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Lie Powers of Infinite-Dimensional Modules

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Abstract. We consider Lie powers of group-modules over fields of prime characteristic and generalise some recent results for finite-dimensional modules to modules of arbitrary dimension.

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1. Introduction

Let G be a group and F a field. For any FG-module V, let $L(V)$ be the free Lie algebra on V (the free Lie algebra freely generated by any basis of V), and regard $L(V)$ as an FG-module by extending the action of G on V so that G acts on $L(V)$ by Lie algebra automorphisms. For each positive integer n, the nth homogeneous component $L^n(V)$ is a submodule of $L(V)$, called the nth Lie power of V.

In the case where V is finite-dimensional the modules $L^n(V)$ have been studied in considerable depth: see [3] and the papers cited there. The results are best when F has characteristic 0, but there has recently been substantial progress in the case of prime characteristic p. In [3], a general decomposition theorem was obtained which reduces the study of arbitrary Lie powers of V to the study of Lie powers of the form $L^{p^i}(B_r)$, where, for each r, B_r is a certain direct summand of the rth tensor power $V^{\otimes r}$. This is a reduction to Lie powers of p-power degree. Information about the isomorphism types of the modules B_r is given in [4] and [2].

Recently, Marianne Johnson and Ralph Stöhr have studied torsion in certain 'free central extensions' of groups and have found that they can make striking use of the decomposition theorem of [3] and the results of [4] and [2] provided that these results are available for infinite-dimensional modules V (see [9] and [10]). The purpose of the present paper is to derive such results by utilising the results in the finite-dimensional case and some facts about modules for Schur algebras. One attractive consequence of the arguments here is a reformulation and sharpening of the previous results in terms of idempotents of the group algebras of the symmetric groups. This gives results that are uniform for all fields of the same characteristic.

2. Preliminaries

All modules considered in this paper will be right modules, except for left modules arising from the action of symmetric groups on tensor powers, as explained below. We shall use the Schur algebras associated with the general linear group, as defined in [6]. However, [6] treats only infinite fields and uses left modules. Thus we give a self-contained summary of the basic facts (following the treatment in [3, §2]).

Let F be a field. Let n and r be positive integers, and let $I(n,r)$ be the set of all ordered r-tuples $\mathbf{i} = (i_1,\dots,i_r)$, where $i_1,\dots,i_r \in \{1,\dots,n\}$. Let $A_F(n,r)$ be the homogeneous component of degree r in the polynomial ring over F in n^2 commuting indeterminates c_{ij} ($1 \leqslant i,j \leqslant n$). Thus $A_F(n,r)$ has an F-basis consisting of the monomials of degree r. For $\mathbf{i},\mathbf{j} \in I(n,r)$, where $\mathbf{i} = (i_1,\dots,i_r)$ and $\mathbf{j} = (j_1,\dots,j_r)$, we write $c_{\mathbf{i},\mathbf{j}}$ for the monomial $c_{i_1j_1}\cdots c_{i_rj_r}$. The $c_{\mathbf{i},\mathbf{j}}$ are not distinct (when $n, r > 1$) but they give a basis (with repetitions) of $A_F(n,r)$.

Let $S_F(n,r) = \mathrm{Hom}_F(A_F(n,r),F)$. Therefore $S_F(n,r)$ has a basis (with repetitions) consisting of the elements $\xi_{\mathbf{i},\mathbf{j}}$ (with $\mathbf{i},\mathbf{j} \in I(n,r)$), where $\xi_{\mathbf{i},\mathbf{j}}(c_{\mathbf{i},\mathbf{j}}) = 1$ and $\xi_{\mathbf{i},\mathbf{j}}(c_{\mathbf{i}',\mathbf{j}'}) = 0$ if $c_{\mathbf{i}',\mathbf{j}'} \neq c_{\mathbf{i},\mathbf{j}}$. Multiplication in $S_F(n,r)$ may be defined as in [6, §2.3]: for $\xi,\eta \in S_F(n,r)$,

$$(\xi\eta)(c_{\mathbf{i},\mathbf{j}}) = \sum_{\mathbf{k}\in I(n,r)} \xi(c_{\mathbf{i},\mathbf{k}})\eta(c_{\mathbf{k},\mathbf{j}}).$$

In this way $S_F(n,r)$ becomes an associative F-algebra with identity element. This is the Schur algebra of degree r. If E is an extension field of F then we usually identify $E \otimes_F S_F(n,r)$ with $S_E(n,r)$ in the obvious way.

For $g = (a_{ij}) \in \mathrm{GL}(n,F)$, define $\zeta_g \in S_F(n,r)$ by

$$\zeta_g(c_{i_1j_1}\cdots c_{i_rj_r}) = a_{i_1j_1}\cdots a_{i_rj_r} \in F.$$

Then (see [6, §2.4]) the map $g \mapsto \zeta_g$ extends to an algebra homomorphism

$$F\mathrm{GL}(n,F) \longrightarrow S_F(n,r) \tag{2.1}$$

that is surjective if F is infinite. If U is a (right) $S_F(n,r)$-module then U may be regarded as a (right) $F\mathrm{GL}(n,F)$-module by means of (2.1), and such $F\mathrm{GL}(n,F)$-modules are called polynomial modules of degree r.

Let V be an n-dimensional F-space (that is, vector space over F) with basis $\{x_1,\dots,x_n\}$ and consider the rth tensor power $V^{\otimes r}$. For $\mathbf{i} \in I(n,r)$, where $\mathbf{i} = (i_1,\dots,i_r)$, write $x_{\mathbf{i}} = x_{i_1} \otimes \cdots \otimes x_{i_r} \in V^{\otimes r}$. Thus the elements $x_{\mathbf{i}}$ form a basis of $V^{\otimes r}$. With respect to the basis $\{x_1, \dots, x_n\}$, the identity representation $\mathrm{GL}(n, F) \to \mathrm{GL}(n, F)$ gives V the structure of an $F\mathrm{GL}(n, F)$-module, called the 'natural' module. Hence $V^{\otimes r}$ becomes an $F\mathrm{GL}(n, F)$-module under the 'diagonal' action of $\mathrm{GL}(n, F)$. It is straightforward to check (see [6, §2.6]) that $V^{\otimes r}$ is a polynomial module of degree r. Indeed, for all $\xi \in S_F(n, r)$, we have

$$x_{\mathbf{i}}\xi = \sum_{\mathbf{j}} \xi(c_{\mathbf{i},\mathbf{j}})x_{\mathbf{j}}. \tag{2.2}$$

In particular, with $r = 1$, V is an $S_F(n, 1)$-module, called the natural module.

We take the symmetric group Σ_r to act on the right on $\{1, \dots, r\}$. Then, if V is any F-space (of finite or infinite dimension), $V^{\otimes r}$ may be given the structure of a left $F\Sigma_r$-module by making Σ_r act on $V^{\otimes r}$ by 'place permutations'; that is, for $\sigma \in \Sigma_r$ and $v_1, \dots, v_r \in V$, we take

$$\sigma(v_1 \otimes \cdots \otimes v_r) = v_{1\sigma} \otimes \cdots \otimes v_{r\sigma}. \tag{2.3}$$

It is easily seen that the action of $F\Sigma_r$ on $V^{\otimes r}$ is faithful when $\dim V \geqslant r$.

