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A Notion of Functional Completeness for First Order Structures II: Quasiprimality

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Abstract. Quasi-varieties of first-order structures were studied by N. Weaver [7] to generalize varieties of algebras; he also established some Malcev like conditions for these classes of structures. Following this line we extend some results of functional completeness of algebras to first-order structures. Specifically, we formulate and characterize a notion of quasiprimality for first-order structures.

Keywords: quasiprimality, first-order structure, $\star$-congruences

1. Introduction

Functional completeness of algebras has been studied by many authors ([1], [2], [3]). In [7], N. Weaver extends some of the results of this study to classes of first-order structures.

This paper is an attempt to extend a result of A. F. Pixley [3] to first-order structures. Let us recall some basic definitions.

Definition 1.1. *Let E be a nonempty set, and $g : E^3 \to E$ be a ternary function.*

(i) *g is called a Malcev function if $g(a, b, b) = a = g(b, b, a)$ for all $a, b \in E$.*

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(ii) *g is called a majority function if $g(a,a,b) = g(a,b,a) = g(b,a,a) = a$ for all $a, b \in E$.*

(iii) *g is called a Pixley function if $g(a,a,b) = g(b,a,b) = g(b,a,a) = b$ for all $a, b \in E$.*

(iv) *The discriminator function on E is the ternary function $\mathbf{d}$ defined by*

$$\mathbf{d}\,(a,b,c) = \begin{cases} a & \text{if } a \neq b; \\ c & \text{if } a = b. \end{cases}$$

Let $\mathcal{A} = (A; F^{\mathcal{A}})$ be an algebra of type F. $\mathcal{A}$ is called *nontrivial* if A contains at least two elements; we will say that $\mathcal{A}$ is *minimal* if it has no proper subalgebra. Consider the type $F_A := F \cup \{c_a; a \in A\}$ obtained from the type F by adding a constant symbol c_a for each element $a \in A$. Then the terms of type F_A are called the *polynomials* of $\mathcal{A}$.

A function $f : A^n \to A$, $n \geq 1$, is called:

- a *term function* of $\mathcal{A}$ if there is some n-ary term t of type F such that $f = t^{\mathcal{A}}$,
- a *polynomial function* of $\mathcal{A}$ if there is some n-ary term p of type F_A such that $f = p^{\mathcal{A}}$.

A ternary term $t(x,y,z)$ in the language of $\mathcal{A}$ (i.e., of type F) is called a *Malcev term* (resp. a *majority term*, a *Pixley term*, a *discriminator term*) for $\mathcal{A}$ if the induced function $t^{\mathcal{A}}$ is a Malcev function (resp. a majority function, a Pixley function, a discriminator function) on A.

Definition 1.2. *Let $\mathcal{A} = (A; F^{\mathcal{A}})$ be a finite nontrivial algebra.*

(i) *$\mathcal{A}$ is called functionally complete if every finitary function $f : A^n \to A$, $n \geq 1$, is a polynomial function of $\mathcal{A}$.*

(ii) *$\mathcal{A}$ is called quasiprimal if every finitary function $f : A^n \to A$, $n \geq 1$, which preserves the subuniverses of $\mathcal{A}^2$ consisting of (the graphs of) isomorphisms between subalgebras of $\mathcal{A}$ is a term function.*

A quasiprimal algebra $\mathcal{A}$ is characterized by any of the following two facts (cf. [1], p. 175):

Fact 1. $\mathcal{A}$ has a discriminator term.

Fact 2. $\mathcal{A}$ has a Pixley term, and every subalgebra of $\mathcal{A}$ is simple.

When $\mathcal{A}$ is quasiprimal, it can be shown that every subuniverse of $\mathcal{A}^2$ is either the product of two subuniverses of $\mathcal{A}$ or (the graph of) an isomorphism between two subalgebras of $\mathcal{A}$.

An n-ary function $f : A^n \to A$ which preserves isomorphisms between subalgebras of $\mathcal{A}$ must preserve the identity map $id_B = \triangle_B$ for each subalgebra $\mathcal{B}$ of $\mathcal{A}$; so h must preserve the subuniverses of $\mathcal{A}$, and consequently any product $B \times D$ of two subuniverses.

Now, a lemma of Baker and Pixley (cf. [1], page 172) states that if a finite algebra $\mathcal{A}$ has a majority term, a function $h : A^n \to A$, $n \geq 1$, is a term function iff h preserves the subuniverses of $\mathcal{A}^2$. This will be our main tool, noting that $A^2 = \nabla_A$ is the largest congruence of $\mathcal{A}$, and in the case of a quasiprimal algebra, h preserves the subuniverses of $\mathcal{A}^2$ iff h preserves the (graphs of) isomorphisms between subalgebras of $\mathcal{A}$.

The work of N. Weaver [7] is based on the notion of $\star$-*congruence* on first-order structures.

Definition 1.3. ([7]) *Let $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$ and $\mathfrak{B} = (B; F^{\mathfrak{B}}; R^{\mathfrak{B}})$ be first-order structures of the same type $(F; R)$.*

(i) *A morphism $\lambda : \mathfrak{A} \to \mathfrak{B}$ is called a $\star$-morphism if for any m-ary $r \in R$, and $a_1, \ldots, a_m \in A$, $\langle a_1, \ldots, a_m\rangle \in r^{\mathfrak{A}}$ iff $\langle \lambda(a_1), \ldots, \lambda(a_m)\rangle \in r^{\mathfrak{B}}$. In this case, the substructure $\lambda(\mathfrak{A})$ of $\mathfrak{B}$ is called a $\star$-image of $\mathfrak{A}$.*

(ii) *A congruence θ of $\mathfrak{A}$ is called a $\star$-congruence if it is compatible with the relations of $\mathfrak{A}$; that is, for any m-ary $r \in R$ and $\langle u_i, v_i\rangle \in \theta$ for $1 \le i \le m$, $\langle u_1, \ldots, u_m\rangle \in r^{\mathfrak{A}}$ iff $\langle v_1, \ldots, v_m\rangle \in r^{\mathfrak{A}}$.*

(iii) *A class $\mathcal{K}$ of structures of the same type is called a $\star$-variety if it is closed under products, substructures and $\star$-images.*

So, $\star$-congruences are exactly the kernels of $\star$-morphisms.

The set $Con_\star(\mathfrak{A})$ of $\star$-congruences of $\mathfrak{A}$ is a sublattice of $Con(\mathfrak{A})$; in fact $Con_\star(\mathfrak{A})$ is a complete lattice with smallest element $id_A = \triangle_A$, and largest element denoted by $1_{\mathfrak{A}}$; in general $1_{\mathfrak{A}} \neq A^2 = \nabla_A$.

Using $\star$-congruences, N. Weaver established some Malcev like conditions, and a structure theorem for $\star$-varieties. In [5] we gave a notion of functionally complete first-order structure relative to $\star$-congruences. Our aim here is to follow this line and formulate a notion of quasiprimality for a first order structure $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$ where ∇_A is replaced by $1_{\mathfrak{A}}$.

Notation. Let $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$ be a first-order structure.

(i) Let θ be an equivalence relation on A and a be an element of A; then $[a]_\theta$ is the θ equivalence class of a. When $\theta = 1_{\mathfrak{A}}$, we simply denote the equivalence class of a by $\overline{a}$.

(ii) For a subset X of A, $Sg(X)$ denotes the subuniverse of $\mathfrak{A}$ generated by X; if $X = \{x_1, \ldots, x_n\}$, we simply denote $Sg(X)$ by $Sg(x_1, \ldots, x_n)$.

(iii) Given two elements $a := \langle a(1), \ldots, a(n)\rangle$ and $b := \langle b(1), \ldots, b(n)\rangle$ of $\mathfrak{A}^n$, let $H(a, b)$ denote the subuniverse of $\mathfrak{A}^2$ defined as follows:

$$H(a, b) = Sg(\langle a(1), b(1)\rangle, \ldots, \langle a(n), b(n)\rangle).$$

(iv) For a nonzero natural number m, let $\underline{m}$ be the set $\{1, 2, \ldots, m\}$.

