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The Endomorphisms of the Lattice of Closed Convex Cones

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Abstract. The set of closed convex cones in $\mathbb{R}^d$ ($d \geq 3$), with the operations of intersection and closed sum, is a lattice. We determine all endomorphisms of this lattice. As a consequence, we obtain a characterization of the duality of convex cones.

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1. Introduction

By $\mathbb{R}^d$ we denote the d-dimensional real Euclidean vector space, equipped with its standard scalar product $\langle \cdot, \cdot \rangle$. We assume throughout that $d \geq 3$. Let $\mathcal{C}^d$ be the set of closed convex cones in $\mathbb{R}^d$ (including $\{0\}$ and $\mathbb{R}^d$). The set $\mathcal{C}^d$ together with the two operations $\cap$ (intersection) and $\vee$, defined by

$$C \vee D := \mathrm{cl}\,\mathrm{conv}\,(C \cup D) = \mathrm{cl}\,(C + D), \qquad C, D \in \mathcal{C}^d,$$

is a lattice. The following theorem determines all endomorphisms of this lattice.

Theorem. *Let $d \geq 3$. Let $\varphi : \mathcal{C}^d \to \mathcal{C}^d$ be a mapping satisfying*

$$\varphi(C \cap D) = \varphi(C) \cap \varphi(D), \tag{1}$$

$$\varphi(C \vee D) = \varphi(C) \vee \varphi(D) \tag{2}$$

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for all $C, D \in \mathcal{C}^d$. *Then either* φ *is constant, or there exists a linear transformation* $g \in GL(d)$ *such that* $\varphi(C) = gC$ *for all* $C \in \mathcal{C}^d$.

This theorem fits into a series of similar results (where the assumptions on the dimension d may differ). Gruber [6] has determined the endomorphisms of the lattice $(\mathcal{K}^d, \cap, \vee)$, where $\mathcal{K}^d$ is the set of compact convex sets in $\mathbb{R}^d$ (including the empty set) and $K \vee L := \mathrm{conv}(K \cup L)$. Gruber [7] has further classified the endomorphisms of the lattice $(\mathcal{B}^d, \cap, \vee)$, where $\mathcal{B}^d$ is the set of unit balls of norms on $\mathbb{R}^d$. This was extended in [4] to a determination of the endomorphisms of the lattices $(\mathcal{K}_0^d, \cap, \vee)$ and $(\mathcal{K}_{(0)}^d, \cap, \vee)$, where $\mathcal{K}_0^d$ and $\mathcal{K}_{(0)}^d$ denote, respectively, the set of compact convex sets in $\mathbb{R}^d$ containing 0 or containing 0 in the interior. The latter result was used to answer a question by Vitali Milman, asking for a characterization of the mapping that associates with every convex body in $\mathcal{K}_{(0)}^d$ its polar body.

That the theorem does not hold for $d = 2$, can be seen as follows. Let $h : \mathbb{R} \to \mathbb{R}$ be a strictly increasing, continuous function satisfying $h(\alpha + \pi) = h(\alpha) + \pi$ for $\alpha \in \mathbb{R}$. Using Cartesian coordinates in $\mathbb{R}^2$, let $R_\alpha := \{\lambda(\cos\alpha, \sin\alpha) : \lambda \geq 0\}$. Define $\varphi(\mathbb{R}^2) := \mathbb{R}^2$ and $\varphi(\{0\}) := \{0\}$. Every cone $C \in \mathcal{C}^2 \setminus \{\mathbb{R}^2, \{0\}\}$ is of the form $C = \bigcup_{\beta \leq \alpha \leq \gamma} R_\alpha$ with $\beta \leq \gamma \leq \beta + \pi$, where the angle β is unique mod 2π. If we define $\varphi(C) := \bigcup_{h(\beta) \leq \alpha \leq h(\gamma)} R_\alpha$, then φ is an endomorphism of the lattice $(\mathcal{C}^2, \cap, \vee)$.