Let V be an F-space with basis $\{x_i : i \in I\}$, where I is some index set. The free associative algebra freely generated by $\{x_i : i \in I\}$ is denoted by $T(V)$ and, for each r, the rth homogeneous component is denoted by $T^r(V)$. However, we shall write products in $T(V)$ as tensor products so that $T(V)$ is thought of as the 'tensor algebra' on V and $T^r(V) = V^{\otimes r}$. Let $\psi \in \mathrm{End}_F(V)$. Then, since $T(V)$ is free on $\{x_i : i \in I\}$, there is a unique algebra endomorphism ψ^* of $T(V)$ such that $v\psi^* = v\psi$ for all $v \in V$. The restriction of ψ^* to $V^{\otimes r}$ gives $\psi^{\otimes r} \in \mathrm{End}_F(V^{\otimes r})$, and it is easy to verify that $\psi^{\otimes r}$ commutes with the action of σ on $V^{\otimes r}$ for all $\sigma \in \Sigma_r$.

Let G be a group and let V be an FG-module. Then, for each $g \in G$, the action of g on V is given by an (invertible) map $\psi_g \in \mathrm{End}_F(V)$. We can make $T(V)$ into an FG-module by taking the action of g to be the algebra automorphism ψ_g^*. Thus $V^{\otimes r}$ is a submodule on which g acts as $\psi_g^{\otimes r}$. (This is the 'diagonal' action we have already met.) Hence the actions of FG and $F\Sigma_r$ on $V^{\otimes r}$ commute; in other words, $V^{\otimes r}$ is an $(F\Sigma_r, FG)$-bimodule.

In particular, if V is the natural $F\mathrm{GL}(n, F)$-module, we see that $V^{\otimes r}$ is an $(F\Sigma_r, F\mathrm{GL}(n, F))$-bimodule. Indeed, from (2.2) and (2.3), it is straightforward to verify the stronger fact that $V^{\otimes r}$ is an $(F\Sigma_r, S_F(n, r))$-bimodule. For $u \in F\Sigma_r$, the right ideal $uF\Sigma_r$ of $F\Sigma_r$ may be regarded as a (right) $F\Sigma_r$-module.

Lemma 2.1. *Let F be a field and r a positive integer. Let $e_1, \dots, e_s, f_1, \dots, f_t$ be idempotent elements of $F\Sigma_r$ such that there is an isomorphism*

$$e_1F\Sigma_r \oplus \cdots \oplus e_sF\Sigma_r \cong f_1F\Sigma_r \oplus \cdots \oplus f_tF\Sigma_r$$

of $F\Sigma_r$-modules. (*These are 'external' direct sums: we do not assume that the ideals span their direct sums within $F\Sigma_r$.*) *Then, if A is an F-algebra and M is an $(F\Sigma_r, A)$-bimodule, there is an isomorphism of A-modules*

$$e_1M \oplus \cdots \oplus e_sM \cong f_1M \oplus \cdots \oplus f_tM.$$

Proof. There is an isomorphism of A-modules $\alpha : F\Sigma_r \otimes_{F\Sigma_r} M \to M$ given by $u \otimes v \mapsto uv$ for all $u \in F\Sigma_r$, $v \in M$. Let e be an idempotent of $F\Sigma_r$. Since $eF\Sigma_r$ is a direct summand of $F\Sigma_r$, the A-module $eF\Sigma_r \otimes_{F\Sigma_r} M$ may be regarded as a direct summand (and hence submodule) of $F\Sigma_r \otimes_{F\Sigma_r} M$ and α restricts to give an isomorphism $eF\Sigma_r \otimes_{F\Sigma_r} M \cong eM$. Thus

$$\begin{aligned} e_1M \oplus \cdots \oplus e_sM &\cong (e_1F\Sigma_r \oplus \cdots \oplus e_sF\Sigma_r) \otimes_{F\Sigma_r} M \\ &\cong (f_1F\Sigma_r \oplus \cdots \oplus f_tF\Sigma_r) \otimes_{F\Sigma_r} M \cong f_1M \oplus \cdots \oplus f_tM, \end{aligned}$$

as required. □

The next lemma is a version of the well-known fact that the Schur functor has a 'right inverse': see [6, §6]. However, we have found no reference that gives exactly what is needed here (with F arbitrary and $\dim V \geqslant r$), so we sketch a short self-contained proof.

Let F be a field and let n and r be positive integers, where $n \geqslant r$. Let $\phi : F\Sigma_r \to S_F(n,r)$ be the linear map satisfying $\sigma\phi = \xi_{(1,\ldots,r),(1\sigma,\ldots,r\sigma)}$ for all $\sigma \in \Sigma_r$, and write $\xi_1 = 1\phi$. It can be checked from the definition of multiplication in $S_F(n,r)$ that $(\sigma\tau)\phi = (\sigma\phi)(\tau\phi)$ for all $\sigma, \tau \in \Sigma_r$. If M is an $S_F(n,r)$-module then it is easily seen that $M\xi_1$ is invariant under $(F\Sigma_r)\phi$. Thus $M\xi_1$ becomes a right $F\Sigma_r$-module, denoted by $\mathrm{S}(M)$ (where S indicates the Schur functor).

Lemma 2.2. *Let F be a field and r a positive integer. Let V be an F-space of finite dimension n, where $n \geqslant r$, and regard $V^{\otimes r}$ as an $(F\Sigma_r, S_F(n,r))$-bimodule. Then, for any right $F\Sigma_r$-module U, we have $\mathrm{S}(U \otimes_{F\Sigma_r} V^{\otimes r}) \cong U$.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis of V and let Z be the subspace of $V^{\otimes r}$ spanned by $\{x_{1\sigma} \otimes \cdots \otimes x_{r\sigma} : \sigma \in \Sigma_r\}$. Clearly there is an isomorphism of F-spaces $\theta : F\Sigma_r \to Z$ given by $\sigma\theta = x_{1\sigma} \otimes \cdots \otimes x_{r\sigma}$ for all $\sigma \in \Sigma_r$. It is easily checked that $\mathrm{S}(V^{\otimes r}) = (V^{\otimes r})\xi_1 = Z$ and that θ is an isomorphism of $(F\Sigma_r, F\Sigma_r)$-bimodules.