Throughout, $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$ will be a finite nontrivial first-order structure.

In Section 2 we give some necessary and sufficient conditions under which $1_{\mathfrak{A}}$ compatible functions are interpolated by terms on $1_{\mathfrak{A}}$ classes. Section 3 is devoted to formulating and characterizing a notion of quasiprimality for $\mathfrak{A}$. An example is given in Section 4.

2. Term interpolation of functions

In this section we examine some necessary and sufficient conditions for term interpolation of $1_{\mathfrak{A}}$ compatible function. For this we need to adapt some definitions on algebras to the situation of first-order structures we want to investigate.

Definition 2.1. *Let $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$ be a first-order structure and $h : A^n \to A$, $n \geq 1$, be a function.*

(i) *h is said to be termal on classes if for each element $a \in A$, there is an n-ary term t such that $t^{\mathfrak{A}}$ and h coincide on $([a]_{1_{\mathfrak{A}}})^n = \overline{a}^n$.*

(ii) *h is said to be weakly termal if for any subset E of A^n satisfying property*

$$a, b \in E \text{ implies } \langle a(i), b(i)\rangle \in 1_{\mathfrak{A}} \text{ for } 1 \leq i \leq n, \tag{P}$$

there is an n-ary term t such that $t^{\mathfrak{A}}$ and h coincide on E.

(iii) *An n-ary term t represents h on classes if for each $a \in A$, $t^{\mathfrak{A}}$ and h coincide on $\overline{a}^n$. In this case we say that h is term representable on classes by t.*

Remark 2.1. We note that:

(i) Weakly termal functions and functions which are term representable on classes are termal on classes.

(ii) For a unary function h, weakly termal and termal on classes are equivalent, and h is term representable on classes means h is a term function.

(iii) A constant function h is termal on classes iff it is a term function.

Example 2.1. Let $\mathcal{A} = (Z_{40}; +, -, \cdot, 0)$ be the ring of integers modulo 40, θ be the congruence associated to the ideal $5Z_{40}$.

Consider the relations $r := [0]_\theta \times [2]_\theta$ and $s := [1]_\theta \times [3]_\theta \times [4]_\theta$ on $A = Z_{40}$. For the structure $\mathfrak{A} = (\mathcal{A}; r, s) = (Z_{40}; +, -, \cdot, 0; r, s)$, one can see that $1_{\mathfrak{A}} = \theta$.

We use the following terms to define the functions we need: $t_1(x, y, z) := x + y + z$, $t_2(x, y, z) := xy + z$, and $t_3(x, y, z) := xyz$.

(i) Define the function $f : A^3 \to A$ by $f(a,b,c) = \begin{cases} a+b+c & \text{if } [a]_\theta = [b]_\theta; \\ ab + c & \text{if } [a]_\theta \neq [b]_\theta = [c]_\theta; \\ abc & \text{elsewhere.} \end{cases}$

Then the term t_1 represents f on classes, and f is also weakly termal.

(ii) Define the function $g : A^3 \to A$ by $g(a,b,c) = \begin{cases} a+b+c & \text{if } [a]_\theta = [b]_\theta; \\ abc & \text{if } [a]_\theta \neq [b]_\theta \text{ and } a \neq 0; \\ ab + c & \text{elsewhere.} \end{cases}$

Then t_1 represents g on classes.

Let $u = \langle 0, 2, 3\rangle \in A^3$ and $E(u) := \{x \in A^3 : \langle x(i), u(i)\rangle \in \theta \text{ for } 1 \leq i \leq 3\}$.

For each $x \in E(u)$, $g(x) = \begin{cases} x(3) & \text{if } x(1) = 0; \\ x(1) \cdot x(2) \cdot x(3) & \text{if not.} \end{cases}$

So there is no term representing g on $E(u)$, and g is not weakly termal.

(iii) Define the function $h : A^3 \to A$ by $h(a,b,c) = \begin{cases} a+b+c & \text{if } [a]_\theta = [0]_\theta; \\ abc & \text{if not.} \end{cases}$

Then, h is weakly termal; but there is no term representing h on classes.

(iv) Define the function $\lambda : A^3 \to A$ by $\lambda(a,b,c) = \begin{cases} a \text{ if } [a]_\theta = [0]_\theta; \\ b \text{ if } [a]_\theta \neq [0]_\theta \text{ and } b \neq 0; \\ ab + c \text{ elsewhere.} \end{cases}$

The first projection $\pi_1^{(3)}$ represents λ on $[0]_\theta = \overline{0}$, the second projection $\pi_2^{(3)}$ represents λ on the other classes; so λ is termal on classes. But there is no term representing λ on classes, and λ is not weakly termal as it can be verified on $E(u)$ where $u = \langle 1, 0, 4 \rangle \in A^3$.

(v) Define the function $\mu : A^3 \to A$ by $\mu(a,b,c) = \begin{cases} a \text{ if } [a]_\theta = [0]_\theta; \\ b \text{ if } [a]_\theta \neq [0]_\theta \text{ and } b = c; \\ c \text{ elsewhere.} \end{cases}$

Then μ is not termal on classes.

We will use the following version of the lemma of Baker and Pixley.

Lemma 2.1. *Suppose that $\mathfrak{A}$ has a majority function which is termal on classes, and let $h : A^n \to A$, $n \geq 1$, be a function.*

(i) *The following conditions are equivalent for an element $u \in A$.*

  (i_1) *For any elements $e_1, e_2, \ldots, e_n$ of $\overline{u}^2$, the element $h^{\mathfrak{A}^2}(e_1, \ldots, e_n)$ belongs to $Sg(e_1, \ldots, e_n)$ in $\mathfrak{A}^2$.*

  (i_2) *For any subset E of $\overline{u}^n$ there is an n-ary term t such that $t^{\mathfrak{A}}$ and h coincide on E.*

(ii) *h is weakly termal if and only if h preserves the subuniverses of $1_{\mathfrak{A}}$.*

Proof. (i): We only need to prove that (i_1) implies (i_2).

Let $a, b \in \overline{u}^n$; then $\langle a(1), b(1) \rangle, \ldots, \langle a(n), b(n) \rangle$ are elements of $\overline{u}^2$. By hypothesis, $h^{\mathfrak{A}^2}(\langle a(1), b(1) \rangle, \ldots, \langle a(n), b(n) \rangle)$ belongs to $Sg(\langle a(1), b(1) \rangle, \ldots, \langle a(n), b(n) \rangle)$; so for some n-ary term t, $h^{\mathfrak{A}^2}(\langle a(1), b(1) \rangle, \ldots, \langle a(n), b(n) \rangle) = t^{\mathfrak{A}^2}(\langle a(1), b(1) \rangle, \ldots, \langle a(n), b(n) \rangle)$; that is $h(a) = t^{\mathfrak{A}}(a)$ and $h(b) = t^{\mathfrak{A}}(b)$.

Suppose that for any $E_0 \subseteq \overline{u}^n$ with $2 \leq card(E_0) \leq k$, there is an n-ary term t which coincides with h on E_0. If $card(E) > k$, let $E_1 \subseteq E$ with $card(E_1) = k+1$, and choose three distinct elements a_1, a_2, a_3 in E_1. Then there are terms t_1, t_2, t_3 such that $t_i^{\mathfrak{A}}$ coincides with h on $E_1 \backslash \{a_i\}$ for $1 \leq i \leq 3$.

There is some $b \in A$ such that $t_1^{\mathfrak{A}}(E_1) \subseteq \overline{b}$; but $t_1^{\mathfrak{A}}(a_2) = h(a_2) = t_3^{\mathfrak{A}}(a_2)$ and $t_1^{\mathfrak{A}}(a_3) = h(a_3) = t_2^{\mathfrak{A}}(a_3)$; so $t_i^{\mathfrak{A}}(E_1) \subseteq \overline{b}$ for each i. Now let t_4 be a term interpolating M on $\overline{b}^3$, and consider the term $\sigma(x) := t_4(t_1(x), t_2(x), t_3(x))$, where $x = (x_1, \ldots, x_n)$. Then h coincides with at least two of the $t_i^{\mathfrak{A}}$ on each $e \in E_1$ for $1 \leq i \leq 3$, so that $\sigma^{\mathfrak{A}}(e) = h(e)$ for each $e \in E_1$. Since E is finite, we can iterate the process and obtain a term which coincides with h on E.

