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Preservers of the Rank of Matrices over a Field

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Abstract. In 2001, Li and Pierce characterized the linear operators that preserve the g rank of real matrices. The present paper describes all rank preserving maps $F : M_{m,n}(K) \longrightarrow M_{m,n}(K)$ of the $m \times n$ matrices over an arbitrary field K which are of the form $F(a_{i,j}) = (f_{i,j}(a_{i,j}))$. The linearity of F is not a priorily assumed, and it turns out that if $\min\{m, n\} \leq 2$, then nonlinear rank preserving maps indeed exist.

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In [7] Pap introduced the so-called g-calculus. Marková [6] observed that functions preserving the rank of matrices play a fundamental role in this calculus. The problem of rank preservation was studied in [3] and [4]. This note generalizes the results of [3] and [4] as well as of [2] and [5] and it may also be viewed as an extension of the results of [1], which concern linear or additive preserver problems.

Let K be a field. Let $M_{m,n}(K)$ denote the set of $m \times n$ matrices over K and put $M_n(K) = M_{n,n}(K)$. Throughout what follows we assume that $F : M_{m,n}(K) \longrightarrow M_{m,n}(K)$ is a map of the form

$$F(a_{i,j})_{i=1,\, j=1}^{m,n} = (f_{i,j}(a_{i,j}))_{i=1,\, j=1}^{m,n}$$

with maps $f_{i,j} : K \longrightarrow K$. We say that F is rank preserving if $\operatorname{rank} F(A) = \operatorname{rank} A$ for all $A \in M_{m,n}(K)$. It is easily seen that if $\min\{m, n\} = 1$, then F is rank preserving if and only if $f_{i,j}(0) = 0$, $f_{i,j}(x) \neq 0$ for all i, j and all $x \neq 0$. Here is our result.

Theorem.

(a) *If* $\min\{m,n\} = 2$*, then* F *is rank preserving on* $M_{m,n}(K)$ *if and only if there exist nonzero* $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in K$ *and an injective function* $g : K \longrightarrow K$ *satisfying* $g(0) = 0$ *and* $g(xy) = g(x)g(y)$ *for all* $x, y \in K$ *such that* $f_{i,j}(x) = u_i v_j g(x)$ *for all* $x \in K$.

(b) *If* $\min\{m,n\} \geq 3$*, then* F *preserves the rank on* $M_{m,n}(K)$ *if and only if there are nonzero* $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n \in K$ *and an injective function* $g : K \longrightarrow K$ *satisfying* $g(xy) = g(x)g(y)$ *and* $g(x+y) = g(x) + g(y)$ *for all* $x, y \in K$ *such that* $f_{i,j}(x) = u_i v_j g(x)$ *for all* $x \in K$.

Thus, for $\min\{m,n\} \geq 3$ the rank preserving maps F on $M_{m,n}(K)$ may be written in the form $F(A) = U[g(a_{i,j})]V$, where $U = \mathrm{diag}(u_1, u_2, \ldots, u_m)$ and $V = \mathrm{diag}(v_1, v_2, \ldots, v_n)$ are invertible diagonal matrices and g is an injective endomorphism of K. For $\min\{m,n\} = 2$, the additivity of g may even be relaxed to the sole requirement that $g(0) = 0$. Note that the maps of part (a) may be nonlinear: for example, one can take $g(x) = x^3$.

Proof. Let $\min\{m,n\} \geq 2$ and suppose F is rank preserving on $M_{m,n}(K)$. We denote by $E_{j,k}$ the matrix whose j,k entry is 1 and the remaining entries of which are 0. Since rank $F(xE_{i,j}) =$ rank $xE_{i,j}$, we see that $f_{i,j}(0) = 0$ and $f_{i,j}(x) \neq 0$ for all i,j and all $x \neq 0$. The matrix all entries of which are 1 has rank 1, and since F preserves the rank of this matrix, it follows that rank $(f_{i,j}(1)) = 1$. This implies that there are $u_i, v_j \in K$ such that $f_{i,j}(1) = u_i v_j$ for all i,j. Because $f_{i,j}(1) \neq 0$, we obtain that $u_i \neq 0$ and $v_j \neq 0$ for all i,j. Put $g_{i,j}(x) = u_i^{-1} v_j^{-1} f_{i,j}(x)$. Clearly, $g_{i,j}(0) = 0$ and $g_{i,j}(1) = 1$ for all i,j.

For $1 \leq i \neq r \leq m$ and $1 \leq k \neq l \leq n$, let $A = E_{i,k} + xE_{i,l} + E_{r,k} + xE_{r,l}$. As rank $F(A) =$ rank $A = 1$, we get

$$0 = f_{i,k}(1)f_{r,l}(x) - f_{r,k}(1)f_{i,l}(x) = u_i v_k u_r v_l g_{r,l}(x) - u_r v_k u_i v_l g_{i,l}(x),$$

that is, $g_{r,l}(x) = g_{i,l}(x)$ for all $x \in K$. Consequently, the matrix $G = (g_{i,j})$ is constant along its column. Analogously one can show that G is constant along the rows. This implies that all $g_{i,j}$ are one and the same function g and that therefore $f_{i,j}(x) = u_i v_j g(x)$ for all i,j and all x. Note that $g(0) = 0$ and $g(1) = 1$.