In particular, Z is injective as a left $F\Sigma_r$-module, hence a direct summand of $V^{\otimes r}$. Thus $U \otimes_{F\Sigma_r} Z$ is isomorphic to a direct summand W of $U \otimes_{F\Sigma_r} V^{\otimes r}$, where W is the subspace of $U \otimes_{F\Sigma_r} V^{\otimes r}$ spanned by $\{u \otimes z : u \in U, z \in Z\}$. Also,

$$\mathrm{S}(U \otimes_{F\Sigma_r} V^{\otimes r}) = (U \otimes_{F\Sigma_r} V^{\otimes r})\xi_1 = W, \tag{2.4}$$

since $(V^{\otimes r})\xi_1 = Z$. However, it is easily seen that $U \otimes_{F\Sigma_r} Z$ and W are isomorphic as right $F\Sigma_r$-modules. Therefore

$$W \cong U \otimes_{F\Sigma_r} Z \cong U \otimes_{F\Sigma_r} F\Sigma_r \cong U.$$

Thus the result follows from (2.4). □

Corollary 2.3. *Let F, r and V be as in Lemma 2.2. Let $e_1, \ldots, e_s, f_1, \ldots, f_t$ be idempotents of $F\Sigma_r$ such that there is an isomorphism of $S_F(n,r)$-modules*

$$e_1V^{\otimes r} \oplus \cdots \oplus e_sV^{\otimes r} \cong f_1V^{\otimes r} \oplus \cdots \oplus f_tV^{\otimes r}.$$

Then, if A is an F-algebra and M is an $(F\Sigma_r, A)$-bimodule, there is an isomorphism of A-modules

$$e_1M \oplus \cdots \oplus e_sM \cong f_1M \oplus \cdots \oplus f_tM.$$

Proof. Let $U_1 = e_1F\Sigma_r \oplus \cdots \oplus e_sF\Sigma_r$ and $U_2 = f_1F\Sigma_r \oplus \cdots \oplus f_tF\Sigma_r$. Then, as in the proof of Lemma 2.1,

$$\begin{aligned} U_1 \otimes_{F\Sigma_r} V^{\otimes r} &\cong e_1V^{\otimes r} \oplus \cdots \oplus e_sV^{\otimes r} \\ &\cong f_1V^{\otimes r} \oplus \cdots \oplus f_tV^{\otimes r} \cong U_2 \otimes_{F\Sigma_r} V^{\otimes r}. \end{aligned}$$

Thus, by Lemma 2.2, $U_1 \cong U_2$. Hence the result follows by Lemma 2.1. □

Let F be a field and suppose that V is an F-space of finite dimension n. Let r be a positive integer, and regard $V^{\otimes r}$ as an $(F\Sigma_r, S_F(n,r))$-bimodule. Thus there are maps $\alpha : F\Sigma_r \to \mathrm{End}_F(V^{\otimes r})$ and $\beta : S_F(n,r) \to \mathrm{End}_F(V^{\otimes r})$, where $(F\Sigma_r)\alpha$ and $S_F(n,r)\beta$ are subalgebras of $\mathrm{End}_F(V^{\otimes r})$. Let $\mathrm{End}_{F\Sigma_r}(V^{\otimes r})$ and $\mathrm{End}_{S_F(n,r)}(V^{\otimes r})$ denote the centralizers in $\mathrm{End}_F(V^{\otimes r})$ of $(F\Sigma_r)\alpha$ and $S_F(n,r)\beta$, respectively. We require a version of 'Schur–Weyl duality'. However, this is usually stated only for infinite fields: see, for example, [11, Theorem 1.2]. Thus we give the simple extra argument needed to deduce the result for an arbitrary field F.

Lemma 2.4 (Schur-Weyl duality) *In the above notation,*

$$\mathrm{End}_{S_F(n,r)}(V^{\otimes r}) = (F\Sigma_r)\alpha, \quad \mathrm{End}_{F\Sigma_r}(V^{\otimes r}) = S_F(n,r)\beta.$$

Proof. Let E be an infinite extension field of F and write $V_E = E \otimes_F V$. We identify $V_E^{\otimes r}$ with $E \otimes_F V^{\otimes r}$. The analogues of α and β over E are

$$\alpha_E : E\Sigma_r \longrightarrow \mathrm{End}_E(V_E^{\otimes r}), \quad \beta_E : S_E(n,r) \longrightarrow \mathrm{End}_E(V_E^{\otimes r}).$$

Identifying $\mathrm{End}_E(V_E^{\otimes r})$ with $E \otimes_F \mathrm{End}_F(V^{\otimes r})$ we find that

$$(E\Sigma_r)\alpha_E = E \otimes_F (F\Sigma_r)\alpha, \quad S_E(n,r)\beta_E = E \otimes_F S_F(n,r)\beta. \tag{2.5}$$

Clearly,

$$(F\Sigma_r)\alpha \subseteq \mathrm{End}_{S_F(n,r)}(V^{\otimes r}), \quad S_F(n,r)\beta \subseteq \mathrm{End}_{F\Sigma_r}(V^{\otimes r}). \tag{2.6}$$

However, by [11, Theorem 1.2],

$$\mathrm{End}_{S_E(n,r)}(V_E^{\otimes r}) = (E\Sigma_r)\alpha_E, \quad \mathrm{End}_{E\Sigma_r}(V_E^{\otimes r}) = S_E(n,r)\beta_E. \tag{2.7}$$

By (2.7) and (2.5), we have $\mathrm{End}_{S_E(n,r)}(V_E^{\otimes r}) = E \otimes_F (F\Sigma_r)\alpha$. However,

$$\mathrm{End}_{S_E(n,r)}(V_E^{\otimes r}) \cong E \otimes_F \mathrm{End}_{S_F(n,r)}(V^{\otimes r}),$$

by [5, (29.5)], since $S_E(n,r)$ may be identified with $E \otimes_F S_F(n,r)$. Hence

$$E \otimes_F \mathrm{End}_{S_F(n,r)}(V^{\otimes r}) \cong E \otimes_F (F\Sigma_r)\alpha.$$

Therefore, from (2.6) and consideration of dimension, $\mathrm{End}_{S_F(n,r)}(V^{\otimes r}) = (F\Sigma_r)\alpha$. A similar argument gives the result for $\mathrm{End}_{F\Sigma_r}(V^{\otimes r})$. □

We require some background from [3], and for this we follow [3, §2] with only minor variations of notation and terminology.

Let F be a field and recall that, for any F-space V, we write $T(V)$ for the tensor algebra on V. Let $F(\infty)$ denote an F-space with a countably infinite basis $\{x_1, x_2, \ldots\}$ and, for each positive integer n, let $F(n)$ denote the subspace of $F(\infty)$ with basis $\{x_1, \ldots, x_n\}$. Then, with the obvious identifications,

$$T(F(1)) \subseteq T(F(2)) \subseteq \cdots \subseteq T(F(\infty)).$$

For positive integers n_1 and n_2, where $n_1 \leqslant n_2$, define $\pi_{n_2,n_1} \in \mathrm{End}_F(F(n_2))$ by

$$x_i\pi_{n_2,n_1} = \begin{cases} x_i & \text{for } i \in \{1, \ldots, n_1\}, \\ 0 & \text{for } i \in \{n_1+1, \ldots, n_2\}. \end{cases}$$

This extends to an endomorphism of $T(F(n_2))$ with image $T(F(n_1))$, and the restriction of this to $F(n_2)^{\otimes r}$ gives $\pi^{\otimes r}_{n_2,n_1} \in \mathrm{End}_F(F(n_2)^{\otimes r})$ with image $F(n_1)^{\otimes r}$.

For each n we regard $F(n)$ as the natural $S_F(n,1)$-module, so that $F(n)^{\otimes r}$ is an $S_F(n,r)$-module. Suppose that $\{W(n) : n \in \mathbb{N}\}$ is a family of modules such that, for all n, $W(n)$ is an $S_F(n,r)$-submodule of $F(n)^{\otimes r}$ and $W(n_2)\pi^{\otimes r}_{n_2,n_1} = W(n_1)$ for all n_1 and n_2 with $n_1 \leqslant n_2$. Then we say that the family $\{W(n) : n \in \mathbb{N}\}$ is a *uniform submodule family* of $\{F(n)^{\otimes r} : n \in \mathbb{N}\}$.

