27, G is a right S-group of type-0. Clearly, G0 = {0}. Let P be the largest ideal of S contained in $(0 : G)_S$ = {s $\in$ S | Gs = {0}}. Let 0 $\neq$ g $\in$ G, s_1, s_2 $\in$ S and gxs_1 = gxs_2 for all x$\in$ S. Let r $\in$ R. Fix x $\in$ S. We have g(rx)s_1 = g(rx)s_2. So gr(xs_1) = gr(xs_2). Since G is a right R-group of type-0(e), by Proposition 3.12, xs_1 - xs_2 $\in$ (0 : G) = {r $\in$ R | Gr = {0}} which is an ideal of R. Let g_0 be a generator of the right S-group G. Now g_0(xs_1 - xs_2) = 0 and hence g_0xs_1 = g_0xs_2. Since g_0S = G, we have g_0R = G. So g_0rs_1 = g_0rs_2, for all r $\in$ R. Since G is a right R-group of type-0(e), by Proposition 3.12, s_1 - s_2 $\in$ (0 : G). We have $(0 : G)_S$ = (0 : G) $\cap$ S is an ideal of S and hence P = $(0 : G)_S$. Now s_1 - s_2 $\in$ (0 : G) $\cap$ S = P. Therefore, G is a right S-group of type-0(e). □

Corollary 3.29. *If R is a right 0(e)-primitive near-ring and I is a nonzero ideal (or a nonzero invariant subnear-ring and a right ideal) of R, then I is a right 0(e)-primitive near-ring.*

Corollary 3.30. *The class of all right 0(e)-primitive near-rings is hereditary.*

Corollary 3.31. *The class of all right 0(e)-primitive near-rings is regular.*

Theorem 3.32. *Suppose that S is an invariant subnear-ring of R. If G is a right S-group of type-0, then G is also a right R-group of type-0.*

Proof. Suppose that G is a right S-group of type-0 and g_0 is a generator. We have that g_0 is distributive over S and $g_0S = G$. For $g \in G$ and $r \in R$, define $gr := g_0(sr)$ if $g = g_0s$, $s \in S$. We show now that this operation is well defined. Suppose that $g = g_0s = g_0t$, s, $t \in S$. Let $r \in R$ and $h := g_0(sr) - g_0(tr)$. Now $hk = (g_0(sr) - g_0(tr))k = g_0((sr)k) - g_0((tr)k) = g_0(s(rk)) - g_0(t(rk)) = g(rk) - g(rk) = 0$, for all $k \in S$. Therefore, $hS = \{0\}$ and hence $h = 0$, that is, $g_0(sr) = g_0(tr)$. We show that G is a right R-group of type-0. It is clear that G is a right R-group. $g_0 = g_0e$ for some $e \in S$. Now $G \supseteq g_0R = g_0(eR) \supseteq g_0(eS) = g_0S = G$. So $g_0R = G$. Let p, $q \in R$ and $x = g_0(p + q) - (g_0p + g_0q)$. Then $xs = (g_0(p + q) - (g_0p + g_0q))s = (g_0(p + q))s - (g_0p + g_0q)s = g_0(ps + qs) - (g_0ps + g_0qs) = (g_0(ps) + g_0(qs)) - (g_0(ps) + g_0(qs)) = 0$, for all $s \in S$. Therefore, $x = 0$ and hence g_0 is a generator of the right R-group G. It can be easily verified that the action of R on G is an extension of the action of S on G. So, an ideal of the right R-group G is also an ideal of the right S-group G. Since the right S-group G has no non-trivial ideals, the right R-group G also has no non-trivial ideals. Therefore, G is also a right R-group of type-0. □

Theorem 3.33. *Let I be an essential left invariant ideal of R. If I is a right 0(e)-primitive near-ring, then R is also a right 0(e)-primitive near-ring.*

Proof. Let I be a right 0(e)-primitive near-ring and G be a faithful right I-group of type-0(e). Let $r \in R$. Let g_0 be a generator of the right I-group G. Define $gr := g_0(ar)$ if $g = g_0a$, $a \in I$. By Theorem 3.32, G is a right R-group of type-0. Suppose that $0 \neq g \in G$, r, $s \in R$ and $gxr = gxs$, for all $x \in R$. Fix $a \in I$. Now $g((ba)r) = g((ba)s)$ and hence $g(b(ar)) = g(b(as))$ for all $b \in I$. Since G is a faithful right I-group of type-0(e), $ar - as = 0$, that is $ar = as$. Now $ar = as$ for all $a \in I$. Since I is a right 0(e)-primitive near-ring, by Theorem 3.24, I is an equiprime near-ring. Also, since I is an essential left invariant ideal of R, by Proposition 2.12, we get that R is an equiprime near-ring. Since R is equiprime and $ar = as$ for all $a \in I$ and I is a left invariant ideal of R, we get that $r = s$. So, $0 = r - s \in P$, P is the largest ideal of R contained in $(0 : G) = \{r \in R \mid Gr = \{0\}\}$. Therefore, G is a right R-group of type-0(e). Let $r \in (0 : G)$. Now $Gr = 0$. So $g_0(ar) = 0$, for all $a \in I$ and hence $0 = g_0((ba)r) = g_0(b(ar)) = (g_0b)ar$ for all a, $b \in I$. Since $g_0I = G$, we have $G(ar) = 0$ for all $a \in I$ and hence $Ir = 0$, as $(0 : G)_I = 0$. Also, since $ar = 0 = a0$ for all $a \in I$ and I is an invariant subnear-ring of R and R is an equiprime near-ring, we get that $r = 0$. Therefore, G is a faithful right R-group of type-0(e) and hence R is a right 0(e)-primitive near-ring. □

