Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 49 (2008), No. 1, 277-283.

Associated Prime Ideals of Skew Polynomial Rings

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Abstract. In this paper, it has been proved that for a Noetherian ring R and an automorphism σ of R, an associated prime ideal of $R[x,\sigma]$ or $R[x,x^{-1},\sigma]$ is the extension of its contraction to R and this contraction is the intersection of the orbit under σ of some associated prime ideal of R. The same statement is true for minimal prime ideals also.
It has also been proved that for a Noetherian $\mathbb{Q}$-algebra ($\mathbb{Q}$ the field of rational numbers) and a derivation δ of R, an associated prime ideal of $R[x,\delta]$ is the extension of its contraction to R and this contraction is an associated prime ideal of R.

MSC 2000: 16-XX (primary), 16S36, 16P40, 16P50, 16U20 (secondary)
Keywords: automorphism, associated prime, minimal prime, derivation, skew polynomial ring

1. Introduction

All rings are associative with identity and all modules unitary. Let R be a ring, σ be an automorphism of R and δ be a right σ-derivation of R ($\delta : R \to R$ is an additive map such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$). For example let σ be an automorphism of a ring R and $\delta : R \to R$ any map.
Let $\phi : R \to M_2(R)$ be defined by

*Sincere thanks to the referee for his suggestions to give this manuscript a real shape.

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a right σ-derivation of R.

Let R be a ring. Let σ be an automorphism of R and δ be a right σ-derivation of R. Then the Ore extension $O(R) = R[x, \sigma, \delta] = \{f = \sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$ subject to the relation ax $= x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x, \sigma, \delta]$ by $O(R)$. In case σ is the identity map (i.e. δ is just a derivation), we denote the differential operator ring $R[x, \delta]$ by $D(R)$ and in case δ is the zero map, we denote $R[x, \sigma]$ by $S(R)$.

Skew Laurent polynomial ring $R[x, x^{-1}, \sigma] = \{\sum_{i=-m}^{n} x^i a_i,\ a_i \in R\}$ subject to the relation $ax = x\sigma(a)$ for all $a \in R$. We denote $R[x, x^{-1}, \sigma]$ by L(R). For more details and related results on skew polynomial rings, the reader is referred to chapter (1) of Goodearl and Warfield [6].

Let I be an ideal of a ring R such that $\sigma^m(I) = I$ for some integer $m \geq 1$, we denote $\cap_{i=1}^{m} \sigma^i(I)$ by I^0. The field of rational numbers, the ring of integers and the set of positive integers are denoted by $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ respectively unless otherwise stated. Let R be a ring. Then the set of prime ideals of R is denoted by Spec(R), the set of associated prime ideals of R (viewed as a right R-module over itself) is denoted by $Ass(R_R)$, and the set of minimal prime ideals of R is denoted by $Min.Spec(R)$. $Assas(M)$ denotes the assassinator of a uniform R-module M, and for any subset J of an R-module M, annihilator of J is denoted by $Ann(J)$. $C(0)$ denotes the set of regular elements of a ring R, and $C(I)$ denotes the set of elements regular modulo I, where I is an ideal of R. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J.

Carl Faith proved in [3] that if R is a commutative ring, then the associated prime ideals of the usual polynomial ring $R[x]$ (viewed as a module over itself) are precisely the ideals of the form P[x], where P is an associated prime ideal of R. Goodearl and Warfield proved in (2ZA) of [6] that if R is a commutative Noetherian ring, and if σ is an automorphism of R, then an ideal I of R is of the form $P \cap R$ for some prime ideal P of $R[x, x^{-1}, \sigma]$ if and only if there is a prime ideal S of R and a positive integer m with $\sigma^m(S) = S$, such that $I = \cap_{i=1}^{m} \sigma^i(S)$. Gabriel proved in [4] that if R is a right Noetherian ring, which is also an algebra over $\mathbb{Q}$ and P is a prime ideal of $R[x, \delta]$, then $P \cap R$ is a prime ideal of R. In Theorem (2.2) of [1], S. Annin has proved the following:

Theorem (2.2) of Annin [1]: *Let R be a ring and M be a right R-module. Let σ be an endomorphism of R and $S = R[x, \sigma]$. Let M_R be σ-compatible. Then $Ass(M[x]_S) = \{P[x]$ such that $P \in Ass(M_R)\}$.*

H. Nordstrom has proved the following result in [10]:

Theorem (1.2) of [10]: *Let R be a ring with identity and σ be a surjective endomorphism of R. Then for any right R-module M, $Ass(M[x, \sigma]) = \{I[x, \sigma],$ $I \in \sigma - Ass(M)\}$.*

In Corollary (1.5) of [10] Nordstrom has proved that if R is a Noetherian ring and σ is an automorphism of R, then $Ass(M[x,\sigma]_S) = \{P_\sigma[x,\sigma],\ P \in Ass(M)\}$, where $P_\sigma = \cap_{i\in N}\sigma^{-i}(P)$ and $S = R[x,\sigma]$.

Concerning associated prime ideals of full Ore extensions $R[x,\sigma,\delta]$, S. Annin generalizes the above in the following way:

Definition (2.1) of Annin [2]: *Let R be a ring and M_R be a right R-module. Let σ be an endomorphism of R and δ be a σ-derivation of R. M_R is said to be σ-compatible if for each $m \in M$, $r \in R$, we have $mr = 0 \Leftrightarrow m\sigma(r) = 0$. Moreover M_R is said to be δ-compatible if for each $m \in M$, $r \in R$, we have $mr = 0 \Rightarrow m\delta(r) = 0$. If M_R is both σ-compatible and δ-compatible, M_R is said to be $(\sigma - \delta)$-compatible.*

Theorem (2.3) of Annin [2]: *Let R be a ring. Let σ be an endomorphism of R and δ be a σ-derivation of R and M_R be a right R-module. If M_R is $(\sigma - \delta)$-compatible, then $Ass(M[x]_S) = \{P[x] \mid P \in Ass(M_R)\}$.*

In [8] Leroy and Matczuk have investigated the relationship between the associated prime ideals of an R-module M_R and that of the induced S-module M_S, where $S = R[x,\sigma,\delta]$ (σ is an automorphism and δ is a σ-derivation of a ring R). They have proved the following:

Theorem (5.7) of [8]: *Suppose M_R contains enough prime submodules and let for $Q \in Ass(M_S)$. If for every $P \in Ass(M_R)$, $\sigma(P) = P$, then $Q = PS$ for some $P \in Ass(M_R)$.*

Motivated by these developments, I investigated the nature of associated prime ideals of certain skew polynomial rings over a Noetherian ring R and their relation with those of the coefficient ring R.

In this paper a structure of associated prime ideals and minimal prime ideals of the skew polynomial rings $S(R) = R[x,\sigma]$ and $L(R) = R[x,x^{-1},\sigma]$ is given, where σ is an automorphism of a right Noetherian ring R. This structure is also given for $R[x,\delta]$, where δ is a derivation of a right Noetherian $\mathbb{Q}$-algebra R.

Let now R be a Noetherian ring and σ be an automorphism of R. Let $K(R)$ be any of $S(R)$ or $L(R)$. Then we prove the following:

$P \in Ass(K(R)_{K(R)})$ if and only if $P = K(P_1)$, where $P_1 = \cap_{i=1}^m \sigma^i(P_2)$; for some $P_2 \in Ass(R_R)$ with $\sigma^m(P_2) = P_2$. This is proved in Theorem 2.4.

We also prove that if R is a Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $P \in Ass(D(R)_{D(R)})$ if and only if $P = D(P_1)$ for some $P_1 \in Ass(R_R)$. This is proved in Theorem 3.7. We prove similar results for minimal prime ideals also.