Lemma 2.5. *Suppose that $\{W(n) : n \in \mathbb{N}\}$ is a uniform submodule family of $\{F(n)^{\otimes r} : n \in \mathbb{N}\}$ such that, for some $m \geqslant r$, $W(m)$ is a direct summand of $F(m)^{\otimes r}$. Then there exists an idempotent e of $F\Sigma_r$ such that $W(n) = eF(n)^{\otimes r}$ for all n.*

Proof. By the hypothesis on $W(m)$, there is an idempotent $S_F(m,r)$-module homomorphism $\rho : F(m)^{\otimes r} \to F(m)^{\otimes r}$ with image $W(m)$. By Lemma 2.4, there exists $e \in F\Sigma_r$ such that $u\rho = eu$ for all $u \in F(m)^{\otimes r}$. Thus $W(m) = eF(m)^{\otimes r}$. Also, since ρ is idempotent and $F\Sigma_r$ acts faithfully on $F(m)^{\otimes r}$, e is an idempotent.

Now consider the family of modules $\{eF(n)^{\otimes r} : n \in \mathbb{N}\}$. Since $\pi^{\otimes r}_{n_2,n_1}$ commutes with the action of $F\Sigma_r$, we have $(eF(n_2)^{\otimes r})\pi^{\otimes r}_{n_2,n_1} = eF(n_1)^{\otimes r}$, for all n_1 and n_2 with $n_1 \leqslant n_2$. Thus $\{eF(n)^{\otimes r}\}$ is a uniform submodule family of $\{F(n)^{\otimes r}\}$. Since $W(m) = eF(m)^{\otimes r}$, we may apply $\pi^{\otimes r}_{m,r}$ to obtain $W(r) = eF(r)^{\otimes r}$. Therefore, by [3, Lemma 2.5], $W(n) = eF(n)^{\otimes r}$ for all n. □

3. Lie powers

Let V be a vector space over a field F with basis $\{x_i : i \in I\}$. The tensor algebra $T(V)$ has the structure of a Lie algebra over F under the multiplication given by $[u,v] = u \otimes v - v \otimes u$ for all $u, v \in T(V)$, and, by a theorem of Witt, the Lie subalgebra generated by $\{x_i : i \in I\}$ is a free Lie algebra, freely generated by $\{x_i : i \in I\}$, which we denote by $L(V)$. For each positive integer r, we write

$$L^r(V) = L(V) \cap T^r(V) = L(V) \cap V^{\otimes r}.$$

If G is a group and V is an FG-module then, as seen in §2, $T(V)$ is an FG-module, and it is easily verified that $L(V)$ and $L^r(V)$ are submodules. We call the module $L^r(V)$ the rth Lie power of V. If V has finite dimension n and is regarded as the natural $S_F(n,1)$-module then $L^r(V)$ is an $S_F(n,r)$-submodule of $V^{\otimes r}$ (see [3, §2]).

In [3, §2, page 177] it was observed that if B is a subspace of $L^s(V)$, for some s, then the Lie subalgebra of $L(V)$ generated by B may be identified with the free Lie algebra $L(B)$, and then, for all r, $L^r(B)$ is a subspace of $L^{rs}(V)$. We make such identifications in the statement of the 'Decomposition Theorem' from [3].

Theorem 3.1. [3, Theorem 4.4] *Let F be a field of prime characteristic p. Let G be a group and V a finite-dimensional FG-module. For each positive integer r there is a submodule B_r of $L^r(V)$ such that B_r is a direct summand of $V^{\otimes r}$ and, for $m \geqslant 0$ and k not divisible by p,*

$$L^{p^m k}(V) = L^{p^m}(B_k) \oplus L^{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L^1(B_{p^m k}). \tag{3.1}$$

We shall extend this to modules V that may be infinite-dimensional.

Let p be a prime and let $\mathbb{F}_p$ be a field of p elements. Let k be a positive integer not divisible by p. By [3, Theorem 4.2], for each positive integer s there exists a uniform submodule family $\{B_{sk}(n) : n \in \mathbb{N}\}$ of $\{\mathbb{F}_p(n)^{\otimes sk} : n \in \mathbb{N}\}$ such that, for all n, $B_{sk}(n) \subseteq L^{sk}(\mathbb{F}_p(n))$, $B_{sk}(n)$ is a direct summand of $\mathbb{F}_p(n)^{\otimes sk}$, and there is an equality of subspaces of $L(\mathbb{F}_p(n))$,

$$\begin{aligned} &L^k(\mathbb{F}_p(n)) \oplus L^{2k}(\mathbb{F}_p(n)) \oplus L^{3k}(\mathbb{F}_p(n)) \oplus \cdots \\ &\qquad = L(B_k(n)) \oplus L(B_{2k}(n)) \oplus L(B_{3k}(n)) \oplus \cdots . \end{aligned} \tag{3.2}$$

Therefore, for all $m \geqslant 0$, $\{B_{p^m k}(n)\}$ is a uniform submodule family of $\{\mathbb{F}_p(n)^{\otimes p^m k}\}$ such that, for all n,

$$B_{p^m k}(n) \subseteq L^{p^m k}(\mathbb{F}_p(n)) \tag{3.3}$$

and $B_{p^m k}(n)$ is a direct summand of $\mathbb{F}_p(n)^{\otimes p^m k}$. Furthermore, from (3.2), by comparing terms of degree $p^m k$ within $L(\mathbb{F}_p(n))$, we have

$$L^{p^m k}(\mathbb{F}_p(n)) = L^{p^m}(B_k(n)) \oplus L^{p^{m-1}}(B_{pk}(n)) \oplus \cdots \oplus L^1(B_{p^m k}(n)). \tag{3.4}$$

Lemma 2.5, applied to the families $\{B_{p^m k}(n)\}$, gives the following result.

Lemma 3.2. *For each $m \geqslant 0$ there exists an idempotent $b_{p^m k}$ of $\mathbb{F}_p\Sigma_{p^m k}$ such that, for all n,*

$$B_{p^m k}(n) = b_{p^m k}\left(\mathbb{F}_p(n)^{\otimes p^m k}\right). \tag{3.5}$$

We now ignore module structure and regard $\mathbb{F}_p(n)$ simply as an $\mathbb{F}_p$-space. Thus, by (3.3), (3.4) and (3.5), if W is any finite-dimensional $\mathbb{F}_p$-space, we have

$$b_{p^m k} W^{\otimes p^m k} \subseteq L^{p^m k}(W) \tag{3.6}$$

and

$$L^{p^m k}(W) = L^{p^m}(b_k W^{\otimes k}) \oplus L^{p^{m-1}}(b_{pk} W^{\otimes pk}) \oplus \cdots \oplus L^1(b_{p^m k} W^{\otimes p^m k}). \tag{3.7}$$

Let F be any extension field of $\mathbb{F}_p$. By tensoring the terms of (3.6) and (3.7) with F we find that (3.6) and (3.7) hold for any finite-dimensional F-space W. We now generalise to arbitrary dimension.

