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Strong Commutativity Preserving Maps on Lie Ideals of Semiprime Rings

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Abstract. Let R be a 2-torsion free semiprime ring and U a nonzero square closed Lie ideal of R. In this paper it is shown that if f is either an endomorphism or an antihomomorphism of R such that $f(U) = U$, then f is strong commutativity preserving on U if and only if f is centralizing on U.

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1. Introduction

Throughout the present paper R will denote a unitary associative ring. As usual, for x, y in R, we write $[x, y] = xy - yx$, and we will use the identities $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = [x, y]z + y[x, z]$. For any $a \in R$, d_a will denote the inner-derivation defined by $d_a(x) = [a, x]$ for all $x \in R$.

A ring R is said to be *semiprime* if $aRa = 0$ implies that a=0. An ideal P of R is prime if $aRb \subseteq P$ implies that $a \in P$ or $b \in P$. Recall that a ring R is semiprime if and only if its zero ideal is the intersection of its prime ideals. Moreover, if the zero ideal of R is prime, then R is said to be a prime ring. An additive subgroup U of a ring R is a Lie ideal if $[U, R] \subseteq U$. Moreover, if $u^2 \in U$ for all $u \in U$, then U is called a *square closed Lie ideal.* Since $(u + v)^2 \in U$ and $[u, v] \in U$, we see that $2uv \in U$ for all $u, v \in U$. For a subset S of R, denote

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by $ann_R(S)$ the two-sided annihilator of S, i.e. $\{x \in R/Sx = xS = \{0\}\}$. For every ideal J of a semiprime ring R, it is known that $ann_R(J)$ is invariant under all derivations and $J \cap ann_R(J) = 0$.

A map $f : R \longrightarrow R$ is centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$; in particular if $[f(x), x] = 0$ for all $x \in S$, then f is called *commuting* on S.

A map $f : R \longrightarrow R$ is called *commutativity preserving* on S if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$, for all $x, y \in S$. In particular, if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$, then f is called *strong commutativity preserving* on S. Recently, M. S. Samman [4] proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing on the entire ring. Moreover, he proved that if R is a 2-torsion free semiprime ring, then a centralizing anti-homomorphism of R onto itself must be strong commutativity preserving. The purpose of this paper is to extend the results of [4] to square closed Lie ideals.

2. Preliminaries and results

In order to prove our main theorems, we shall need the following results.

Lemma 1. *Let R be a 2-torsion free semiprime ring and U a nonzero Lie ideal of R. If $[U, U] = 0$, then $U \subseteq Z(R)$.*

Proof. Let $u \in U$; since $[u, rt] \in U$ for all $r, t \in R$, then $[u, [u, rt]] = 0$. Hence $u[u, rt] = [u, rt]u$. Therefore

$$ur[u, t] + u[u, r]t = r[u, t]u + [u, r]tu.$$

As $u[u, r] = [u, r]u$ and $[u, t]u = u[u, t]$, then

$$ur[u, t] + [u, r]ut = ru[u, t] + [u, r]tu.$$

It follows that $2[u, r][u, t] = 0$ for all $u \in U$ and $r, t \in R$. Since R is 2-torsion free, thus

$$[u, r][u, t] = 0, \quad \text{for all } u \in U \text{ and } r, t \in R. \tag{1}$$

Replace t by sr in (1) to get $[u, r]R[u, r] = 0$ for all $u \in U, r, t \in R$. The fact R is semiprime implies that $U \subseteq Z(R)$. □

In all that follows U will be a square closed Lie ideal of R and M will denote the ideal of R generated by $[U, U]$, that is $M = R[U, U]R$.

Lemma 2. *Let R be a 2-torsion free semiprime ring and d a derivation of R. If a in R satisfies $ad(U) = 0$, then $ad(M) = 0$.*

Proof. Let P be an arbitrary prime ideal of R, and note that $\overline{R} = \dfrac{R}{P}$ is prime. If $[U, U] \subseteq P$ or $char(\overline{R}) = 2$, then $2ad(R)M \subseteq P$ and $2Mad(R) \subset P$. Assume now that $[U, U] \not\subseteq P$ and $char(\overline{R}) \neq 2$. The fact that R is 2-torsion free and $ad(U) = \{0\}$ implies that $aUd(v) = \{0\}$ for all $v \in U$ and thus $\bar{a}\overline{U}\overline{d(U)} = \bar{0}$.