Theorem 3.34. *The class of all right 0(e)-primitive near-rings is closed under essential left invariant extensions.*

Remark 3.35. By Proposition 2.13, the class of all equiprime near-rings satisfies condition F_l. So, the class of all right 0(e)-primitive near-rings which is a class of equiprime near-rings also satisfies condition F_l.

By Theorem 2.10, Corollaries 3.26 and 3.31, Theorem 3.34 and Remark 3.35, we get the following:

Theorem 3.36. *Let $\mathcal{E}$ be the class of all right 0(e)-primitive near-rings and $\mathcal{UE}$ be the upper radical class determined by $\mathcal{E}$. Then $\mathcal{UE}$ is a c-hereditary Kurosh-Amitsur radical class in the variety of all near-rings with hereditary semisimple class $\mathcal{SUE} = \overline{\mathcal{E}}$. So, $J^r_{0(e)}$ is a KA-radical in the class of all near-rings and for any ideal I of R, $J^r_{0(e)}(I) \subseteq J^r_{0(e)}(R) \cap I$ with equality, if I is left invariant.*

Corollary 3.37. *$J^r_{0(e)}$ is an ideal-hereditary KA-radical in the class of all zero-symmetric near-rings.*

Corollary 3.38. *$J^r_{0(e)}$ is a special radical in the class of all near-rings.*

4. Relations with other radicals

In this section we see the relations of the radical $J^r_{0(e)}$ with other known radicals of near-rings.

P and P_e denote the prime and equiprime radicals of near-rings respectively. In view of Corollary 3.25, we have the following:

Proposition 4.1. *Let R be a near-ring. Then $P(R) \subseteq P_e(R) \subseteq J^r_{0(e)}(R)$.*

Let (S, +) be a group containing more than two elements. Define a trivial multiplication in S by rs = r if s $\neq$ 0 and 0 if s = 0 for all r, s $\in$ S. Now S is a zero-symmetric right near-ring. Clearly, S is a left S-group of type-2 and hence S is a simple near-ring. Therefore, S is 2-primitive on the left S-group S. So $J_2(S) = \{0\} = P(S)$. It is clear that S is not equiprime and hence $J^r_{0(e)}(R) = P_e(S) = S \neq \{0\} = P(S) = J_2(S)$.

Now we give an example of a centralizer simple near-ring with identity which is an equiprime, right 0(e)-primitive and left 3-primitive near-ring.

Example 4.2. Let G := $Z_5 \times Z_5$, Z_5 is the additive group of integers modulo 5. Define t : G $\to$ G by t(x) = 2x. t is an automorphism of the abelian group (G, +), where (a, b) + (c, d) = (a + c, b + d), (a, b), (c, d) $\in$ G. Now order of t $\in$ Aut G is four. So, A := $\{I, t, t^2, t^3\}$ is a fixed point free cyclic subgroup of Aut G. By Corollary 7.4 of Veldsman [14], the centralizer near-ring U := C(A, G) = $\{f \in M_0(G) \mid fs = sf \text{ for all } s \in A\}$ is a simple zero-symmetric near-ring with identity. Moreover, U is equiprime and left 2-primitive. So U is a left 3-primitive near-ring. Clearly, $M_1 := Z_5 \times \{0\}$, $M_2 := \{0\} \times Z_5$ and $M_3 := \{(a, a) \mid a \in Z_5\}$ are the maximal (minimal) subgroups (normal subgroups) of G. Note that t maps M_i onto M_i, i = 1, 2, 3. We show now that $K_i := \{f \in U \mid f(G) \subseteq M_i\}$, i = 1, 2, 3 are the maximal right ideals of U. Clearly, K_i is a proper right ideal of U. Let K be a maximal right ideal of U. Let $G_K = \{x \in G \mid x = f(y), \text{ for some } f \in K, y \in G\}$. Let x, z $\in$ G. We get $f_1, f_2 \in K$ such that $f_1(y_1) = x$ and $f_2(y_2) = z$ for some $y_1, y_2 \in G$. If $y_2 = 0$, then z = 0 and hence x - z = x $\in G_K$. Suppose that $y_2 \neq 0$. We get h $\in$ U such that $h(y_2) = y_1$ and h = 0 on G $\setminus$ Ay_2. Now $f_1h \in K$ and hence $f_1h - f_2 \in K$. So x - z = $f_1(y_1) - f_2(y_2) = (f_1h - f_2)(y_2) \in G_K$. Therefore, G_K is a (normal) subgroup of G. We show that $G_K \neq G$. Suppose that $G_K = G$.