Proposition 3.3. *Let p be a prime and let $\mathbb{F}_p$ be a field of p elements. Let k be a positive integer not divisible by p. Then, for each $m \geqslant 0$, there is an idempotent $b_{p^m k}$ of $\mathbb{F}_p\Sigma_{p^m k}$ such that if F is any extension field of $\mathbb{F}_p$ and V is any F-space* (*of finite or infinite dimension*) *we have*

$$b_{p^m k}V^{\otimes p^m k} \subseteq L^{p^m k}(V) \tag{3.8}$$

and

$$L^{p^m k}(V) = L^{p^m}(b_k V^{\otimes k}) \oplus L^{p^{m-1}}(b_{pk}V^{\otimes pk}) \oplus \cdots \oplus L^1(b_{p^m k}V^{\otimes p^m k}). \tag{3.9}$$

Proof. We use the idempotents $b_{p^m k}$ of Lemma 3.2. Suppose that (3.8) does not hold. Then there exist $v_1, \ldots, v_{p^m k} \in V$ such that $b_{p^m k}(v_1 \otimes \cdots \otimes v_{p^m k}) \notin L^{p^m k}(V)$. Let W be the subspace of V spanned by $v_1, \ldots, v_{p^m k}$. Then $b_{p^m k}W^{\otimes p^m k} \nsubseteq L^{p^m k}(W)$, contrary to (3.6) over the field F. Thus (3.8) holds.

Write S_V for the subspace of $L(V)$ defined by

$$S_V = L^{p^m}(b_k V^{\otimes k}) + L^{p^{m-1}}(b_{pk}V^{\otimes pk}) + \cdots + L^1(b_{p^m k}V^{\otimes p^m k}).$$

If the sum on the right is not a direct sum then there exists a finite-dimensional subspace W of V such that the corresponding sum S_W is not a direct sum, contrary to (3.7) over F. Similarly, if $L^{p^m k}(V) \nsubseteq S_V$ then there exists a finite-dimensional subspace W of V such that $L^{p^m k}(W) \nsubseteq S_W$, again contrary to (3.7) over F. By (3.8), $S_V \subseteq L^{p^m k}(V)$. Thus (3.9) holds. $\square$

In the rest of this paper, whenever F is a field of characteristic p, we assume that F is an extension field of $\mathbb{F}_p$ (rather than a field isomorphic to $\mathbb{F}_p$) so that the idempotents of Proposition 3.3 are available. This is essentially a notational issue. We can now derive our first main result.

Theorem 3.4. *Theorem* 3.1 *holds for an FG-module V of arbitrary dimension, where we take $B_{p^m k} = b_{p^m k}V^{\otimes p^m k}$ for all $m \geqslant 0$ and all k not divisible by p.*

Proof. Since $V^{\otimes p^m k}$ is an $(F\Sigma_{p^m k}, FG)$-bimodule, $B_{p^m k}$ is an FG-submodule of $V^{\otimes p^m k}$. It is a direct summand, since $b_{p^m k}$ is an idempotent, and, by (3.8), it is a submodule of $L^{p^m k}(V)$. Finally, (3.9) gives (3.1). $\square$

It is easily seen, by induction, that the modules $B_{p^m k}$ satisfying (3.1), for a given finite-dimensional module V, are determined uniquely up to isomorphism: thus we may take them to be the modules $b_{p^m k}V^{\otimes p^m k}$. Some information about the isomorphism types was given in [4]. In stating the result of interest here we use the notation $\bigoplus^r B$ for the direct sum of r copies of an F-space B.

Theorem 3.5. [4, Theorem 4.2]. *In the notation of Theorem* 3.1, *for each* $m \geqslant 0$ *there is an isomorphism of* FG*-modules*

$$(\bigoplus^{p^m} B_{p^m k}) \oplus (\bigoplus^{p^{m-1}} B_{p^{m-1}k}^{\otimes p}) \oplus \cdots \oplus (\bigoplus^{p} B_{pk}^{\otimes p^{m-1}}) \oplus B_k^{\otimes p^m} \cong L^k(V^{\otimes p^m}). \tag{3.10}$$

We shall extend this to modules V that may be infinite-dimensional, taking B_i to be $b_i V^{\otimes i}$ for all i. However, we need some preliminary facts.

Let k be any positive integer and let ℓ_k be the element of $\mathbb{Z}\Sigma_k$ defined, using the cycles $(2,1)$, $(3,2,1)$, …, $(k,\dots,2,1)$ of Σ_k, by

$$\ell_k = (1-(k,\dots,2,1))\cdots(1-(3,2,1))(1-(2,1)).$$

Let V be a vector space over a field F, and interpret ℓ_k as an element of $F\Sigma_k$. Then it is well known and straightforward to verify that

$$\ell_k(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = [\cdots[[v_1,v_2],v_3],\dots,v_k]\,, \tag{3.11}$$

for all $v_1, v_2, \dots, v_k \in V$. It follows that $L^k(V) = \ell_k V^{\otimes k}$.

Suppose, for the moment, that $\operatorname{char} F = 0$. Then, by (3.11) and the Dynkin-Specht-Wever criterion for Lie elements [7, Theorem V.8], we have

$$\ell_k^2(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = k\ell_k(v_1 \otimes v_2 \otimes \cdots \otimes v_k).$$

However, $F\Sigma_k$ acts faithfully on $V^{\otimes k}$ when $\dim V \geqslant k$. Hence $\ell_k^2 = k\ell_k$.

Now let F be any field such that $\operatorname{char} F \nmid k$. Thus $1/k$ exists in the prime subfield $\mathbb{F}$ of F and we can define an idempotent ω_k of $\mathbb{F}\Sigma_k$ (sometimes called the 'Dynkin idempotent') by $\omega_k = (1/k)\ell_k$. Then, for any F-space V,

$$L^k(V) = \omega_k V^{\otimes k}. \tag{3.12}$$

Let r and s be positive integers, and partition the set $\{1,\dots,rs\}$ as

$$\{1,\dots,s\} \cup \{s+1,\dots,2s\} \cup \cdots \cup \{(r-1)s+1,\dots,rs\}. \tag{3.13}$$

This gives, in an obvious way, an embedding $\mu : \Sigma_s \times \cdots \times \Sigma_s \to \Sigma_{rs}$, where the direct product has r factors. Let $\delta : \Sigma_s \to \Sigma_s \times \cdots \times \Sigma_s$ be the 'diagonal' embedding given by $\sigma\delta = (\sigma,\dots,\sigma)$ for all $\sigma \in \Sigma_s$. Then, by composition of δ and μ, we obtain an embedding $\lambda : \Sigma_s \longrightarrow \Sigma_{rs}$. Consider the partition

$$\{1, s+1, \dots, (r-1)s+1\} \cup \{2, s+2, \dots, (r-1)s+2\} \cup \cdots \cup \{s, 2s, \dots, rs\}, \tag{3.14}$$

with s parts. It is easily checked that, for all $\sigma \in \Sigma_s$, $\sigma\lambda$ is the permutation that permutes these s parts according to σ, while preserving the order of the numbers within each part.