These results may be useful to examine the primary decomposition and existence of artinian quotient rings of these rings. These properties form the fundamental edifice on which a ring is studied. For example, for a commutative Noetherian ring R, we know that the ideal (0) has a reduced primary decomposition say $(0) = \cap_{j=1}^n Ij$. For this see Theorem (4) of Zariski and Samuel [13]. Let $\sqrt{Ij} = Pj$, where Pj is a prime ideal belonging to Ij. Now by Theorem (23) of Zariski and Samuel [13] there exists a positive integer k such that $Pj^{(k)} \subseteq Ij$, $1 \leq j \leq n$. Therefore $\cap_{j=1}^{n} Pj^{(k)} = 0$. Now by the first uniqueness Theorem $Pj \in Ass(R)$, $1 \leq j \leq n$. Now it can be seen that $R/Pj^{(k)}$ has an artinian quotient ring and that $\delta(Pj^{(k)}) \subseteq Pj^{(k)}$ and ultimately we get a primary decomposition for $R[x, \delta]$. The cases of $S(R)$ and $L(R)$ can be treated similarly. For the definition of symbolic power $P^{(k)}$ of a semiprime ideal P of a commutative Noetherian ring and the related results, the reader is referred to page number 236 of Zariski and Samuel [13].

The next step is to investigate these results for the Ore extension $R[x, \sigma, \delta]$, where σ is an automorphism of a right Noetherian ring R and δ is a σ-derivation of R. For the time being we are unable to generalize these results for $R[x, \sigma, \delta]$.

2. Associated primes of $R[x, \sigma]$ and $R[x, x^{-1}, \sigma]$

We begin with the following:

Proposition 2.1. *Let R be a right Noetherian ring and σ be an automorphism of R. Then there exists an integer $m \geq 1$ such that $\sigma^m(P) = P$ for all $P \in Ass(R_R)$.*

Proof. We know that $Ass(R_R)$ is finite and $\sigma(P) \in Ass(R_R)$ for any $P \in Ass(R_R)$, therefore there exists an integer $m \geq 1$ such that $\sigma^m(P) = P$ for all $P \in Ass(R_R)$. □

Proposition 2.2. *Let R be a semiprime right Goldie ring. Let σ be an automorphism of R and δ be an σ-derivation of R. Let $O(R) = R[x, \sigma, \delta]$. If $f \in O(R)$ is a regular element, then there exists $g \in O(R)$ such that gf has leading coefficient regular in R.*

Proof. Let $S = \{a_m \in R$ such that $x^m a_m + \cdots + a_0 \in O(R)f$, some $m\} \cup \{0\}$. Then since $O(R)$ is semiprime and Noetherian, $O(R)f$ is an essential left ideal of $O(R)$, and therefore S is an essential left ideal of R. So S contains a left regular element, and since R is semiprime, Proposition (3.2.13) of Rowen [11] implies that S contains a regular element. Therefore there exists $g \in O(R)$ such that gf has leading coefficient regular in R. □

Proposition 2.3. *Let R be right Noetherian ring and σ be an automorphism of R. Let $K(R)$ be any of $S(R)$ or $L(R)$. Let $P \in Ass(K(R)_{K(R)})$. Then there exists $Q \in Ass(R_R)$ with $\sigma^m(Q) = Q$ for some integer $m \geq 1$ such that $P \cap R = Q^0 = \cap_{i=1}^{m} \sigma^i(Q)$. Also $K(P \cap R) = P$.*

Proof. Choose a right ideal I of $K(R)$ with $P = Ann(I) = Assas(I)$, and choose $f \in I$ to be nonzero of minimal degree (with leading coefficient a_n). Without loss of generality, $Q = Ann(a_n R) = Assas(a_n R)$. This implies that $fQ = 0$. Therefore $fK(R)Q^0 \subseteq fQK(R) = 0$. So $Q^0 \subseteq (P \cap R)$. But it is clear that $(P \cap R) \subseteq Q$, and $(P \cap R)$ is σ-invariant. Thus $P \cap R \subseteq Q^0$. Now by Jategaonkar [7], $K(P \cap R)$ is a prime ideal of $K(R)$. Suppose $K(P \cap R) \neq P$. Then by Proposition 2.2 there exists $g \in C(K((P \cap R))$, and $h_1 \in K(R)$ such that $h_1 g$ has leading co-efficient regular modulo $P \cap R$. Let $h_1 g = \sum_{i=0}^{k} x^i d_i$. Now $P \subseteq Ann(frR)$, $r \in R$ and since $h_1 g \in P$, we have $frRh_1 g = 0$. Therefore $x^{n+k}\sigma^k(a_n)\sigma^k(r)Rd_k + \cdots + a_0 rRd_0 = 0$. So $\sigma^k(a_n)\sigma^k(r)Rd_k = 0$; i.e. $\sigma^{-k}(d_k) \in Ann(a_n rR) = Q$, but $d_k \in C(P \cap R)$, therefore $d_k \in C(\sigma^j(Q))$ for all $j \geq 1$ which is a contradiction. Hence $K(P \cap R) = P$. □