Now G $\setminus$ {0} = $\cup_i^n$ Ab_i for some $b_i \in$ G and positive integer n, Ab_i are pair wise disjoint. Since $b_i \in G_K$, we get $h_i \in$ K such that $b_i = h_i(c_i)$ for some $c_i \in$ G. We also get $s_i \in$ U such that $c_i = s_i(b_i)$ and $s_i = 0$ on G $\setminus$ Ab_i. Clearly, the identity mapping I of G is given by I = $h_1s_1 + h_2s_2 + \cdots + h_ns_n \in$ K and that K = U, a contradiction to K $\neq$ U. Therefore, $G_K \neq$ G. We get a maximal normal subgroup M of G containing G_K. Now K $\subseteq$ (M : G) = {f $\in$ U | f(G) $\subseteq$ M} $\neq$ U. Since K is maximal, K = (M : G). Now it follows that each K_i is a maximal right ideal of U. We show now that U/K is a right U-group of type-0(e) under the operation (f + K)h := fh + K, f, h $\in$ U. Since U has the identity and K is maximal in U, U/K is a right U-group of type-0. Obviously, (U/K)0 = {0}. Let I $\neq$ v $\in$ A and let d $\in$ G $\setminus$ M. We show that d - v(d) $\notin$ M. We get a normal subgroup N of G such that G = M + N and M $\cap$ N = {0}. Note that v(M) = M and v(N) = N. Let d = $m_1 + n_1$ and v(d) = $m_2 + n_2$, m_1, m_2, $\in$ M and n_1, $n_2 \in$ N. If d - v(d) $\in$ M, then n_1 - $n_2 \in$ N $\cap$ M = {0} and hence $n_1 = n_2$. Since $m_2 + n_2$ = v(d) = v(m_1 + n_1) = v(m_1) + v(n_1) = v(m_1) + v(n_2), we have n_2 = v(n_2), a contradiction to the fact that v is fixed point free and $n_2 \neq 0$. Therefore, d - v(d) $\notin$ M. Since U is simple, {0} is the largest ideal of U contained in (0 : U/K) = {f $\in$ U | Uf $\subseteq$ K}. Let q $\in$ U $\setminus$ K, r, s $\in$ U and qfr - qfs $\in$ K for all f $\in$ U. Now q(w) $\notin$ M for some w $\in$ G. Suppose that r $\neq$ s. We get e $\in$ G such that r(e) $\neq$ s(e). Let r(e) $\neq$ 0. Suppose that s(e) $\notin$ Ar(e). Define f_0 on G by f_0(r(e)) = w and f_0 = 0 on G $\setminus$ Ar(e). Clearly, $f_0 \in$ U. Now (qf_0r - qf_0s)(e) = q(w) $\notin$ M, a contradiction. Assume now that s(e) $\in$ Ar(e), that is, s(e) = v(r(e)) for some v $\in$ A. Now (qf_0r - qf_0s)(e) = q(w) - v(q(w)) $\notin$ M, a contradiction. Therefore, r = s and U/K is a right U-group of type-0(e). Hence U is a right 0(e)-primitive near-ring.

Proposition 4.3. *Let G be a finite group and A be a fixed point free cyclic subgroup of Aut G. Suppose that for each maximal normal subgroup M of G there is an element $a_M \in G \setminus M$ and $I \neq t \in A$ such that $s(a_M)$ - $s(t(a_M)) \in M$ for all $s \in A$. Then the simple left* 3*-primitive near-ring C(A, G) is a $J^r_{0(e)}$-radical near-ring with identity.*

Proof. We have that U := C(A, G) is a simple near-ring with identity. Let T be a right U-group of type-0(e). Now T is U-isomorphic to U/K for some maximal right ideal K of U. By the same arguments used in Example 4.2 one can easily get that K = (M : G) = {f $\in$ U | f(G) $\subseteq$ M} for some maximal normal subgroup M of G. By our assumption we get $a_M \in$ G $\setminus$ M and I $\neq$ t $\in$ A such that s(a_M) - s(t(a_M)) $\in$ M for all s $\in$ A. Define h_1 on G by $h_1(a_M) = a_M$ and h_1 = 0 on G $\setminus$ Aa_M. Also define $h_2(a_M)$ = t(a_M) and h_2 = 0 on G $\setminus$ Aa_M. Now h_1, $h_2 \in$ U $\setminus$ K. Now (h_1fh_1 - h_1fh_2)(s(a_M)) = s(p(a_M)) - s(p(t(a_M))) if f(a_M) = p(a_M) for some p $\in$ A and 0 if f(a_M) $\in$ G $\setminus$ Aa_M. Therefore, h_1fh_1 - h_1fh_2 $\in$ K for all f $\in$ U, but h_1 - $h_2 \notin$ {0} which is the largest ideal of U contained in (0 : U/K). This is a contradiction to the fact that U/K is a right U-group of type-0(e). Therefore, U has no right U-group of type-0(e) and hence U is a $J^r_{0(e)}$-radical near-ring. □

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