Let F be any field and identify $F(\Sigma_s \times \cdots \times \Sigma_s)$ with $F\Sigma_s \otimes_F \cdots \otimes_F F\Sigma_s$. Then we may extend μ and λ linearly to obtain embeddings

$$\mu : F\Sigma_s \otimes \cdots \otimes F\Sigma_s \longrightarrow F\Sigma_{rs}, \quad \lambda : F\Sigma_s \longrightarrow F\Sigma_{rs}.$$

For $u, u_1, \dots, u_r \in F\Sigma_s$ we write $u_1 \# \cdots \# u_r$ to denote $(u_1 \otimes \cdots \otimes u_r)\mu$ and $u^{\# r}$ for $u \# \cdots \# u$. Also, for $u \in F\Sigma_s$, we write $u^{[r]}$ to denote $u\lambda$. (In general, when u is a proper linear combination of group elements, $u^{[r]} \neq u^{\# r}$.) Note that, if $u, u_1, \dots, u_r$ are idempotents of $F\Sigma_s$ then $u_1 \# \cdots \# u_r$, $u^{\# r}$ and $u^{[r]}$ are idempotents of $F\Sigma_{rs}$.

Let G be a group and let V be an FG-module. As usual, we index the factors of $V^{\otimes rs}$ by the set $\{1, \dots, rs\}$: more formally, $V^{\otimes rs}$ is identified with $\bigotimes_{i=1}^{rs} V_i$ where, for each i, there is a fixed isomorphism from V_i to V. Let Σ_{rs} act on $V^{\otimes rs}$ by place permutations. Now consider $V^{\otimes r} \otimes \cdots \otimes V^{\otimes r}$, that is $(V^{\otimes r})^{\otimes s}$, where the rs factors V are indexed by $\{1, \dots, rs\}$ in the order given by the partition (3.14). Let Σ_{rs} act on $(V^{\otimes r})^{\otimes s}$ by place permutations according to this indexing. It follows that $V^{\otimes rs}$ and $(V^{\otimes r})^{\otimes s}$ are isomorphic as $(F\Sigma_{rs}, FG)$-bimodules. Note that $F\Sigma_s$ also acts on $(V^{\otimes r})^{\otimes s}$ by place permutations, with $V^{\otimes r}$ instead of V in the usual construction. Then, for $w_1, \dots, w_s \in V^{\otimes r}$ and $\sigma \in \Sigma_s$ we find that

$$(\sigma\lambda)(w_1 \otimes \cdots \otimes w_s) = w_{1\sigma} \otimes \cdots \otimes w_{s\sigma} = \sigma(w_1 \otimes \cdots \otimes w_s).$$

Thus $\sigma\lambda$ and σ act in the same way on $(V^{\otimes r})^{\otimes s}$. Hence, for $u \in F\Sigma_s$, $u^{[r]}$ and u act in the same way on $(V^{\otimes r})^{\otimes s}$.

Suppose that k is a positive integer not divisible by $\operatorname{char} F$. By (3.12), $L^k(V^{\otimes r}) = \omega_k((V^{\otimes r})^{\otimes k})$. Thus (taking $s = k$ in the above analysis), we have $L^k(V^{\otimes r}) = \omega_k^{[r]}((V^{\otimes r})^{\otimes k})$. Hence we obtain an FG-module isomorphism

$$L^k(V^{\otimes r}) \cong \omega_k^{[r]} V^{\otimes rk}. \tag{3.15}$$

We next identify $V^{\otimes rs}$ with $V^{\otimes s} \otimes \cdots \otimes V^{\otimes s}$, that is $(V^{\otimes s})^{\otimes r}$, where the rs factors V are indexed in natural order, as given by (3.13). We also take $F\Sigma_s$ to act on $V^{\otimes s}$ by place permutations. Then, for $u_1, \dots, u_r \in F\Sigma_s$, we find that

$$(u_1 \# \cdots \# u_r)(V^{\otimes s})^{\otimes r} = (u_1 V^{\otimes s}) \otimes \cdots \otimes (u_r V^{\otimes s}).$$

Thus there is an FG-module isomorphism

$$(u_1 \# \cdots \# u_r) V^{\otimes rs} \cong (u_1 V^{\otimes s}) \otimes \cdots \otimes (u_r V^{\otimes s}). \tag{3.16}$$

In particular, for $u \in F\Sigma_s$,

$$u^{\# r} V^{\otimes rs} \cong (u V^{\otimes s})^{\otimes r}. \tag{3.17}$$

Suppose that V is a finite-dimensional FG-module, as in Theorem 3.5, where we take $B_i = b_i V^{\otimes i}$ for all i. Thus, by (3.10), (3.15) and (3.17), we have

$$\begin{aligned}&\Big(\bigoplus^{p^m} b_{p^m k}^{\# 1} V^{\otimes p^m k}\Big) \oplus \Big(\bigoplus^{p^{m-1}} b_{p^{m-1}k}^{\# p} V^{\otimes p^m k}\Big) \oplus \cdots \\ &\qquad \cdots \oplus \Big(\bigoplus^{p} b_{pk}^{\# p^{m-1}} V^{\otimes p^m k}\Big) \oplus b_k^{\# p^m} V^{\otimes p^m k} \;\cong\; \omega_k^{[p^m]} V^{\otimes p^m k}.\end{aligned} \tag{3.18}$$

Let W be an F-space of finite dimension n, where $n \geqslant p^m k$, and regard W as the natural $S_F(n,1)$-module. Let E be an infinite extension field of F. Thus we may regard $E \otimes_F W$ as the natural $S_E(n,1)$-module or, equivalently, the natural $E\mathrm{GL}(n,E)$-module. By (3.18) there is an isomorphism of $E\mathrm{GL}(n,E)$-modules

$$(\bigoplus^{p^m} b^{\#1}_{p^m k}(E \otimes W)^{\otimes p^m k}) \oplus \cdots \oplus b^{\#p^m}_k (E \otimes W)^{\otimes p^m k} \cong \omega^{[p^m]}_k (E \otimes W)^{\otimes p^m k}. \tag{3.19}$$

Since E is infinite, this is an isomorphism of $S_E(n, p^m k)$-modules (see [6, §2.4]). The spaces $b^{\#p^{m-i}}_{p^i k} W^{\otimes p^m k}$ and $\omega^{[p^m]}_k W^{\otimes p^m k}$ are $S_F(n, p^m k)$-modules, and we make the identifications $S_E(n, p^m k) = E \otimes S_F(n, p^m k)$,

$$b^{\#p^{m-i}}_{p^i k}(E \otimes W)^{\otimes p^m k} = E \otimes b^{\#p^{m-i}}_{p^i k} W^{\otimes p^m k}, \quad \omega^{[p^m]}_k (E \otimes W)^{\otimes p^m k} = E \otimes \omega^{[p^m]}_k W^{\otimes p^m k}.$$

Hence, by (3.19) and the Noether-Deuring theorem [5, (29.11)], there is an $S_F(n, p^m k)$-module isomorphism

$$(\bigoplus^{p^m} b^{\#1}_{p^m k} W^{\otimes p^m k}) \oplus \cdots \oplus b^{\#p^m}_k W^{\otimes p^m k} \cong \omega^{[p^m]}_k W^{\otimes p^m k}. \tag{3.20}$$

We can now derive our second main result.

Theorem 3.6. *Theorem* 3.5 *holds for an FG-module V of arbitrary dimension, where we take $B_{p^i k} = b_{p^i k} V^{\otimes p^i k}$ for all $i \geqslant 0$.*

Proof. By (3.20) and Corollary 2.3, there is an isomorphism of the form (3.18) for arbitrary V. Thus, by (3.15) and (3.17), we obtain (3.10) for arbitrary V. □

4. Decomposition of the modules $B_{p^m k}$

Let F be a field and let k be a positive integer not divisible by char F. Let E be the field obtained from F by adjoining (if necessary) a primitive kth root of unity ε, and let $\langle \varepsilon \rangle$ denote the multiplicative group generated by ε, consisting of all kth roots of unity in E. For $\xi \in \langle \varepsilon \rangle$ write $|\xi|$ for the multiplicative order of ξ.