To prove that g is injective, suppose $x \neq y$ and consider $B = E_{1,1} + E_{1,2} + xE_{2,1} + yE_{2,2}$. We have rank $F(B) =$ rank $B = 2$ and hence

$$0 \neq f_{1,1}(1)f_{2,2}(y) - f_{1,2}(1)f_{2,1}(x) = u_1 v_1 u_2 v_2 g(y) - u_1 v_2 u_2 v_1 g(x),$$

which implies that $g(x) \neq g(y)$, as desired.

To show that $g(xy) = g(x)g(y)$, take $C = E_{1,1} + xE_{1,2} + yE_{2,1} + xyE_{2,1}$. Since rank $F(C) =$ rank $C = 1$, we obtain that

$$0 = f_{1,1}(1)f_{2,2}(xy) - f_{1,2}(x)f_{2,1}(y) = u_1 v_1 u_2 v_2 g(xy) - u_1 v_2 u_2 v_1 g(x)g(y),$$

that is, we arrive at the equality $g(xy) = g(x)g(y)$.

At this point we have proved the "only if" part of (a). To get the "only if" part of (b), assume $\min\{m,n\} \geq 3$ and consider

$$D = xE_{1,1} + E_{1,2} + yE_{2,1} + E_{2,3} + (x+y)E_{3,1} + E_{3,2} + E_{3,3}.$$

As rank $D = 2$, we conclude that the determinant of the upper-left 3×3 submatrix of $F(D)$ must be zero, which means that

$$0 = u_1u_2u_3v_1v_2v_3(-g(x) - g(y) + g(xy)).$$

Thus, $g(x+y) = g(x) + g(y)$. The proof of the "only if" part of (b) is also complete.

We now prove the "if" part of (a). Clearly, F maps the zero matrix to itself. So assume rank $A \geq 1$. Let

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any submatrix of order 2 of A. The determinant of the corresponding submatrix of $F(A)$ is

$$\begin{aligned} f_{i,r}(a)f_{k,l}(d) - f_{i,l}(b)f_{k,r}(c) &= u_iv_ru_kv_lg(a)g(d) - u_iv_lu_kv_rg(b)g(c) \\ &= u_iu_kv_rv_l(g(ad) - g(bc)). \end{aligned}$$

Since g is injective, we deduce that $\det P = ad - bc = 0$ if and only if $g(ad) - g(bc) = 0$. This proves that A and $F(A)$ have the same rank.

We finally prove the "if" part of (b). Let $A \in M_{m,n}(K)$, let $D = (d_{i,j})\,_{i,j=1}^{k}$ be any submatrix of the order k, and put $R = (f_{i,j}(d_{i,j}))\,_{i,j=1}^{k}$. The assertion will follow as soon as we have shown that $\det Q = 0$ if and only if $\det R = 0$. We have

$$\det R = \sum_{\pi \in S_k} (-1)^{\operatorname{sgn}\pi} \prod_{i=1}^{k} f_{i,\pi(i)}(d_{i,\pi(i)}) = \sum_{\pi \in S_k} (-1)^{\operatorname{sgn}\pi} \prod_{i=1}^{k} u_iv_{\pi(i)}g(d_{i,\pi(i)}) =$$

$$= \prod_{i=1}^{k}(u_iv_i) \sum_{\pi \in S_k} (-1)^{\operatorname{sgn}\pi} \prod_{i=1}^{k} g(d_{i,\pi(i)}) = \prod_{i=1}^{k}(u_iv_i) g\left(\sum_{\pi \in S_k} (-1)^{\operatorname{sgn}\pi} \prod_{i=1}^{k} d_{i,\pi(i)} \right) =$$

$$= cg(\det Q), \text{ where } c := \prod_{i=1}^{k}(u_iv_i).$$

Since $c \neq 0$ and since $g(x) = 0$ if and only if $x = 0$, we arrive at the desired conclusion that $\det R = 0$ if and only if $\det Q = 0$. □

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References

[1] Li, C.-K.; Pierce S.: *Linear preserver problems*. Am. Math. Mon. **108** (2001), 591–605. Zbl 0991.15001

[2] Hao, L. Z.; Zhang, X.: *Functions preserving rank-k matrices of order n over fields*. Miskolc Math. Notes **6** (2005), 197–200. Zbl 1094.15001

[3] Kalinowski, J.: *On rank equivalence and preserving rank operators*. Novi Sad J. Math. **32**(1) (2002), 133–139.

[4] Kalinowski, J.: *On functions preserving rank of matrices*. Miskolc Math. Notes **4** (2003), 35–37. Zbl 1026.15001

[5] Kalinowski, J.: *Rank of matrices and the Pexider equation*. Beitr. Algebra Geom. **47**(2) (2006), 519–526. Zbl 1117.15002

[6] Marková, A.: *Some remarks on the pseudo-linear algebra*. Tatra Mt. Math. Publ. **6** (1995), 123–130. Zbl 0860.15002

[7] Pap, E.: *g-calculus*. Zb. Rad., Prir.-Mat. Fak., Univ. Novom Sadu, Ser. Mat. **23** (1993), 145–156. Zbl 0823.28011

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