(ii): The proof is essentially the same, using property (P), and can be found in [6]. □

Recall that for $a, b \in A^n$, $H(a,b)$ is the subuniverse $Sg(\langle a(1), b(1)\rangle, \dots, \langle a(n), b(n)\rangle)$ of $\mathfrak{A}^2$. An argument similar to that used in the above lemma gives the following.

Corollary 2.2. *Suppose that $\mathfrak{A}$ has a majority function which is term representable on classes, and there is some $u \in A$ such that $card(\overline{u}) \geq 3$. Let $h : A^n \to A$, $n \geq 1$, be a function such that for any elements $a, b \in \bigcup_{u\in A} \overline{u}^n$, $\langle h(a), h(b)\rangle \in H(a,b)$. Then h is term representable on classes.* □

Definition 2.2. *Let $h : A^n \to A$, $n \geq 1$, be a function.*

(i) *h is said to be $1_{\mathfrak{A}}$ compatible if for any pairs $\langle a_i, b_i\rangle \in 1_{\mathfrak{A}}$, $1 \leq i \leq n$, we have $h^{A^2}(\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle) = \langle h^A(a_1, \dots, a_n), h^A(b_1, \dots, b_n)\rangle \in 1_{\mathfrak{A}}$.*

(ii) *h is said to be compatible on a class $\overline{u}$ if there is some $b \in A$ such that $h(\overline{u}^n) \subseteq \overline{b}$.*

For example, consider the following ternary functions $\mathbf{q}$ and $\mathbf{d}$ on A:

$$\text{(a):}\quad \mathbf{q}(x,y,z) = \begin{cases} x & \text{if } \overline{x} = \overline{y} = \overline{z} \text{ and } x \neq y; \\ z & \text{elsewhere.} \end{cases}$$

$$\text{(b):}\quad \mathbf{d}(x,y,z) = \begin{cases} x & \text{if } x \neq y; \\ z & \text{elsewhere.} \end{cases}$$

Then, $\mathbf{q}$ is $1_{\mathfrak{A}}$ compatible, but is a Pixley function only on classes; the discriminator function $\mathbf{d}$ is compatible on classes.

A weakly termal function $h : A^n \to A$ is $1_{\mathfrak{A}}$ compatible.

Let m and n be nonzero natural numbers, and $a^1, \dots, a^m \in \bigcup_{u\in A} \overline{u}^n$, and define the elements a_i of A^m by $a_i := \langle a^1(i), \dots, a^m(i)\rangle$ for $1 \leq i \leq n$. Consider the set $K(a_1, \dots, a_n) := \{x \in A^m;\ \langle x(i), x(j)\rangle \in H(a^i, a^j) \text{ for } i, j \in \underline{m}\}$. Then $Sg(a_1, \dots, a_n) \subseteq K(a_1, \dots, a_n)$ as subuniverses of $\mathfrak{A}^m$.

Theorem 2.3. *Consider the following properties on $\mathfrak{A}$:*

(i) *Every n-ary function which preserves the subuniverses of $1_{\mathfrak{A}}$, $n \geq 1$, is term representable on classes.*

(ii) *Let m and n be nonzero natural numbers:*

  (ii_1) *If $a, b \in \bigcup_{u\in A} \overline{u}^n$, then $H(a,b) = Sg(a(1), \dots, a(n)) \times Sg(b(1), \dots, b(n))$ or $H(a,b) \subseteq 1_{\mathfrak{A}}$.*

  (ii_2) *If $a^1, \dots, a^m \in \bigcup_{u\in A} \overline{u}^n$, and $a_i := \langle a^1(i), \dots, a^m(i)\rangle$ for $1 \leq i \leq n$, then $Sg(a_1, \dots, a_n) = K(a_1, \dots, a_n)$.*

Then (i)⇒(ii_1) *and* (ii)⇒(i).

Proof. (i) $\Rightarrow$ (ii$_1$): Obviously, $H(a,b) \subseteq Sg(a(1),\ldots,a(n)) \times Sg(b(1),\ldots,b(n))$. Assume $H(a,b) \nsubseteq 1_{\mathbf{A}}$; then there is some $i \in \underline{n}$ such that $\langle a(i), b(i)\rangle \notin 1_{\mathfrak{A}}$; W.l.o.g. let $i=1$. To show that $Sg(a(1),\ldots,a(n)) \times Sg(b(1),\ldots,b(n)) \subseteq H(a,b)$ let $\langle u,v\rangle \in Sg(a(1),\ldots,a(n)) \times Sg(b(1),\ldots,b(n))$. There are terms t_1 and t_2 such that $u = t_1^{\mathfrak{A}}(a)$ and $v = t_2^{\mathfrak{A}}(b)$. Consider the function $h : A^n \to A$ defined by

$$h(x) = \begin{cases} t_1^{\mathfrak{A}}(x) & \text{if } \ x \in \overline{a(1)}^n; \\ t_2^{\mathfrak{A}}(x) & \text{if } \ x \in \overline{b(1)}^n; \\ x(1) & \text{elsewhere.} \end{cases}$$

Then h preserves the subuniverses of $1_{\mathfrak{A}}$. By (i) let t be a term representing h on classes; we have $\langle u,v\rangle = \langle h(a), h(b)\rangle = \langle t^{\mathfrak{A}}(a), t^{\mathfrak{A}}(b)\rangle = t^{\mathfrak{A}^2}(\langle a,b\rangle) \in H(a,b)$.

So $H(a,b) = Sg(a(1),\ldots,a(n)) \times Sg(b(1),\ldots,b(n))$ holds.

(ii) $\Rightarrow$ (i): Let $h :\ A^n \to A$ be a function which preserves the subuniverses of $1_{\mathfrak{A}}$; let $a^1,\ldots,a^m$ be the distinct elements of $\bigcup\limits_{u\in A} \overline{u}^n$, and $a_i := \langle a^1(i),\ldots,a^m(i)\rangle$ for $1 \le i \le n$.

Let $k,l \in \underline{m}$ be arbitrary. If $\langle a^k(1), a^l(1)\rangle \in 1_{\mathfrak{A}}$, then $\langle h(a^k), h(a^l)\rangle \in H(a^k,a^l)$; if not $H(a^k,a^l) = Sg(a^k(1),\ldots,a^k(n)) \times Sg(a^l(1),\ldots,a^l(n))$ by (ii$_1$), so that $\langle h(a^k), h(a^l)\rangle \in H(a^k,a^l)$.

Thus $\langle h(a^1),\ldots,h(a^m)\rangle \in K(a_1,\ldots,a_n) = Sg(a_1,\ldots,a_n)$ by (ii$_2$). So there is an n-ary term t such that $\langle h(a^1),\ldots,h(a^m)\rangle = t^{\mathfrak{A}^m}(a_1,\ldots,a_n)$; that is $\langle h(a^1),\ldots, h(a^m)\rangle = \langle t^{\mathfrak{A}}(a^1),\ldots,t^{\mathfrak{A}}(a^m)\rangle$, and t represents h on classes. $\square$

If $\mathfrak{A}$ is a minimal structure (i.e. $\mathfrak{A}$ has no proper substructure), property (ii$_1$) of the above theorem implies that A^2 is the only subuniverse of $\mathfrak{A}^2$ that may not be contained in $1_{\mathfrak{A}}$.

Definition 2.3. $\mathfrak{A}$ *satisfies the class subuniverse property (briefly CSP), if it satisfies the condition* (ii$_2$) *of Theorem* 2.3.