Let V be an F-space and write $V_E = E \otimes_F V$. Let σ_k be the k-cycle $(1, 2, \ldots, k)$ of Σ_k, and, for each $\xi \in \langle \varepsilon \rangle$, let e_ξ be the element of $E\Sigma_k$ defined by

$$e_\xi = \frac{1}{k} \sum_{i=0}^{k-1} \xi^{-i} \sigma_k^i. \tag{4.1}$$

It is easy to verify that e_ξ is an idempotent of $E\Sigma_k$ and that $e_\xi V_E^{\otimes k}$ is the ξ-eigenspace of $V_E^{\otimes k}$ under the action of σ_k. Thus $V_E^{\otimes k} = \bigoplus_{\xi \in \langle \varepsilon \rangle} e_\xi V_E^{\otimes k}$.

If A is an E-algebra such that $V_E^{\otimes k}$ is an $(E\Sigma_k, A)$-bimodule, then each $e_\xi V_E^{\otimes k}$ is an A-submodule of $V_E^{\otimes k}$. For l prime to k, σ_k and σ_k^l are conjugate in Σ_k. It follows easily that there is an isomorphism of A-modules

$$e_\xi V_E^{\otimes k} \cong e_{\xi'} V_E^{\otimes k} \text{ when } |\xi| = |\xi'|. \tag{4.2}$$

Now suppose that G is a group and that V is a finite-dimensional FG-module. As shown in [2, §2], by means of [1], there exist FG-modules $(V^{\otimes k})_\xi$ such that

$$E \otimes_F (V^{\otimes k})_\xi \cong e_\xi V_E^{\otimes k}, \tag{4.3}$$

$V^{\otimes k} \cong \bigoplus_{\xi \in \langle \varepsilon \rangle} (V^{\otimes k})_\xi$, and $(V^{\otimes k})_\xi \cong (V^{\otimes k})_{\xi'}$ when $|\xi| = |\xi'|$. As in [2], for each (positive) divisor d of k, let $U_{k,d}$ denote an FG-module satisfying

$$U_{k,d} \cong (V^{\otimes k})_\xi \text{ when } |\xi| = d. \tag{4.4}$$

Theorem 4.1 [2, Theorem 4.2]. *Let F be a field of prime characteristic p, G a group, and V a finite-dimensional FG-module. Let k be a positive integer not divisible by p and let m be a non-negative integer. Let $B_{p^m k}$ be the module given by Theorem* 3.1 *and, for each divisor d of k, let $U_{k,d}$ be the module of* (4.4). *Then there is an index set Λ_0, and, for each $\lambda \in \Lambda_0$, a p^m-tuple $(\lambda(1), \dots, \lambda(p^m))$ of divisors of k, such that*

$$B_{p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} U_{k,\lambda(1)} \otimes \cdots \otimes U_{k,\lambda(p^m)}.$$

The purpose of this section is to generalise this theorem to modules V that are allowed to be infinite-dimensional. We begin with a general result.

Lemma 4.2. *Let k be a positive integer and let E be a field with prime field $\mathbb{F}$. Let w be an idempotent of $E\Sigma_k$. Then there exists an idempotent w_0 of $\mathbb{F}\Sigma_k$ such that the ideals $wE\Sigma_k$ and $w_0 E\Sigma_k$ of $E\Sigma_k$ are isomorphic as $E\Sigma_k$-modules.*

Proof. Suppose that P is a principal indecomposable $\mathbb{F}\Sigma_k$-module and let R denote the radical of $\mathbb{F}\Sigma_k$. Thus P/PR is irreducible (see [5, (54.11)]). We have

$$E \otimes_{\mathbb{F}} (P/PR) \cong (E \otimes_{\mathbb{F}} P)/(E \otimes_{\mathbb{F}} P)(E \otimes_{\mathbb{F}} R).$$

Since $\mathbb{F}$ is a splitting field for Σ_k (see [8, Theorem 11.5]), $E \otimes_{\mathbb{F}} (P/PR)$ is irreducible. Hence $(E \otimes_{\mathbb{F}} P)/(E \otimes_{\mathbb{F}} P)(E \otimes_{\mathbb{F}} R)$ is irreducible. However, $E \otimes_{\mathbb{F}} R$ is the radical of $E \otimes_{\mathbb{F}} \mathbb{F}\Sigma_k$ (see [5, (29.22)]). It follows that $E \otimes_{\mathbb{F}} P$ is indecomposable.

In the rest of the proof we identify $E \otimes_{\mathbb{F}} \mathbb{F}\Sigma_k$ with $E\Sigma_k$. Write $\mathbb{F}\Sigma_k$ as a direct sum of principal indecomposables, $\mathbb{F}\Sigma_k = \bigoplus_{j=1}^r P_j$. Thus $E\Sigma_k = \bigoplus_{j=1}^r E \otimes_{\mathbb{F}} P_j$, and, by what was proved above, each $E \otimes_{\mathbb{F}} P_j$ is indecomposable. Since w is an idempotent, the right ideal $wE\Sigma_k$ is a direct summand of $E\Sigma_k$. Hence, by the Krull-Schmidt theorem, it is isomorphic, as a right $E\Sigma_k$-module, to the direct sum of some subset of the modules $E \otimes_{\mathbb{F}} P_j$. Hence there is a direct summand U of $\mathbb{F}\Sigma_k$ such that $wE\Sigma_k \cong E \otimes_{\mathbb{F}} U$. Since U is a direct summand, we may write $U = w_0 \mathbb{F}\Sigma_k$ for some idempotent w_0 of $\mathbb{F}\Sigma_k$. It follows that $wE\Sigma_k \cong w_0 E\Sigma_k$. □

Recall from Section 2 that if F is a field and n is a positive integer then $F(n)$ denotes an F-space of dimension n regarded as the natural $S_F(n,1)$-module.

Let $\mathbb{F}$ be a prime field and let k be a positive integer not divisible by $\operatorname{char} \mathbb{F}$. Let C be the (cyclotomic) field obtained from $\mathbb{F}$ by adjoining a primitive kth root of unity ε_k. For each divisor d of k write $\varepsilon_d = \varepsilon_k^{k/d}$, so that $|\varepsilon_d| = d$, and let $e_{\varepsilon_d} \in C\Sigma_k$ be defined as in (4.1) with $\xi = \varepsilon_d$. By Lemmas 4.2 and 2.1 there exists an idempotent $u_{k,d}$ of $\mathbb{F}\Sigma_k$ such that, for any positive integer n, $e_{\varepsilon_d} C(n)^{\otimes k}$ and

$u_{k,d}C(n)^{\otimes k}$ are isomorphic $S_C(n,k)$-modules. Hence, by (4.2), if η is any element of $\langle \varepsilon_k \rangle$ of order d, there is an isomorphism of $S_C(n,k)$-modules

$$e_\eta C(n)^{\otimes k} \cong u_{k,d}C(n)^{\otimes k}. \tag{4.5}$$

Let E be any extension field of $\mathbb{F}$ such that E contains a primitive kth root of unity ε, and write $C' = \mathbb{F}(\varepsilon)$. Let χ be an isomorphism from C to C'. Of course, χ is the identity on $\mathbb{F}$. Applying χ to (4.5) and recalling that $u_{k,d} \in \mathbb{F}\Sigma_k$, we get an $S_{C'}(n,k)$-module isomorphism $e_\xi C'(n)^{\otimes k} \cong u_{k,d}C'(n)^{\otimes k}$, for all $\xi \in C'$ with $|\xi| = d$. Hence, by tensoring with E, we get an $S_E(n,k)$-module isomorphism

$$e_\xi E(n)^{\otimes k} \cong u_{k,d}E(n)^{\otimes k}, \tag{4.6}$$

for all $\xi \in E$ with $|\xi| = d$.