As in [4], the knowledge of the endomorphisms leads to a characterization of duality. For $C \in \mathcal{C}^d$, the *dual cone* is defined by

$$C^* := \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 0 \text{ for all } y \in C\};$$

it is again in $\mathcal{C}^d$. The mapping $C \mapsto C^*$ is an involution that interchanges the lattice operations and reverses the order.

In a series of papers, Artstein-Avidan and Milman [1], [2], [3] have distilled the essential properties of an abstract duality from various classical duality operations for functions and convex sets. They have established the surprising fact that the involution property together with the sole condition of order reversion is in several cases sufficient for the essentially unique characterization of a duality. For example, the Legendre transform is, up to obvious linear modifications, the only involution on the set of lower semi-continuous convex functions that reverses the order. Our condition (4) in the subsequent corollary follows their scheme, and this provides yet another example of the phenomenon they discovered.

Corollary 1. *Let* $d \geq 3$. *Let* $\psi : \mathcal{C}^d \to \mathcal{C}^d$ *be a mapping satisfying*

$$\psi(\psi(C)) = C, \qquad \text{for all } C \in \mathcal{C}^d \tag{3}$$

and

$$C \subset D \ \Rightarrow\ \psi(C) \supset \psi(D) \qquad \text{for all } C, D \in \mathcal{C}^d. \tag{4}$$

Then there exists a selfadjoint linear transformation $g \in GL(d)$ *such that* $\psi(C) = gC^*$ *for all* $C \in \mathcal{C}^d$.

As one of many outcomes of their approach, Artstein-Avidan and Milman [3, Theorem 13] have also arrived at the following unexpected result. Let $\mathcal{C}_0^d$ denote the system of closed convex sets in $\mathbb{R}^d$ containing 0. If $\varphi : \mathcal{C}_0^d \to \mathcal{C}_0^d$ is a bijective map satisfying $K_1 \subset K_2 \Leftrightarrow \varphi(K_1) \subset \varphi(K_2)$ for all $K_1, K_2 \in \mathcal{C}_0^d$, then there exists a linear transformation $g \in GL(d)$ such that $\varphi(K) = gK$ for $K \in \mathcal{C}_0^d$. Our theorem implies a similar result for convex cones.

Corollary 2. *Let $d \geq 3$. Let $\varphi : \mathcal{C}^d \to \mathcal{C}^d$ be a bijective mapping satisfying*

$$C_1 \subset C_2 \Leftrightarrow \varphi(C_1) \subset \varphi(C_2) \qquad \text{for all } C_1, C_2 \in \mathcal{C}^d. \tag{5}$$

Then there exists a linear transformation $g \in GL(d)$ such that $\varphi(C) = gC$ for $C \in \mathcal{C}^d$.

In fact, as will be clear from the proof, analogous assertions are true (for $d \geq 2$) for each of the systems $\mathcal{K}^d$, $\mathcal{B}^d$, $\mathcal{K}_0^d$, $\mathcal{K}_{(0)}^d$, since in each case the lattice endomorphisms are known to be induced either by an affine transformation (in the first case, according to Gruber [6]) or by a linear transformation (in the other three cases). Only in the case of $\mathcal{C}_0^d$ treated by Artstein-Avidan and Milman, no corresponding classification of the lattice endomorphisms is known, so that their proof proceeds in a different way.

2. Proof of the theorem

The main line of reasoning is similar to that in [4], but there are some characteristic differences (which is already seen from the fact that the two-dimensional case is exceptional).

Let $\varphi : \mathcal{C}^d \to \mathcal{C}^d$ satisfy the assumptions (1) and (2). Then φ is inclusion preserving, that is, $C \subset D$ for $C, D \in \mathcal{C}^d$ implies $\varphi(C) \subset \varphi(D)$.

We write $\{0\} =: \bar{0}$. A *ray* is a set $R^x := \{\lambda x : \lambda \geq 0\}$, where $x \in \mathbb{R}^d \setminus \{0\}$. By $\mathcal{R}$ we denote the set of all rays in $\mathbb{R}^d$.

Following the procedure in [6], we distinguish several cases.

Case 1: $\varphi(R) = \varphi(\bar{0})$ for all $R \in \mathcal{R}$.