As $[U,U] \not\subset P$, then $\overline{U} \not\subset Z(\bar{R})$. Since $[\overline{U},\overline{U}] \neq \bar{0}$ from [4, Lemma 4] either $\overline{d(U)} = \bar{0}$ or $\bar{a} = \bar{0}$, that is $d(U) \subseteq P$ or $a \in P$. If $d(U) \subseteq P$, then $d[r,u] \in P$ for all $r \in R$ and $u \in U$. Replace r by rv, where $v \in U$, to get $d(R)[U,U] \subseteq P$. Thus $d(R)R[U,U] \subseteq P$ which yields $d(R) \subseteq P$ because $[U,U] \not\subset P$. In conclusion $ad(R) \subseteq P$. Consequently, $ad(R)M \subseteq P$ and $Mad(R) \subseteq P$. We now know that $2ad(R)M \subseteq P$ and $2Mad(R) \subseteq P$ for all prime ideals P of R, hence $2ad(R)M = 2Mad(R) = \{0\}$. By 2-torsion-freeness we conclude that $ad(R)M = Mad(R) = \{0\}$. If we set $J = ann_R(ann_R(M))$, then obviously $ad(R)J = 0$. Since R is semiprime, then $d(J) \subseteq J$ so that $ad(J) \subseteq J \bigcap ann_R(J)$. Once again using the semiprimeness of R, we conclude that $J \bigcap ann_R(J) = 0$ so that $ad(J) = 0$. Since $M \subseteq J$, this leads us to $ad(M) = 0$. □

Lemma 3. *Let R be a 2-torsion free semiprime ring. If $z \in U$ is such that $z[U,U] = 0$, then $[z,U] = 0$.*

Proof. If $[U,U] = 0$, then $U \subseteq Z(R)$ by Lemma 1 and therefore $[z,U] = 0$. Now suppose that $[U,U] \neq 0$; from $z[U,U] = 0$ we get $zd_u(v) = 0$ for all $u,v \in U$. Using Lemma 2, we find that $zd_u(x) = 0$ for all $u \in U,\ x \in M = R[U,U]R$. But $zd_u(x) = 0$ assures that $zd_x(u) = 0$ for all $u \in U$, $x \in M$ and once again using Lemma 2, we get $zd_x(M) = 0$, for all $x \in M$. Hence $zd_x(y) = 0$ for all $x,y \in M$ and thus

$$z[x,y] = 0 \text{ for all } x,y \in M.$$

Replace y by yz to get $zy[x,z] = 0$, so that $zM[x,z] = 0$. In view of $zxM[x,z] = 0$, we then obtain $[x,z]M[x,z] = 0$. Since an ideal of a semiprime ring is semiprime, $[x,z] = 0$ for all $x \in M$. As $R[U,U] \subseteq M$, then $[z, r[u,v]] = 0$ for all $r \in R, u,v \in U$. Using $[u,v] \in M$, it then follows that $[z,r][u,v] = 0$. Replace r by rs in the least equality, we find that $[z,r]s[u,v] = 0$ so that $[z,r]R[u,v] = 0$, for all $u,v \in U,\ r \in R$. In particular $[z,v]R[z,v] = 0$, proving $[z,v] = 0$ for all $v \in U$ and thus $[z,U] = 0$. □

Now we are ready for our first theorem.

Theorem 1. *Let R be a 2-torsion free semiprime ring and U a nonzero square closed Lie ideal of R. Suppose that f is an endomorphism of R such that $f(U) = U$. Then f is strong commutativity preserving on U if and only if f is centralizing on U.*

Proof. From $[x,2xy] = [f(x), f(2xy)]$ for all $x,y \in U$, it follows that $(x - f(x))[x,y] = 0$ for all $x,y \in U$. Replacing y by $2uy$ where $u,y \in U$, we get

$$(x - f(x))U[x,y] = 0 \text{ for all } x,u \in U. \tag{2}$$

As $2[U,U]R \subseteq U$ (because $2[u,v]r = 2[u,vr] - 2v[u,r]$), then (2) implies that