Now suppose that F is any extension field of $\mathbb{F}$ and let E be obtained from F by adjoining (if necessary) a primitive kth root of unity. Let G be a group and let V be an FG-module of finite dimension n. For each divisor d of k let $U_{k,d}$ be an FG-module satisfying (4.4). We regard V_E as the natural $S_E(n,1)$-module. Let $\xi \in E$ with $|\xi| = d$. Then, by (4.6), there is an $S_E(n,k)$-module isomorphism

$$\alpha : e_\xi V_E^{\otimes k} \longrightarrow u_{k,d}V_E^{\otimes k}. \tag{4.7}$$

If we think of V_E as the natural $E\mathrm{GL}(n,E)$-module, the map α of (4.7) is an isomorphism of $E\mathrm{GL}(n,E)$-modules. Also, since V is an FG-module, V_E is an EG-module, and there is an associated homomorphism $\rho : G \to \mathrm{GL}(n,E)$. The action of G on $V_E^{\otimes k}$ is given by the composition of ρ and the action of $\mathrm{GL}(n,E)$ on $V_E^{\otimes k}$. It follows that α is an isomorphism of EG-modules.

By (4.3) and (4.4), $E \otimes U_{k,d}$ and $e_\xi V_E^{\otimes k}$ are isomorphic EG-modules. Thus, since (4.7) is an EG-module isomorphism, $E \otimes U_{k,d}$ and $u_{k,d}V_E^{\otimes k}$ are isomorphic EG-modules. Hence, by the Noether-Deuring theorem, $U_{k,d}$ and $u_{k,d}V^{\otimes k}$ are isomorphic FG-modules. Thus we have proved the following proposition.

Proposition 4.3. *Let $\mathbb{F}$ be a prime field, k a positive integer not divisible by* $\operatorname{char}\mathbb{F}$*, and d a divisor of k. Then there is an idempotent $u_{k,d}$ of $\mathbb{F}\Sigma_k$ such that if G is any group, F is any extension field of $\mathbb{F}$, and V is any finite-dimensional FG-module, then the FG-module $U_{k,d}$ of* (4.4) *satisfies $U_{k,d} \cong u_{k,d}V^{\otimes k}$.*

From now on we assume that p is a prime and F is an extension field of $\mathbb{F}_p$, the field of p elements. We take k to be a positive integer not divisible by p. For each divisor d of k we use the idempotent $u_{k,d}$ of $\mathbb{F}_p\Sigma_k$ given by Proposition 4.3.

Let m be a non-negative integer and let W be an F-space of dimension $p^m k$. Let E be an infinite extension field of F and write $W_E = E \otimes_F W$, where we regard W_E as the natural $E\mathrm{GL}(p^m k, E)$-module.

We apply Theorem 4.1 but with E replacing F, $G = \mathrm{GL}(p^m k, E)$ and $V = W_E$. Recall that $B_{p^m k}$ may be written up to isomorphism as $b_{p^m k}W_E^{\otimes p^m k}$ and, by Proposition 4.3, for each divisor d of k, $U_{k,d}$ may be written up to isomorphism as $u_{k,d}W_E^{\otimes k}$. Hence Theorem 4.1 gives an isomorphism of $E\mathrm{GL}(p^m k, E)$-modules

$$b_{p^m k}W_E^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} u_{k,\lambda(1)}W_E^{\otimes k} \otimes \cdots \otimes u_{k,\lambda(p^m)}W_E^{\otimes k}.$$

Therefore, by (3.16),

$$b_{p^m k} W_E^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} (u_{k,\lambda(1)} \# \cdots \# u_{k,\lambda(p^m)}) W_E^{\otimes p^m k}. \tag{4.8}$$

Since E is infinite, (4.8) is an isomorphism of $S_E(p^m k, p^m k)$-modules. Hence, by the Noether-Deuring theorem, there is an $S_F(p^m k, p^m k)$-module isomorphism

$$b_{p^m k} W^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} (u_{k,\lambda(1)} \# \cdots \# u_{k,\lambda(p^m)}) W^{\otimes p^m k}. \tag{4.9}$$

Let V be an FG-module of arbitrary, possibly infinite, dimension. Then, by (4.9) and Corollary 2.3, there is an isomorphism of FG-modules

$$b_{p^m k} V^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} (u_{k,\lambda(1)} \# \cdots \# u_{k,\lambda(p^m)}) V^{\otimes p^m k}.$$

In other words, by (3.16),

$$b_{p^m k} V^{\otimes p^m k} \cong \bigoplus_{\lambda \in \Lambda_0} u_{k,\lambda(1)} V^{\otimes k} \otimes \cdots \otimes u_{k,\lambda(p^m)} V^{\otimes k}.$$

Thus we have proved the following result.

Theorem 4.4. *Theorem* 4.1 *holds for an FG-module V of arbitrary dimension, where we take $B_{p^m k} = b_{p^m k} V^{\otimes p^m k}$ and $U_{k,d} = u_{k,d} V^{\otimes k}$ for every divisor d of k.*

We conclude with an observation on the idempotent $u_{k,k}$.

Proposition 4.5. *Under the hypotheses of Theorem* 4.4, $u_{k,k} V^{\otimes k} \cong L^k(V)$.

Proof. We use the idempotent ω_k of $\mathbb{F}_p \Sigma_k$ defined in Section 3. Let E be an infinite extension field of F and let W be an F-space of dimension k. Write $W_E = E \otimes_F W$ and regard W_E as the natural $E\mathrm{GL}(k, E)$-module. By [2, Lemma 2.3] combined with Proposition 4.3, $u_{k,k} W_E^{\otimes k}$ and $L^k(W_E)$ are isomorphic as $E\mathrm{GL}(k, E)$-modules. Thus, by (3.12), $u_{k,k} W_E^{\otimes k} \cong \omega_k W_E^{\otimes k}$. Since E is infinite, this is an isomorphism of $S_E(k, k)$-modules. Hence, by the Noether-Deuring theorem, $u_{k,k} W^{\otimes k}$ and $\omega_k W^{\otimes k}$ are isomorphic as $S_F(k, k)$-modules. By Corollary 2.3, if V is an FG-module of arbitrary dimension, $u_{k,k} V^{\otimes k}$ and $\omega_k V^{\otimes k}$ are isomorphic as FG-modules. Thus, by (3.12), $u_{k,k} V^{\otimes k} \cong L^k(V)$. $\square$

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