Theorem 2.4. *Let R be a Noetherian ring and σ be an automorphism of R. Let $K(R)$ be any of $S(R)$ or $L(R)$. Then:*

1. *$P \in Ass(K(R)_{K(R)})$ if and only if there exists $Q \in Ass(R_R)$ such that $K(P \cap R) = P$ and $(P \cap R) = Q^0$.*
2. *$P \in Min.Spec(K(R))$ if and only if there exists $Q \in Min.Spec(R)$ such that $K(P \cap R) = P$ and $P \cap R = Q^0$.*

Proof. 1. Let $Q = Ann(cR) = Assas(cR)$, $c \in R$. Now $\sigma^m(Q) = Q$ for some integer $m \geq 1$ by Proposition 2.1. Now using Proposition 2.2 as used in Proposition 2.3, we have $K(Q^0) = Ann(chK(R))$ for all $h \in K(R)$. Therefore $K(Q^0) = Ann(cK(R)) = Assas(cK(R))$.

Converse is true by Proposition 2.3.

2. Let $Q \in Min.Spec(R)$. Then $\sigma^m(Q) = Q$ for some integer $m \geq 1$. Let $Q_1 = Q^0$. Then by Proposition (10.6.12) of McConnell and Robson [9] and Theorem (7.27) of Goodearl and Warfield [6], $Q_2 = K(Q_1) \in Min.Spec(K(R))$.

Conversely suppose that $P \in Min.Spec(K(R))$. Then $P \cap R = Q^0$ for some $Q \in Spec(R)$ and Q contains a minimal prime Q_1. Then $P \supseteq K(R)Q_1^0$, which is a prime ideal of $K(R)$. Hence $P = K(R)Q_1^0$. □

3. Associated primes of $R[x, \delta]$

We begin with the following:

Definition 3.1. *Let R be ring and δ be a derivation of R. Then $D(R) = R[x, \delta]$ is the usual differential operator ring. Let I be an ideal of R such that $\delta(I) \subseteq I$. Then as usual $D(I) = I[x, \delta]$.*

We recall that given a ring R, the formal power series ring $R[[t]] = \{\sum_{i=0}^{\infty} x^i a_i$, $a_i \in R\}$. We denote $R[[t]]$ by T.

Proposition 3.2. *Let R be a Noetherian $\mathbb{Q}$-algebra and δ be a derivation of R. Then $e^{t\delta}$ is an automorphism of $T = R[[t]]$.*

Proof. The proof is on the same lines as in Seidenberg [12], and a sketch in the non-commutative case is provided in Blair and Small [3]. □

Lemma 3.3. *Let R be a Noetherian $\mathbb{Q}$-algebra and δ be a derivation of R. Let T be as usual. Then an ideal I of R is δ-invariant if and only if IT is $e^{t\delta}$-invariant.*

The proof of this lemma is obvious.

Proposition 3.4. *Let R be a ring and T as usual. Then:*

1. $Q \in Ass(R_R)$ *implies that* $QT \in Ass(T_T)$.
2. $P \in Ass(T_T)$ *implies that* $(P \cap R) \in Ass(R_R)$ *and* $P = (P \cap R)T$.

The proof of this proposition is obvious.

Proposition 3.5. *Let R be a ring and T be as usual. Then:*

1. $P \in Min.Spec(T)$ *implies that* $(P \cap R) \in Min.Spec(R)$ *and* $P = (P \cap R)T$.
2. $Q \in Min.Spec(R)$ *implies that* $QT \in Min.Spec(T)$.