Corollary 2.4. *Suppose that* $\mathfrak{A}$ *has a majority function which is termal on classes. Then each of the properties* (i) *and* (ii) *of Theorem* 2.3 *is equivalent to this third one:*

(iii) *Every weakly termal function is term representable on classes.*

Proof. From the hypothesis and Lemma 2.1 (ii), properties (i) and (iii) are equivalent. So, we only have to prove that (i) and (ii) are equivalent. For this, we must show that (i) implies (ii$_2$).

Let $a^1,\ldots,a^m$ be elements of $\bigcup\limits_{u\in A} \overline{u}^n$, and $a_i := \langle a^1(i),\ldots,a^m(i)\rangle$ for $1 \le i \le n$. Take an $x \in K(a_1,\ldots,a_n)$. For each $k \in \underline{m}$, let $J_k := \{l \in \underline{m} : \langle a^k(1), a^l(1)\rangle \in 1_{\mathfrak{A}}\}$; furthermore let $\sigma(k)$ be the minimum of J_k. If l_1, l_2 are in J_k, $\langle x(l_1), x(l_2)\rangle$ is an element of $H(a^{l_1}, a^{l_2})$ and the later is $Sg(\langle a^{l_1}(1), a^{l_2}(1)\rangle,\ldots,\langle a^{l_1}(n), a^{l_2}(n)\rangle)$. From the proof of Lemma 2.1 (i), we can find a term $t_{\sigma(k)}$ such that $x(l) = t^{\mathfrak{A}}_{\sigma(k)}(a^l)$ for each $l \in J_k$.

Consider the function $h : A^n \to A$ defined by:

$$h(u) = \begin{cases} t^{\mathfrak{A}}_{\sigma(k)}(u) & \text{if } u \in \overline{a^k(1)}^n \text{ for some } k; \\ u(1) & \text{if not.} \end{cases}$$

Let E be a subuniverse of $1_{\mathfrak{A}}$, and $\langle b(j), c(j)\rangle \in E$ for $1 \leq j \leq n$.
If $b := \langle b(1), \dots, b(n)\rangle \in \overline{a^k(1)}^n$ for some k, then so is $c := \langle c(1), \dots, c(n)\rangle$ and $\langle h(b), h(c)\rangle = \langle t^{\mathfrak{A}}_{\sigma(k)}(b), t^{\mathfrak{A}}_{\sigma(k)}(c)\rangle = t^{\mathfrak{A}^2}_{\sigma(k)}(\langle b(1), c(1)\rangle, \dots, \langle b(n), c(n)\rangle) \in E$. Otherwise, $\langle h(b), h(c)\rangle = \langle b(1), c(1)\rangle \in E$.

So h preserves the subuniverses of $1_{\mathfrak{A}}$, and there is a term t representing h on classes. Now, $x = \langle x(1), \dots, x(m)\rangle = \langle t^{\mathfrak{A}}(a^1), \dots, t^{\mathfrak{A}}(a^m)\rangle = t^{\mathfrak{A}^m}(\langle a_1, \dots, a_n\rangle)$ is an element of $Sg(a_1, \dots, a_n)$; thus we have (ii$_2$). □

3. $\star$-quasiprimality

We begin this section with a version of a Fleischer's lemma on subuniverses of product algebras in a permutable variety (cf. [1], p. 169). First we need a notion of subdirect product suitable for our purpose.

Definition 3.1. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be substructures of $\mathfrak{A}$, and $\pi_i : \mathfrak{B}_1 \times \mathfrak{B}_2 \to \mathfrak{B}_i$, $1 \leq i \leq 2$, be the canonical projections. Let $\mathfrak{B}$ be a substructure of $\mathfrak{B}_1 \times \mathfrak{B}_2$.*

(i) *$\mathfrak{B} \subseteq \mathfrak{B}_1 \times \mathfrak{B}_2$ is called a subdirect product of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ if $\pi_i(B) = B_i$ for $i = 1, 2$.*

(ii) *$\mathfrak{B} \subseteq \mathfrak{B}_1 \times \mathfrak{B}_2$ is called a $\star$-subdirect product if it is a subdirect product and $ker(\pi_i) \cap B^2$ is a $\star$-congruence of $\mathfrak{B}$ for $i = 1, 2$.*

Note that for any $\star$-congruence θ of the $\star$-subdirect product $\mathfrak{B}$ of $\mathfrak{B}_1 \times \mathfrak{B}_2$, $\pi_i(\theta)$ is a $\star$-congruence of $\mathfrak{B}_i$.

Lemma 3.1. *Let $\mathfrak{B}_1, \mathfrak{B}_2$ be substructures of $\mathfrak{A}$ and $\mathfrak{B} \subseteq \mathfrak{B}_1 \times \mathfrak{B}_2$ be a $\star$-subdirect product such that $Con_\star(\mathfrak{B})$ is permutable. Then there are $\star$-epimorphisms $\mathfrak{B}_1 \xrightarrow{h_1} \mathfrak{D} \xleftarrow{h_2} \mathfrak{B}_2$ such that $B = \{\langle b_1, b_2\rangle \in B_1 \times B_2 : h_1(b_1) = h_2(b_2)\}$.*

Proof. Let $\tau_i = \pi_i \restriction_B : B \to B_i$ be the restriction of π_i to B, for $1 \leq i \leq 2$. Since $\mathfrak{B} \subseteq \mathfrak{B}_1 \times \mathfrak{B}_2$ is a $\star$-subdirect product, each τ_i is a $\star$-epimorphism; then $\rho_i = ker(\tau_i) = ker(\pi_i) \cap B^2$ is in $Con_\star(\mathfrak{B})$, and so is $\rho = \rho_1 \vee \rho_2 = \rho_1 \circ \rho_2$. Let $\tau : \mathfrak{B} \to \mathfrak{B}/\rho$ be the canonical $\star$-epimorphism; there are epimorphisms $\mathfrak{B}_1 \xrightarrow{h_1} \mathfrak{B}/\rho \xleftarrow{h_2} \mathfrak{B}_2$ such that $\tau = h_i \circ \tau_i$ for each i; moreover each h_i is a $\star$-morphism.

If $\langle b_1, b_2\rangle \in B$, $h_1(b_1) = h_1 \circ \tau_1(\langle b_1, b_2\rangle) = h_2 \circ \tau_2(\langle b_1, b_2\rangle) = h_2(b_2)$.

Conversely let $\langle b_1, b_2\rangle \in B_1 \times B_2$ such that $h_1(b_1) = h_2(b_2)$; there are elements $a_1 \in B_1$ and $a_2 \in B_2$ such that $\langle b_1, a_2\rangle$ and $\langle a_1, b_2\rangle$ are in B. Then $\tau(\langle b_1, a_2\rangle) = h_1(b_1) = h_2(b_2) = \tau(\langle a_1, b_2\rangle)$ and $(\langle a_1, b_2\rangle, \langle b_1, a_2\rangle) \in \rho = \rho_1 \circ \rho_2$. Let $\langle u, v\rangle$ be an element of B such that $\langle a_1, b_2\rangle \rho_2 \langle u, v\rangle \rho_1 \langle b_1, a_2\rangle$; then $b_2 = v$ and $u = b_1$, so $\langle b_1, b_2\rangle = \langle u, v\rangle \in B$. □

A substructure $\mathfrak{B}$ (or subuniverse B) of $\mathfrak{B}_1 \times \mathfrak{B}_2$ is said to be *rectangular* if whenever $\langle x, y\rangle$, $\langle u, v\rangle$, $\langle x, v\rangle$ are in B then so is $\langle u, y\rangle$. Then Lemma 3.1 shows that every $\star$-subdirect product $\mathfrak{B} \subseteq \mathfrak{B}_1 \times \mathfrak{B}_2$ with $Con_\star(\mathfrak{B})$ permutable is rectangular.

For every substructure $\mathfrak{B}$ of $\mathfrak{A}$, the largest congruence of $\mathfrak{B}$ is the restriction of the largest congruence of $\mathfrak{A}$; that is, $\nabla_B = \nabla_A \cap B^2$. For $\star$-congruences, we want $1_\mathfrak{A}$ to act similarly as ∇_A on substructures. Whence the following definition.