We choose $d+1$ rays $R_1, \dots, R_{d+1}$ with $R_1 \vee \dots \vee R_{d+1} = \mathbb{R}^d$. For any $C \in \mathcal{C}^d$ we have $\bar{0} \subset C \subset R_1 \vee \dots \vee R_{d+1}$ and hence $\varphi(\bar{0}) \subset \varphi(C) \subset \varphi(R_1) \vee \dots \vee \varphi(R_{d+1}) = \varphi(\bar{0})$, hence

$$\varphi(C) = \varphi(\bar{0}) \qquad \text{for } C \in \mathcal{C}^d. \tag{6}$$

Case 2: $\varphi(R) = \varphi(\bar{0})$ for some $R \in \mathcal{R}$, but not for all rays R.

Let $\mathcal{A} := \{R \in \mathcal{R} : \varphi(R) = \varphi(\bar{0})\}$, and let A be the union of all rays in $\mathcal{A}$. If $S, U \in \mathcal{A}$, $T \in \mathcal{R}$ and $T \subset S \vee U$, then $\varphi(\bar{0}) \subset \varphi(T) \subset \varphi(S) \vee \varphi(U) = \varphi(\bar{0})$, hence $T \in \mathcal{A}$; thus A is a convex cone. Since $A \neq \mathbb{R}^d$, the closure of A is contained in a closed halfspace with 0 in the boundary. We can choose rays $R \in \mathcal{A}$ and $T, U \in \mathcal{R} \setminus \mathcal{A}$ with $U \cap T = \bar{0}$ and $T \subset R \vee U$. Then $\varphi(\bar{0}) \subset \varphi(T) \subset [\varphi(R) \vee \varphi(U)] \cap \varphi(T) = \varphi(U) \cap \varphi(T) = \varphi(\bar{0})$, hence $\varphi(T) = \varphi(\bar{0})$, which is a contradiction.

Thus, Case 2 cannot occur, and we are left with the situation where $\varphi(R) \neq \varphi(\bar{0})$ for all $R \in \mathcal{R}$. This implies

$$\dim \varphi(R) \geq 1 \qquad \text{for } R \in \mathcal{R}.$$

In fact, since $\bar{0} \subset R$, the equality $\varphi(R) = \bar{0}$ would imply $\varphi(\bar{0}) \subset \varphi(R) = \bar{0}$ and hence $\varphi(R) = \varphi(\bar{0})$, a contradiction.

Case 3: $\varphi(R) \neq \varphi(\bar{0})$ for all $R \in \mathcal{R}$, $\varphi(\bar{0}) \neq \bar{0}$.

Case 4: $\varphi(R) \neq \varphi(\bar{0})$ for all $R \in \mathcal{R}$, $\varphi(\bar{0}) = \bar{0}$; there exists a ray $P \in \mathcal{R}$ with $\dim \varphi(P) \geq 2$.

With the same proof as in [4] we have:

(P1) *In an n-dimensional affine space, let M be a fixed closed convex set and let $\mathcal{F}$ be a family of n-dimensional closed convex sets such that $K \neq M$ for all $K \in \mathcal{F}$ and $K_1 \cap K_2 = M$ whenever $K_1, K_2 \in \mathcal{F}$ and $K_1 \neq K_2$. Then $\mathcal{F}$ is at most countable.*

By assumption, $d \geq 3$. We show simultaneously that Case 3 and Case 4 both lead to a contradiction. In Case 3, we put $P := \bar{0}$.

Let $B := \varphi(P)$ and $b := \dim B$, then $b \geq 1$ in Case 3 and $b \geq 2$ in Case 4.

A ray R is called *free* if $R \not\subset \operatorname{lin} P$ (so every ray is free in Case 3). Let R be a free ray. Then $\varphi(R) \cap \varphi(P) = \varphi(R \cap P) = \varphi(\bar{0})$. If $\varphi(R \vee P) = B$, then $\varphi(R) \vee \varphi(P) = \varphi(P)$, thus $\varphi(R) \subset \varphi(P)$ and hence $\varphi(R) \cap \varphi(P) = \varphi(R) \neq \varphi(\bar{0})$, a contradiction. This shows that $\varphi(R \vee P) \neq B$.