$$(x - f(x))[U,U]R[x,y] = 0 \text{ for all } x,y \in U. \tag{3}$$

Let P be an arbitrary prime ideal of R. It follows from (3) that for each $x \in U$, either $(x - f(x))[U,U] \subseteq P$ or $[x,U] \subseteq P$. The two sets of elements of U for which these conditions hold are additive subgroups of U whose union is U, hence one must be equal to U. Therefore $(x - f(x))[U,U] \subseteq P$ for all $x \in U$ and all prime ideals P, i.e., $(x - f(x))[U,U] = \{0\}$ for all $x \in U$. Since $f(U) \subseteq U$, then $u - f(u) \in U$ for all $u \in U$ and Lemma 3 yields

$$[u - f(u), v] = 0 \text{ for all } u, v \in U.$$

Consequently, $[f(u), u] = 0$ for all $u \in U$ so that f is commuting on U. Accordingly, f is centralizing on U.

Conversely, suppose that $[f(x), x] \in Z(R)$ for all $x \in U$. By linearization $[x, f(y)] + [y, f(x)] \in Z(R)$ for all x, y in U. Using $[x, f(x^2)] + [x^2, f(x)] \in Z(R)$ together with 2-torsion-freeness, we find that $(x + f(x))[x, f(x)] \in Z(R)$, for all $x \in U$. Hence $[(x + f(x))[x, f(x)], x] = 0$ and therefore $[x, f(x)]^2 = 0$. Since $[x, f(x)]$ in $Z(R)$, this yields $[x, f(x)]R[x, f(x)] = 0$ and the semiprimeness of R forces

$$[x, f(x)] = 0 \text{ for all } x \in U.$$

Thus f is commuting on U and therefore $[f(x), y] = [x, f(y)]$ for all $x, y \in U$. As R is 2-torsion free, then $[f(x), xy] = [x, f(xy)]$ and thereby $(f(x) - x)[f(x), y] = 0$ for all $x, y \in U$. Replacing y by $2uy$ where $u \in U$, we get $(f(x) - x)u[f(x), y] = 0$, so that $(f(x) - x)U[x, f(y)] = 0$. Since $f(U) = U$, then $(f(x) - x)U[x, y] = 0$ for all $x, y \in U$. From $2[U,U]R \subseteq U$, it then follows that

$$(f(x) - x)[U,U]R[x, y] = 0 \text{ for all } x, y \in U.$$

Reasoning as in the first part of the proof, we find that $[f(z) - z, u] = 0$ for all $z, u \in U$, and therefore $[f(z), u] = [z, u]$, for all $z, u \in U$. Consequently, for $y, z \in U$, this leads us to $[f(z), f(y)] = [z, f(y)] = [z, y]$, proving that f is strong commutativity preserving on U. □

Remark. From the proof of Theorem 1, one can easily see that the condition $f(U) \subseteq U$ is sufficient to prove that f is strong commutativity preserving implies that f is commuting on U and therefore centralizing on U.

We easily derive the Proposition 2.1 of [4], for 2-torsion free semiprime rings, as a corollary to Theorem 1.

Corollary 1. *Let f be an epimorphism of a 2-torsion free semiprime ring R. Then f is strong commutativity preserving if and only if f is centralizing.*

In [3] it is proved that if R is a 2-torsion free prime ring and T an automorphism of R which is centralizing on a Lie ideal U of R and nontrivial on U, then U is contained in the center of R. Accordingly, in the special case when $U = R$, Theorem 2 gives a commutativity criterion as follows.

Corollary 2. *Let f be a nontrivial automorphism of a 2-torsion free prime ring R. If f is strong commutativity preserving, then R is commutative.*

To end this paper, the following theorem gives a condition under which an antihomomorphism becomes strong commutativity preserving.

Theorem 2. *Let R be a 2-torsion free semiprime ring and U a square closed Lie ideal of R. If f is an antihomomorphism of R such that $f(U) = U$, then f is centralizing on U if and only if f is strong commutativity preserving on U.*