Definition 3.2. *$1_\mathfrak{A}$ is said to be hereditarily maximal if $1_\mathfrak{B} = 1_\mathfrak{A} \cap B^2$ for each substructure $\mathfrak{B}$ of $\mathfrak{A}$.*

Lemma 3.2. *Let $\mathfrak{A}$ be a structure such that $1_\mathfrak{A}$ is hereditarily maximal, and $\mathfrak{B}$ be a substructure of $\mathfrak{A}^2$ such that B is contained in $1_\mathfrak{A}$.*

(i) *If $\mathfrak{A}$ has a Malcev function which is termal on classes, then $Con_\star(\mathfrak{B})$ is permutable.*

(ii) *If $\mathfrak{A}$ has a Pixley function which is termal on classes, then $Con_\star(\mathfrak{B})$ is arithmetical.*

Proof. (i): For any $\theta, \varphi \in Con_\star(\mathfrak{B})$ and $\langle a, b\rangle \in \theta \circ \varphi$, let $c \in B$ such that $a\varphi c\theta b$; now let $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$ and $c = \langle c_1, c_2\rangle$; then $a_i, b_i, c_i \in \overline{a_1}$ for each i. Let $t(x, y, z)$ be a ternary term representing the given Malcev function on $\overline{a_1}$; then $a = t^{\mathfrak{A}^2}(a, b, b)\theta t^{\mathfrak{A}^2}(a, c, b)\varphi t^{\mathfrak{A}^2}(c, c, b) = b$, and $\langle a, b\rangle \in \varphi \circ \theta$.

(ii): Let θ, φ, ψ be in $Con_\star(\mathfrak{B})$, and $\langle a, b\rangle \in \theta \wedge (\varphi \circ \psi)$; then $a\psi c\varphi b$ for some $c \in B$. Let $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$ and $c = \langle c_1, c_2\rangle$; then $a_i, b_i, c_i \in \overline{a_1}$ for each i. Let $t(x, y, z)$ be a ternary term representing the given Pixley function on $\overline{a_1}$; then $t^{\mathfrak{A}^2}(a, c, b)\theta t^{\mathfrak{A}^2}(a, c, a) = a$ and $t^{\mathfrak{A}^2}(a, c, b)\theta t^{\mathfrak{A}^2}(b, c, b) = b$. So $a = t^{\mathfrak{A}^2}(a, b, b)(\theta \wedge \varphi)t^{\mathfrak{A}^2}(a, c, b)\ (\theta \wedge \psi)t^{\mathfrak{A}^2}(a, a, b) = b$, and $\langle a, b\rangle \in (\theta \wedge \psi) \circ (\theta \wedge \varphi)$, thus $\theta \wedge (\varphi \circ \psi) \subseteq (\theta \wedge \psi) \circ (\theta \wedge \varphi)$. If $\theta = \triangle_B$, we obtain $\varphi \circ \psi \subseteq \psi \circ \varphi$. So $\theta \wedge (\varphi \vee \psi) \subseteq (\theta \wedge \psi) \vee (\theta \wedge \varphi)$, and $Con_\star(\mathfrak{B})$ is arithmetical. □

One of the main properties of a quasiprimal algebra $\mathcal{A} = (A; F^\mathcal{A})$ is that every subalgebra of $\mathcal{A}$ is simple. A notion of quasiprimality for first-order structures may be expected to entail a similar property. Below we give a notion of simplicity which fits with $\star$-congruences.

Definition 3.3. *Let $\mathfrak{B}$ be a substructure of $\mathfrak{A}$.*

(i) *A congruence $\theta \in Con_\star(\mathfrak{B})$ is said to be simple if for every $b \in B$, $[b]_\theta = \{b\}$ or $[b]_\theta = [b]_{1_\mathfrak{B}}$.*

(ii) *$\mathfrak{B}$ is said to be $\star$-simple if every $\star$-congruence of $\mathfrak{B}$ is simple.*

(iii) *$\mathfrak{A}$ is said to be hereditarily $\star$-simple if every substructure of $\mathfrak{A}$ is $\star$-simple.*

It is proved in [6] that if the discriminator function $\mathbf{d}$ is termal on classes, then $\mathfrak{A}$ is $\star$-simple and $Con_\star(\mathfrak{A})$ is arithmetical.

Definition 3.4. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be substructures of $\mathfrak{A}$, $\theta_i \in Con(\mathfrak{B}_i)$ for $1 \le i \le 2$, and $\alpha : \mathfrak{B}_1/\theta_1 \to \mathfrak{B}_2/\theta_2$ be an isomorphism.*

(i) *The set* $E := \{\langle b_1, b_2\rangle \in B_1 \times B_2\colon \alpha(b_1/\theta_1) = b_2/\theta_2\}$ *is called the lifting of* α, *and denoted by lift*(α).

(ii) *If* $E \subseteq 1_{\mathfrak{A}}$ *then* E *is called a* $1_{\mathfrak{A}}$ *lifting, and* α *is called a* $1_{\mathfrak{A}}$ *isomorphism.*

If moreover θ_1 *and* θ_2 *are simple then* E *is called a simple* $1_{\mathfrak{A}}$ *lifting.*

When $\mathfrak{A}$ has a Malcev function which is termal on classes, we see from Lemma 3.1 and Lemma 3.2 that any subuniverse E of $1_{\mathfrak{A}}$ is rectangular, thus is the lifting of some $1_{\mathfrak{A}}$ isomorphism $\alpha : \mathfrak{B}_1/\theta_1 \to \mathfrak{B}_2/\theta_2$, where $B_i = \pi_i(E)$ is a subuniverse of $\mathfrak{A}$ and $\theta_i \in Con_\star(\mathfrak{B}_i)$ for $1 \le i \le 2$. If moreover $\mathfrak{A}$ is hereditarily $\star$-simple then E is a simple $1_{\mathfrak{A}}$ lifting.

Theorem 3.3. *Let* $\mathfrak{A}$ *be a structure with* $1_{\mathfrak{A}}$ *hereditarily maximal. Then the following properties are equivalent:*

(i) *The discriminator function* $\mathbf{d}$ *is termal on classes.*

(ii) $\mathfrak{A}$ *is hereditarily* $\star$*-simple and has a Pixley function which is termal on classes.*

(iii) *Every function* $h : A^n \to A$, $n \ge 1$, *which preserves simple* $1_{\mathfrak{A}}$ *liftings is termal on classes.*

Proof. (i) $\Rightarrow$ (ii): First note that the discriminator function $\mathbf{d}$ is a Pixley function.

For hereditary $\star$-simplicity, let $\mathfrak{B}$ be a substructure of $\mathfrak{A}$, $b \in B$ and $\theta \in Con_\star(\mathfrak{B})$ such that $[b]_\theta \neq \{b\}$. Then there is some $c \in B$ such that $\langle b, c\rangle \in \theta$ and $b \neq c$. Now, for each $u \in [b]_{1_{\mathfrak{B}}}$, $\langle c, u\rangle = \langle \mathbf{d}(c, b, u), \mathbf{d}(b, b, u)\rangle \in \theta$. Thus $[b]_{1_{\mathfrak{B}}} = [b]_\theta$, showing that θ is a simple $\star$-congruence.

(ii) $\Rightarrow$ (iii): A Pixley function is also a Malcev function, thus subuniverses of $1_{\mathfrak{A}}$ are $1_{\mathfrak{A}}$ liftings, which are simple. From the given Pixley function, we can obtain a majority function which is termal on classes. So by Lemma 2.1 (ii), h is weakly termal, hence termal on classes.

(iii) $\Rightarrow$ (i): Consider the function $q : A^3 \to A$ defined by

$$q(a, b, c) = \begin{cases} c & \text{if } \ a = b \text{ and } \ \overline{b} = \overline{c}; \\ a & \text{if } \ a \neq b \ \text{ and } \overline{a} = \overline{b}; \\ b & \text{elsewhere.} \end{cases}$$

Let E be the lifting of a simple $1_{\mathfrak{A}}$ isomorphism $\alpha : \mathfrak{B}_1/\theta_1 \to \mathfrak{B}_2/\theta_2$, and u, v, w be elements of E.