By a *sheet* we understand a set $R \vee \operatorname{lin} P$ with a free ray R. A sheet is called *bad* if it contains a free ray R with $\varphi(P \vee R) \subset \operatorname{lin} B$. If R, S are free rays in different sheets, then $(R \vee P) \cap (S \vee P) = P$, hence $\varphi(R \vee P) \cap \varphi(S \vee P) = B$. Now it follows from (P1) (applied in $\operatorname{lin} B$) that there are at most countably many bad sheets. The other sheets are called *good*.

Suppose that $b \geq d - 1$. Let the set $\mathcal{S}$ contain precisely one free ray from every good sheet, and no other elements, and put $\mathcal{F} := \{\varphi(R \vee P) : R \in \mathcal{S}\}$. If $R \in \mathcal{S}$, then $\varphi(R \vee P) \not\subset \operatorname{lin} B$, hence $\dim \varphi(R \vee P) = d$. It follows from (P1) that the family $\mathcal{F}$ is countable. This is a contradiction, since there are uncountably many good sheets. This proves that $b \leq d - 2$.

Let $k \in \{1, \ldots, d - b\}$. A set $\{x_1, \ldots, x_k\}$ of points in $\mathbb{R}^d \setminus \operatorname{lin} P$, briefly a k-set, is called *full* if $\dim \varphi(R^{x_1} \vee \cdots \vee R^{x_k} \vee P) \geq b + k$. A k-flat $E \subset \mathbb{R}^d$ is called *general* if $0 \notin E$ in Case 3, and if $\dim \operatorname{aff}(E \cup \operatorname{lin} P) = k + 2$ in Case 4.

If x_1 is contained in a good sheet, then $B \subset \varphi(R^{x_1} \vee P) \not\subset \operatorname{lin} B$, hence $\{x_1\}$ is a full 1-set. We assert the following.

(P2) *Let $k \in \{2, \ldots, d - b\}$. In every general $(k-1)$-flat $E \subset \mathbb{R}^d$ there is a full k-set.*

The proof of this proposition can be taken verbally from [4], if expressions like $\bar{x}_1 \vee \cdots \vee \bar{x}_k \vee \bar{p}$ are replaced by $R^{x_1} \vee \cdots \vee R^{x_k} \vee P$. It need, therefore, not be repeated.

Let $k = d - b$, and choose a general k-flat $F \subset \mathbb{R}^d$. Let $\mathcal{E}$ be the family of $(k-1)$-flats contained in F and parallel to a fixed $(k-1)$-flat. Since $E \in \mathcal{E}$ is general, by (P2) there exists a full k-set $\{x_1, \dots, x_k\}$ in E; put $C_E := R^{x_1} \vee \dots \vee R^{x_k} \vee P$. Then $\dim \varphi(C_E) = d$ and $C_{E_1} \cap C_{E_2} = P$ for $E_1, E_2 \in \mathcal{E}$ with $E_1 \neq E_2$, hence $\varphi(C_{E_1}) \cap \varphi(C_{E_2}) = B$. By (P1), this is a contradiction, since $\mathcal{E}$ is uncountable.

Since Cases 3 and 4 cannot occur, we are left with

Case 5: $\varphi(\bar{0}) = \bar{0}$, and $\dim \varphi(R) = 1$ for all $R \in \mathcal{R}$.

In this case, the φ-image of a ray is either a ray or a line. We show first that it is always a ray.

Let $x_1, \dots, x_{d+1} \in \mathbb{R}^d$ be the vertices of a simplex containing 0 in its interior. For each $i \in \{1, \dots, d+1\}$, choose a point $x_i' \in \varphi(R^{x_i}) \setminus \{0\}$. Now the argument used in [4], at the beginning of the treatment of Case 5 (with $\bar{x}_i$ and $\bar{a}$ replaced by R^{x_i} and R^a, respectively) shows that each $\varphi(R)$, $R \in \mathcal{R}$, has 0 as an endpoint and hence is a ray.