Proof. 1. Let $P \in Min.Spec(T)$. Then $(P \cap R) \in Spec(R)$. Let $(P \cap R) \notin Min.Spec(R)$. Suppose $P_1 \subset (P \cap R)$ is a minimal prime ideal of R. Then $P_1T \subset (P \cap R)T \subseteq P$.

2. Let $Q \in Min.Spec(R)$. Then $QT \in Spec(T)$. Let $QT \notin Min.Spec(T)$. Suppose $Q_1 \subset QT$ is a minimal prime ideal of T. Then $(Q_1 \cap R) \subset QT \cap R = Q$. □

Theorem 3.6. *Let R be a Noetherian $\mathbb{Q}$-algebra and δ be a derivation of R. Let $P \in Ass(R_R) \cup Min.Spec(R)$. Then $\delta(P) \subseteq P$. (For minimal prime ideals case see Lemma* (2.20) *of Goodearl and Warfield* [6]*).*

Proof. Let $T = R[[t]]$. Now by Proposition 3.2 $e^{t\delta}$ is an automorphism of T. Let $P \in Ass(R_R) \cup Min.Spec(R)$. Then by Proposition 3.4 and Proposition 3.5 $PT \in Ass(T_T) \cup Min.Spec(T)$. Therefore there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(PT) = PT$; i.e. $e^{nt\delta}(PT) = PT$. But R is a $\mathbb{Q}$-algebra, therefore $e^{t\delta}(PT) = PT$, and now Lemma 3.3 implies that $\delta(P) \subseteq P$. □

Theorem 3.7. *Let R be a Noetherian $\mathbb{Q}$-algebra and δ be a derivation of R. Then:*

1. $P \in Ass(D(R)_{D(R)})$ *if and only if* $P = D(P \cap R)$ *and* $P \cap R \in Ass(R_R)$.
2. $P \in Min.Spec(D(R))$ *if and only if* $P = D(P \cap R)$ *and* $P \cap R \in Min.Spec(R)$.

Proof. 1. Let $P_1 \in Ass(R_R)$. Then $\delta(P_1) \subseteq P_1$ by Theorem 3.6. Let $P_1 = Ann(cR) = Assas(cR)$, $c \in R$. Now by Proposition (14.2.5) (ii) of McConnell and Robson [9] $D(P_1) \in Spec(D(R))$ and for any $h \in D(R)$, $D(P_1) = Assas(ch.D(R))$.

Converse can be proved on the same lines as in Proposition 2.3.

2. Let $P_1 \in Min.Spec(R)$. Then $\delta(P_1) \subseteq P_1$ by Theorem 3.6. Therefore by Proposition (14.2.5) (ii) of McConnell and Robson [9], $D(P_1) \in Spec(D(R))$. Suppose $P_2 \subset D(P_1)$ is a minimal prime ideal of $D(R)$. Then $P_2 = D(P_2 \cap R) \subset D(P_1) \in Min.Spec(D(R))$. So $P_2 \cap R \subset P_1$ which is not possible.

Conversely suppose that $P \in Min.Spec(D(R))$. Then $P \cap R \in Spec(R)$ by Lemma (2.21) of Goodearl and Warfield [6]. Let $P_1 \subset P \cap R$ be a minimal prime ideal of R. Then $D(P_1) \subset D(P \cap R)$ and as in first paragraph $D(P_1) \in Spec(D(R))$, which is a contradiction. Hence $P \cap R \in Min.spec(R)$. □

Remark 3.8. The above results namely Theorem 2.4 and Theorem 3.7 are clearly true if we consider the left version (i.e. when a ring A is viewed as a left A-module and $Ass(A_A)$ replaced accordingly by $Ass({}_AA)$.

The results, namely Theorem 2.4 and Theorem 3.7 are yet to be investigated for the Ore extension $O(R) = R[x, \sigma, \delta]$, where σ is an automorphism of R and δ is a σ-derivation of R. One of the main difficulties is that if $P \in Min.Spec(R) \cup Ass(R_R)$, then P need not be δ-invariant. Also let $a \in P$. We have $ax = x\sigma(a) + \delta(a)$ and the coefficients need not lie in P.

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Received November 26, 2006