• Suppose that $\overline{u(1)} = \overline{v(1)} = \overline{w(1)}$, then $[u(1)]_{\theta_1} = [v(1)]_{\theta_1}$ iff $[u(2)]_{\theta_2} = [v(2)]_{\theta_2}$.

$[(u(1) = v(1)$ iff $(u(2) = v(2))]$ implies $q^{A^2}(u, v, w) \in \{u, w\}$.

$[(u(1) = v(1))$ iff $(u(2) \neq v(2))]$ implies $[u(1)]_{\theta_1} = [v(1)]_{\theta_1} = [w(1)]_{\theta_1} = [u(1)]_{1_{\mathfrak{A}} \cap B_1^2}$, then $q^{A^2}(u, v, w) \in \{\langle u(1), w(2)\rangle, \langle w(1), u(2)\rangle\}$, and the later is a subset of E.

• Suppose that $\overline{u(1)} = \overline{v(1)} \neq \overline{w(1)}$.

If $[(u(1) = v(1))$ iff $(u(2) = v(2))]$, then $q^{A^2}(u, v, w) \in \{u, v\}$.

If $[(u(1) = v(1))$ iff $(u(2) \neq v(2))]$, then $q^{A^2}(u, v, w) = u$.

• Suppose that $\overline{u(1)} \neq \overline{v(1)}$; then $q^{A^2}(u, v, w) = v$.

So q preserves E, and is termal on classes. Since q coincides with $\mathbf{d}$ on classes, $\mathbf{d}$ is also termal on classes. □

Definition 1.2 and the above theorem motivate the following definition.

Definition 3.5. (i) *$\mathfrak{A}$ is called weak $\star$-quasiprimal if every function on A which preserves simple $1_{\mathfrak{A}}$ liftings is termal on classes.*

(ii) *$\mathfrak{A}$ is called $\star$-quasiprimal if every function on A which preserves simple $1_{\mathfrak{A}}$ liftings is term representable on classes.*

So, Theorem 3.3 characterizes weak $\star$-quasiprimality. Below we use the property CSP (see Definition 2.3) in the characterization of $\star$-quasiprimality.

Theorem 3.4. *Suppose that $1_{\mathfrak{A}}$ is hereditarily maximal. Then $\mathfrak{A}$ is $\star$-quasiprimal if and only if the following conditions are satisfied:*

(i) *The discriminator function $\mathbf{d}$ is term representable on classes.*

(ii) *For any nonzero natural number n and $a, b \in \bigcup_{u\in A} \overline{u}^n$, $H(a, b) \subseteq 1_{\mathfrak{A}}$ or $H(a, b) = Sg(a(1), \ldots, a(n)) \times Sg(b(1), \ldots, b(n))$.*

(iii) *$\mathfrak{A}$ satisfies the CSP.*

Proof. (⇒): From the proof of Theorem 3.3, $\mathbf{d}$ is term representable on classes.

Since $\mathbf{d}$ is a Pixley function, we can obtain a majority function which is termal on classes, and subuniverses of $1_{\mathfrak{A}}$ are simple $1_{\mathfrak{A}}$ liftings, thus the result follows from Corollary 2.4.

⇐): Since $\mathbf{d}$ is term representable on classes and $1_{\mathfrak{A}}$ is hereditarily maximal, the subuniverses of $1_{\mathfrak{A}}$ are simple $1_{\mathfrak{A}}$ liftings, so the result follows from Theorem 2.3. □

Note that in $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$, if for each m-ary $r \in R$, $r^{\mathfrak{A}}$ is empty or $r^{\mathfrak{A}} = A^m$, the above results correspond to the standard notion of quasiprimality on algebras.

In particular, condition (i) in the above theorem may be replaced by "$\mathfrak{A}$ is hereditarily $\star$-simple and has a Pixley function which is term representable on classes".

We end the section by examining the case of minimal structures.

Lemma 3.5. *Let $\mathfrak{A}$ be weak $\star$-quasiprimal and minimal. Then:*

(i) *Any subuniverse of $1_{\mathfrak{A}}$ is an automorphism or a simple $\star$-congruence of $\mathfrak{A}$.*

(ii) *If a subuniverse of $1_{\mathfrak{A}}$ is a nontrivial automorphism of $\mathfrak{A}$, then $\triangle_A$ and $1_{\mathfrak{A}}$ are the only $\star$-congruences of $\mathfrak{A}$.*

Proof. (i): Let E be a subuniverse of $1_{\mathfrak{A}}$; then E is the lifting of some isomorphism $\alpha : A/\theta_1 \to A/\theta_2$, where θ_1 and θ_2 are simple $\star$-congruences of $\mathfrak{A}$.

If $\theta_1 = \triangle_A$ then $\theta_2 = \triangle_A$; and α is an isomorphism of $\mathfrak{A}$.

If $\theta_1 \neq \triangle_A$ then $\theta_2 \neq \triangle_A$; and there is some $a \in A$ such that $[a]_{\theta_1} = \overline{a} \neq \{a\}$. Since $E = \mathrm{lift}(\alpha)$ is a simple $1_{\mathfrak{A}}$ lifting, we must have $\alpha(\overline{a}) = \overline{a}$. Let b be any element of A; by minimality $b = t^{\mathfrak{A}}(a)$ for some unary term t. Then $\alpha(b/\theta_1) =$

$\alpha(t^{\mathfrak{A}/\theta_1}(a/\theta_1)) = t^{\mathfrak{A}/\theta_2}(\alpha(a/\theta_1)) = t^{\mathfrak{A}/\theta_2}(a/\theta_2) = t^{\mathfrak{A}}(a)/\theta_2 = b/\theta_2$; so, $\alpha = Id_{A/\theta_1}$ and $\theta_1 = \theta_2 = E$.

(ii): Let α be a nontrivial automorphism with $\alpha \subseteq 1_{\mathfrak{A}}$. If $\varphi \in Con_\star(\mathfrak{A})$ such that $\varphi \neq \triangle_A$, then $[a]_\varphi = \overline{a} \neq \{a\}$, for some a. Also $b = \alpha(a) \neq a$, for otherwise the fix points of α would form a proper subuniverse of $\mathfrak{A}$. Moreover $b \in \overline{a} = [a]_\varphi$.

Let c be any element of A; then $c = t^{\mathfrak{A}}(a)$ for some term t; now, $d = t^{\mathfrak{A}}(b) = t^{\mathfrak{A}}(\alpha(a)) = \alpha(t^{\mathfrak{A}}(a)) = \alpha(c)$, and $\alpha(c) \neq c$. Since $\langle c, d\rangle = \langle t^{\mathfrak{A}}(a), t^{\mathfrak{A}}(b)\rangle = t^{\mathfrak{A}^2}(\langle a, b\rangle) \in \varphi$, we have $d \in [c]_\varphi$ and $[c]_\varphi = \overline{c}$. This shows that $\varphi = 1_{\mathfrak{A}} \neq \triangle_A$. □

Definition 3.6. *The structure $\mathfrak{A}$ is called $\star$-demiprimal (resp. weak $\star$-demiprimal) if it is minimal and every n-ary function h on A, $n \geq 1$, which preserves $1_{\mathfrak{A}}$ automorphisms and simple $\star$-congruences is term representable (resp. termal) on classes.*

By Lemma 3.5, $\mathfrak{A}$ is weak $\star$-demiprimal if and only if $\mathfrak{A}$ is weak $\star$-quasiprimal and minimal.