Let $E \subset \mathbb{R}^d$ be a two-dimensional linear subspace. Let $R, S \subset E$ be different rays. If $\varphi(S) = -\varphi(R)$, we choose a ray $T \subset E$ different from R and S. Then $\varphi(T) \neq -\varphi(R)$, since $\varphi(T) \cap \varphi(S) = \bar{0}$. Hence, we can assume from the beginning that $\varphi(S) \neq -\varphi(R)$. Let E' be the two-dimensional subspace spanned by $\varphi(R)$ and $\varphi(S)$. In the following, A, B, Z are rays in E. If $Z \subset A \vee B$ and $\varphi(A), \varphi(B) \subset E'$, then $\varphi(Z) \subset \varphi(A) \vee \varphi(B) \subset E'$. This yields $\varphi(Z) \subset E'$ for all $Z \subset R \vee S$. Let $Z \subset (R - S) \setminus (-S)$. Then $R \subset Z \vee S$, hence $\varphi(R) \subset \varphi(Z) \vee \varphi(S)$, which implies $\varphi(Z) \subset E'$. Similarly, $Z \subset (S - R) \setminus (-R)$ implies $\varphi(Z) \subset E'$. Finally, if $Z \subset -R - S$, we can choose rays $U \subset R - S$ and $V \subset S - R$ with $Z \subset U \vee V$, which gives $\varphi(Z) \subset E'$. We have proved that $\varphi(R) \subset E'$ for every ray $R \subset E$.

Let $R \in \mathcal{R}$. We assert that $\varphi(-R) = -\varphi(R)$. For the proof, choose $S \in \mathcal{R}$ with $S \neq \pm R$. Let A, B, C be any three of the rays $R, -R, S, -S$. Then $(A \vee B) \cap C = \bar{0}$, hence $(\varphi(A) \vee \varphi(B)) \cap \varphi(C) = \bar{0}$ and thus $\varphi(C) \not\subset \varphi(A) \vee \varphi(B)$. Since this holds for all choices of A, B, C from $\{R, -R, S, -S\}$ and since $\varphi(R), \varphi(-R), \varphi(S), \varphi(-S)$ lie in a two-dimensional subspace, the set $\{\varphi(R), \varphi(-R), \varphi(S), \varphi(-S)\}$ must be of the form $\{\pm U, \pm V\}$ with two rays U, V satisfying $U \neq \pm V$. Suppose that $\varphi(-R) = -\varphi(S)$. Choose a ray $T \subset S - R$ with $T \neq S, -R$. Then $\varphi(T) \subset \varphi(S) \vee \varphi(-R)$ and $\varphi(T) \cap \varphi(-R) = \varphi(T) \cap \varphi(S) = \bar{0}$, a contradiction. Thus, $\varphi(-R) = -\varphi(R)$ is the only possibility.

Let $\mathcal{L}$ be the set of one-dimensional linear subspaces of $\mathbb{R}^d$. For $L \in \mathcal{L}$, there are two rays $R, -R$ with $L = R \vee (-R)$; we call them the *generating rays* of L. Since $\varphi(-R) = -\varphi(R)$, the line $f(L) := \varphi(R) - \varphi(R)$ does not depend on the choice of R. Thus, we have defined a map $f : \mathcal{L} \to \mathcal{L}$. Let $L_1, L_2 \in \mathcal{L}$, $L_1 \neq L_2$. Let R_i be a generating ray of L_i $(i = 1, 2)$. If $f(L_1) = f(L_2)$, then either $\varphi(R_1) = \varphi(R_2)$ or $\varphi(R_1) = -\varphi(R_2)$. But $R_1 \cap R_2 = R_1 \cap (-R_2) = \bar{0}$, hence $\varphi(R_1) \cap \varphi(R_2) = \varphi(R_1) \cap (-\varphi(R_2)) = \bar{0}$, a contradiction. Thus, the map f is injective.