Proof. Suppose $[U, U] \neq 0$ and then $M = R[U, U]R$ is a nonzero ideal of R. If f is centralizing on U, then reasoning as in the proof of Theorem 1 we find that f is commuting on U, so that $[f(x), y] = [x, f(y)]$ for all $x, y \in U$. Since R is 2-torsion free, using $[f(x), 2xy] = [x, f(2xy)]$ together with $f(U) = U$ we get

$$x[x, y] = [x, y]f(x) \text{ for all } x, y \in U. \tag{4}$$

Replace y by $2uy$ in (4), where $u \in U$, and once again using 2-torsion-freeness, we get $[x, u][x, y + f(y)] = 0$. Write $2uv$ instead of u in this equality, with $v \in U$, to find that $[x, u]v[x, y + f(y)] = 0$. Hence

$$[x, u]U[x, y + f(y)] = 0 \text{ for all } x, u, y \in U. \tag{5}$$

Since $f(U) \subseteq U$, replacing u by $y + f(y)$ in (5), we conclude that

$$[x, y + f(y)]U[x, y + f(y)] = 0 \text{ for all } x, y \in U. \tag{6}$$

If we set $T(U) = \{x \in R/[x, R] \subseteq U\}$, then $[T(U), R] \subseteq U \subseteq T(U)$ and from ([2], Lemma 1.4, p. 5) it follows that $T(U)$ is a subring of R. Moreover, $R[T(U), T(U)]R \subseteq T(U)$. Indeed, let $x, y \in T(U)$ and $r \in R$. From $[x, yr] = [x, y]r + y[x, r] \in T(U)$ and $y[x, r] \in T(U)$ it follows that $[x, y]r \in T(U)$. Since $[T(U), R] \subseteq T(U)$, then

$$[[x, y]r, s] = [x, y]rs - s[x, y]r \in T(U) \text{ for all } r, s \in R;$$

and therefore $s[x, y]r \in T(U)$ so that $R[T(U), T(U)]R \subseteq T(U)$. In particular $R[U, U]R \subseteq T(U)$, which proves that $[M, R] \subseteq U$, where $M = R[U, U]R$.

In view of (6), if we set $[x, y + f(y)] = a$ then $aUa = 0$. Let $u \in U$, $m \in M$ and $r \in R$; from $[mau, r] \in [M, R] \subseteq U$ it follows that

$$0 = a[mau, r]a = a[ma, r]ua + ama[u, r]a = a[ma, r]ua = amarua,$$

so that $amaRua = 0$. Using $2am \in 2[U, U]R \subseteq U$, Lemma 1.4, we get $amaRama = 0$, hence $aMa = 0$. Since $a \in M$, we obviously get $a = 0$, which implies that $[f(x), y] = [y, x]$, for all $x, y \in U$. Accordingly,

$$[f(x), f(y)] = [f(y), x] = [x, y] \text{ for all } x, y \in U,$$

proving that f is strong commutativity preserving on U.

Conversely, if f is strong commutativity preserving on U, then

$$[f(x), f(y)] = [x, y], \text{ for all } x, y \in U. \tag{7}$$

Replace y by $2xy$ in (7) we obtain

$$x[x, y] = [x, y]f(x). \tag{8}$$

Write $2uy$ instead of y in (8), where $u \in U$, to find that

$$xu[x, y] + x[x, u]y = u[x, y]f(x) + [x, u]yf(x).$$

Since $x[x, u]y = [x, u]f(x)y$ and $[x, y]f(x) = x[x, y]$, by (8), then

$$xu[x, y] + [x, u]f(x)y = ux[x, y] + [x, u]yf(x)$$

and therefore

$$[x, u][x + f(x), y] = 0 \text{ for all } x, y, u \in U. \tag{9}$$

Replacing y by x in (9), we obtain

$$[x, u][x, f(x)] = 0 \text{ for all } x, u \in U. \tag{10}$$

As $f(U) \subseteq U$, write $2f(x)u$ instead of u in (10) to get $[x, f(x)]u[x, f(x)] = 0$ and thus

$$[x, f(x)]U[x, f(x)] = 0.$$

If we set $a = [x, f(x)]$, then $aUa = 0$ and $a \in M = R[U, U]R$. Reasoning as in the first part of our proof, we conclude that $a = 0$ so that $[x, f(x)] = 0$. Accordingly, f is commuting on U and therefore f is centralizing on U. □

Remark. In the particular case when $U = R$, the implication that f is strong commutativity preserving implying that f is centralizing is still valid without conditions on characteristic of R.

In [4], Proposition 2.4 M. S. Samman proved that if R is a 2-torsion free semiprime ring, then a centralizing antihomomorphism of R onto itself must be strong commutativity preserving. Applying Theorem 2, we obtain a more general result as follows:

Corollary 3. *Let R be a 2-torsion free semiprime ring. If f is an antihomomorphism of R onto itself, then f is centralizing if and only if f is strong commutativity preserving.*

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