Theorem 3.6. *The structure $\mathfrak{A}$ is $\star$-demiprimal if and only if the following conditions are satisfied:*

(i) *$\mathfrak{A}$ is $\star$-quasiprimal.*

(ii) *The only subuniverses of $\mathfrak{A}^2$ are A^2, the simple $\star$-congruences and $1_{\mathfrak{A}}$ automorphisms of $\mathfrak{A}$.*

Proof. ($\Rightarrow$): If $\mathfrak{A}$ is $\star$-demiprimal, then the function q defined in the proof of Theorem 3.3 is term representable, hence **d** is also term representable. Since $\mathfrak{A}$ is minimal, by Lemma 3.5 the subuniverses of $1_{\mathfrak{A}}$ are simple $\star$-congruences and $1_{\mathfrak{A}}$ automorphisms of $\mathfrak{A}$; these are the only simple $1_{\mathfrak{A}}$ liftings of $\mathfrak{A}$. Therefore, $\mathfrak{A}$ is $\star$-quasiprimal. From Theorem 2.3 it is clear that A^2 is the only subuniverse of $\mathfrak{A}^2$ which may not be contained in $1_{\mathfrak{A}}$.

($\Leftarrow$): By (ii), $\mathfrak{A}$ is minimal. From (i), **d**is term representable; and (ii) implies that simple $\star$-congruences and $1_{\mathfrak{A}}$ automorphisms are the only (simple) $1_{\mathfrak{A}}$ liftings. Since $\mathfrak{A}$ is $\star$-quasiprimal, every function which preserves these liftings is term representable on classes; so $\mathfrak{A}$ is $\star$-demiprimal. □

In fact using Lemma 3.5 and the observation after Theorem 2.3, condition (ii) in Theorem 3.6 may be stated as follows:

" The only subuniverses of $\mathfrak{A}^2$ other than A^2 are either $1_{\mathfrak{A}}$, id_A and the $1_{\mathfrak{A}}$ automorphisms, or the $\star$-congruences".

4. An example

We illustrate some of the notions and results of the preceding sections through a finite ring.

4.1. The basic universe

Let p and q be prime natural numbers such that $2 < p < q$; consider the ring $\mathcal{A} = (Z_{p^3q}; +, \cdot, 0)$, its ideal $I = p^3A = p^3Z_{p^3q}$, and the congruence $\theta = \{\langle a, b\rangle \in A^2; a - b \in I\}$.

Define the relations $r^A := [0]_\theta \times [p]_\theta \times [p^2]_\theta$ and $s^A := \prod_{\substack{(1\le a\le p^3-1)\\ a\notin\{0,p,p^2\}}} [a]_\theta$. We obtain a structure $\mathfrak{A} = (\mathcal{A}; r^A, s^A) = (Z_{p^3q}; +, \cdot, 0; r^A, s^A)$.

Let $\langle a, b\rangle, \langle c, d\rangle \in \theta$ such that $\langle a, c\rangle \in r^A$; then $a \in [0]_\theta$ and $c \in [p]_\theta$; so $b \in [0]_\theta$ and $d \in [p]_\theta$; i.e., $\langle b, d\rangle \in r^A$, showing that θ is compatible with r^A. Similarly we verify that θ is compatible with s^A; so θ is a $\star$-congruence of $\mathfrak{A}$.

Now if φ is a congruence of $\mathfrak{A}$ such that $\varphi \nsubseteq \theta$, then φ is not compatible with r^A; so $\theta = 1_{\mathfrak{A}}$, and since $\triangle_A$ is the only congruence contained in θ we obtain $Con_\star(\mathfrak{A}) = \{\triangle_A, 1_{\mathfrak{A}}\}$.

We note that the term $t(x, y, z) := (x - y)^{q-1}(x - z) + z$ represents the discriminator function on classes.

The subuniverses of $\mathfrak{A}$ are of the form $p^\alpha q^\beta A$, where $0 \le \alpha \le 3$ and $0 \le \beta \le 1$. For $C = p^2A = p^2Z_{p^3q}$, we have the substructure $\mathfrak{C} = (p^2Z_{p^3q}; +, \cdot, 0; r^{\mathfrak{C}}, s^{\mathfrak{C}})$, where $r^{\mathfrak{C}} = r^A \cap C^3 = \emptyset$ and $s^{\mathfrak{C}} = s^A \cap C^{p^3-3} = \emptyset$.

So $1_{\mathfrak{C}} = C^2 = \nabla_C \nsubseteq 1_{\mathfrak{A}}$, and $1_{\mathfrak{A}}$ is not hereditarily maximal.

4.2. The structure $(\mathfrak{A}; p)$

We consider the structure $(\mathfrak{A}; p) := (Z_{p^3q}; +, \cdot, 0, p; r^A, s^A)$ where p is a constant. Then $1_{(\mathfrak{A};p)} = \theta = 1_{\mathfrak{A}}$.

The only subuniverse of $(\mathfrak{A}; p)$ is $B := pA = pZ_{p^3q}$, and $\mathfrak{B} = (pZ_{p^3q}; +, \cdot, 0, p; r^{\mathfrak{B}}, s^{\mathfrak{B}})$, where $r^{\mathfrak{B}} = r^A \cap B^3 = r^A$ since $r^A \subseteq B^3$, and $s^{\mathfrak{B}} = s^A \cap B^{(p^2-3)} = \emptyset$.

$I = p^3A$ is an ideal of the underlying ring $\mathcal{B}$ of $\mathfrak{B}$, and $1_{\mathfrak{B}} = \theta \cap B^2 = 1_{\mathfrak{A}} \cap B^2$; so $1_{(\mathfrak{A};p)}$ is hereditarily maximal. Moreover, I is a minimal ideal of $\mathcal{B}$, so we have $Con_\star(\mathfrak{B}) = \{\triangle_B, 1_{\mathfrak{B}}\}$.

Since the discriminator function on A is term representable on classes, $(\mathfrak{A}; p)$ is weak $\star$-quasiprimal.

Let us look for the subuniverses of $1_{(\mathfrak{A};p)}$. Let E be such a subuniverse; since p is a constant of $(\mathfrak{A}; p)$, $\langle p, p\rangle$ is a constant of $(\mathfrak{A}; p)^2$; so $\triangle_B \subseteq E$.

Suppose that $\triangle_B \subsetneq E \subseteq 1_{\mathfrak{B}}$; there is some $\langle u, v\rangle \in 1_{\mathfrak{B}}$ such that $\langle u, v\rangle \in E$ and $u \neq v$. If $b \in B$ such that $b \in \overline{u}$, then $\langle b, u\rangle = \langle t^{\mathfrak{A}}(u, u, b), t^{\mathfrak{A}}(u, v, b) \in E$; in particular, $\langle u + p^3, u\rangle \in E$, and $\langle a + p^3, a\rangle \in E$ for each $a \in B$; so $1_{\mathfrak{B}} \subseteq E$.

Suppose that $\triangle_B \subsetneq E \nsubseteq 1_{\mathfrak{B}}$; there is some $\langle u, v\rangle \in E$ such that $u \notin B$; so p does not divide u, and there are α, β such that $\alpha u + \beta p = 1$. Then $\alpha\langle u, v\rangle + \beta\langle p, p\rangle = \langle 1, 1 + \alpha(v - u)\rangle$.

If $u = v$, we have $\langle 1, 1\rangle = \alpha\langle u, v\rangle + \beta\langle p, p\rangle \in E$, and $\triangle_A \subseteq E$.

If $u \neq v$, then $\langle q, q\rangle = q\langle 1, 1 + \alpha(v - u)\rangle \in E$ since $v - u \in I = p^3A$. But $\langle p, p\rangle \in E$, so $\langle 1, 1\rangle \in E$ and $\triangle_A \subseteq E$. For each $a \in [1]_{1_{\mathfrak{A}}}$, $\langle a, 1\rangle = \mathbf{d}(\langle 1, 1\rangle, \langle 1, 1 + \alpha(v-u)\rangle, \langle a, a\rangle) \in E$; in particular $\langle 1+p^3, 1\rangle \in E$, and $\langle p^3, 0\rangle \in E$; so $I \times \{0\} \subseteq E$, and $1_{\mathfrak{A}} \subseteq E$.

Therefore the only subuniverses of $1_{(\mathfrak{A};p)}$ are $\triangle_B, 1_{\mathfrak{B}}, \triangle_A$ and $1_{\mathfrak{A}}$.