Suppose that $L_1, L_2, L_3 \in \mathcal{L}$ are different and coplanar, that is, contained in a two-dimensional linear subspace of $\mathbb{R}^d$. Then we can choose generating rays R_i of L_i $(i = 1, 2, 3)$ such that $R_2 \subset R_1 \vee R_3$. Then $\varphi(R_2) \subset \varphi(R_1) \vee \varphi(R_2)$. Therefore, the lines $f(L_1), f(L_2), f(L_3)$ are coplanar.

Since $d \geq 3$, then it follows from the Fundamental Theorem of Projective Geometry (see, e.g., Faure [5] for a proof of a general version), together with the injectivity of f, that there exists a linear transformation g of $\mathbb{R}^d$ such that $f(L) = gL$ for all $L \in \mathcal{L}$.

Let $R \in \mathcal{R}$. Then $\varphi(R)$ and $\varphi(-R) = -\varphi(R)$ are rays with $gR - gR = g(R - R) = f(R - R) = \varphi(R) - \varphi(R)$. It follows that either $\varphi(R) = gR$ or $\varphi(R) = -gR$. Let $\mathcal{A}_\pm := \{R \in \mathcal{R} : \varphi(R) = \pm gR\}$, and let $A_\pm$ be the union of the rays in $\mathcal{A}_\pm$. Then A_+ and A_- are convex cones. If both are non-empty, they have a common boundary ray S and we can choose rays $R_+ \in \mathcal{A}_+$ and $R_- \in \mathcal{A}_-$ such that $S \subset R_+ \vee R_-$. Then $\varphi(S) \subset \varphi(R_+) \vee \varphi(R_-)$, hence either $gS \subset gR_+ - gR_-$ or $-gS \subset gR_+ - gR_-$, which is a contradiction. It follows that A_+ or A_- is empty, and we can assume (replacing g by $-g$ if necessary) that $A_- = \emptyset$. Then $\varphi(R) = gR$ for all $R \in \mathcal{R}$.

Let $C \in \mathcal{C}^d$ and $R \in \mathcal{R}$. If $R \subset C$, then $gR = \varphi(R) \subset \varphi(C)$, hence $gC \subset \varphi(C)$. Let S be a ray with $S \not\subset gC$. Then $g^{-1}S \not\subset C$, hence $g^{-1}S \cap C = \bar{0}$. This gives $S \cap \varphi(C) = g(g^{-1}S) \cap \varphi(C) = \varphi(g^{-1}S) \cap \varphi(C) = \bar{0}$ and thus $S \not\subset \varphi(C)$. We have obtained $gC = \varphi(C)$, which completes the proof of the theorem.

3. Proof of the corollaries

First we prove Corollary 2. Let φ satisfy the assumptions of the corollary. For the reader's convenience, we modify (to sets) an argument that is due to Artstein-Avidan and Milman [2, Lemma 4]. Let $C_1, C_2 \in \mathcal{C}^d$. Then (5) gives $\varphi(C_1 \cap C_2) \subset \varphi(C_1) \cap \varphi(C_2)$. If this inclusion is strict, then by the bijectivity of φ there exists $B \in \mathcal{C}^d$ with $\varphi(C_1 \cap C_2) \subsetneqq \varphi(B) \subset \varphi(C_1) \cap \varphi(C_2)$, and (5) gives $C_1 \cap C_2 \subset B \subset C_1 \cap C_2$, a contradiction. In the same way it follows that $\varphi(C_1 \vee C_2) = \varphi(C_1) \vee \varphi(C_2)$, thus φ is a lattice endomorphism. Now the assertion of Corollary 2 follows from the theorem.

For the proof of Corollary 1, we note that the duality mapping $C \mapsto C^*$ reverses inclusions. Therefore, if ψ satisfies the assumptions of Corollary 1, then the map φ defined by $\varphi(C) := \psi(C)^*$ satisfies the assumptions of Corollary 2. Hence, there exists a linear transformation $g \in GL(d)$ such that $\varphi(C) = gC$ for $C \in \mathcal{C}^d$. Then $\psi(C) = (gC)^* = g^{-t}C^*$, and (3) implies $g^t = g$. This proves Corollary 1.

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