Consider the function $h : A^2 \to A$ defined by

$$h(a) = \begin{cases} a(1)\cdot a(2) & \text{if } a \in \overline{0} \cup \overline{p} \cup \overline{p^2}; \\ a(1)+a(2) & \text{elsewhere.} \end{cases}$$

That is, the term $t_1(x,y) = xy$ represents h on $\overline{0} \cup \overline{p} \cup \overline{p^2}$ and the term $t_2(x,y) = x+y$ represents h elsewhere. It is clear that h preserves the subuniverses of $1_{(\mathfrak{A};p)}$; but there is no term representing h on classes: for the elements $a = \langle 1,1\rangle$ and $b = \langle p,p\rangle$ of A^2, we have $h(a) = t_2^{\mathfrak{A}}(a) = 1+1 = 2 \neq 1 = t_1^{\mathfrak{A}}(a)$, and $h(b) = t_1^{\mathfrak{A}}(b) = p^2 \neq p+p = 2p = t_2^{\mathfrak{A}}(b)$. So $(\mathfrak{A};p)$ is not $\star$-quasiprimal.

4.3. The structure $(\mathfrak{A}; h, p)$

Consider the structure $\mathfrak{D} = (\mathfrak{A}; h, p) = (Z_{p^3q}; +, \cdot, h, 0, p)$ where h is the function defined above. Then $B = pZ_{p^3q}$ is still the only subuniverse of $\mathfrak{D}$, and h is compatible with θ; so $1_{\mathfrak{D}} = 1_{\mathfrak{A}}$, and $(\mathfrak{B}; h^{\mathfrak{B}})$ is the only substructure of $\mathfrak{D}$. Moreover, $1_{\mathfrak{D}}$ and $1_{(\mathfrak{A};p)}$ have the same subuniverses.

We use Corollary 2.2 to show that $\mathfrak{D}$ is $\star$-quasiprimal. To this end, we need the subuniverses of $\mathfrak{D}^2$.

Since $\mathcal{A} = (Z_{p^3q}; +, \cdot, 0)$ is a ring (hence has a Malcev term), any subuniverse of $\mathfrak{D}^2$ is the lifting of some isomorphism $\alpha : \mathfrak{D}_1/\theta_1 \to \mathfrak{D}_2/\theta_2$, where D_i is a subuniverse of $\mathfrak{D}$ and $\theta_i \in Con(\mathfrak{D}_i)$ for each i; thus $D_i = B$ or $D_i = A = D$ for each i. Moreover $\triangle_B \subseteq E$.

(i) Suppose that $\triangle_B \subsetneq E \subseteq B^2$; then $D_1 = B = D_2$. Since p is a constant and p generates B, we have $\alpha(p/\theta_1) = p/\theta_2$, and $\alpha(b/\theta_1) = b/\theta_2$ for each $b \in B$, so that $\theta_1 = \theta_2$ and $\alpha = id_{B/\theta_1}$; that is, $E = \theta_1$ is a congruence of $\mathfrak{D}_1 = (\mathfrak{B}; h)$. The congruences of $\mathfrak{B}$ are those associated to the ideals $B = pA$, p^2A, p^3A, pqA, p^2qA and $\{0\} = p^3qA$; h preserves te congruences $\triangle_B$ (associated to $\{0\}$), $1_{\mathfrak{B}}$ (associated to p^3A) and $\nabla_B = B^2$ (associated to $B = pA$). For the congruence φ associated to p^2A, let $a, b \in A^2$ be the constant vectors with values p and $p+p^2$ respectively; then $\langle p, p+p^2\rangle \in \varphi$, but $\langle (h(a), h(b)\rangle = \langle p^2, 2(p+p^2)\rangle \notin \varphi$. So h is not compatible with φ, and φ is not a congruence of $\mathfrak{D}_1 = (\mathfrak{B}; h)$. Similarly, we see that h is not compatible with the congruences of $\mathfrak{B}$ associated to the ideals pqA and p^2qA. So $\triangle_B, 1_{\mathfrak{B}}$ and ∇_B are the only congruences of $\mathfrak{D}_1$ (and hence the only subuniverses of $\mathfrak{D}$ contained in B^2).

(ii) Suppose that $D_1 = A$ and $D_2 = B$. Let b be an element of B such that $\alpha(1/\theta_1) = b/\theta_2$. Since p is a constant in $\mathfrak{D}$, $\alpha(p/\theta_1) = p/\theta_2$; so $p/\theta_2 = pb/\theta_2$, and $p - pb \in \theta_2$. But $\triangle_B$ and $1_{\mathfrak{B}}$ are the only congruences of $\mathfrak{D}_2$ which are different from $\nabla_{D_2} = B^2$, and none of them can contain $p - pb$ since $b \neq 1$; so $\theta_2 = \nabla_{D_2} = B^2$, and $E = A \times B$.

Similarly, $D_1 = B$ and $D_2 = A$ implies $E = B \times A$.

(iii) Suppose that $D_1 = A = D_2$; then $\mathfrak{D}_1 = \mathfrak{D} = \mathfrak{D}_2$.

Let a be an element of A such that $\alpha(1/\theta_1) = a/\theta_2$; since $\alpha(p/\theta_1) = p/\theta_2$, we have $p/\theta_2 = pa/\theta_2$, and $p - pa \in \theta_2$. As in (i), we see that the only congruences of $\mathfrak{D}$ are $\triangle_A, 1_{\mathfrak{D}} = 1_{\mathfrak{A}}$, and $\nabla_D = \nabla_A = A^2$.

If $\theta_2 = \triangle_A$, then $\theta_1 = \triangle_A$, and $p - pa \in \theta_2$ iff $a = 1$; so α is the identity on A and $E = \triangle_A$.

If $\theta_2 = \nabla_A$, then $\theta_1 = \nabla_A$ and $E = \nabla_A$.

If $\theta_2 = 1_{\mathfrak{D}} = 1_{\mathfrak{A}}$, then $\theta_2 = 1_{\mathfrak{A}}$, and $pa - p \in \theta_2$ iff $pa - p = kp^3$ for some k. So $a = 1 + kp^2$; but $\alpha(1/\theta_1)^2 = \alpha(1/\theta_1)$ implies $a^2 - a \in \theta_2$; i.e., p^3 must divide $kp^2 + k^2p^4 = kp^2(1 + kp^2)$. So p divides k and $a/\theta_2 = 1/\theta_2$. Thus $\alpha(x/\theta_1) = x/\theta_2$ for each $x \in D$, and α is the identity; this shows that $E = 1_{\mathfrak{D}} = 1_{\mathfrak{A}}$.

Thus the subuniverses of $\mathfrak{D}^2$ are $\triangle_B$, $1_{\mathfrak{B}}$, B^2, $B \times A$, $A \times B$, $\triangle_A, 1_{\mathfrak{A}}$, and A^2.

Now let $g : A^n \to A$ be a function preserving (simple) $1_{\mathfrak{D}}$ liftings; then g preserves B, and hence all subuniverses of $\mathfrak{D}^2$. For any elements $a, b \in \bigcup\limits_{u\in D} \overline{u}^n$, we have the following possibilities:

- $a, b \in B^n$ and $((\overline{a(1)} = \overline{b(1)})$ or $(\overline{a(1)} \neq \overline{b(1)}))$,
- $(a \in B^n$ and $b \notin B^n)$ or $(a \notin B^n$ and $b \in B^n)$,
- $a, b \notin B^n$ and $((\overline{a(1)} = \overline{b(1)})$ or $\overline{a(1)} \neq \overline{b(1)})$.

By checking for theses cases, we see that $\langle g(a), g(b)\rangle = g^{\mathfrak{D}^2}(\langle a(1), b(1)\rangle, \ldots, \langle a(n), b(n)\rangle)$ is an element of $Sg(\langle a(1), b(1)\rangle, \ldots, \langle a(n), b(n)\rangle) = H(a, b)$. By Corollary 2.2, g is term representable on classes; thus $\mathfrak{D}$ is $\star$-quasiprimal.

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