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A Surface which has a Family of Geodesics of Curvature

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Abstract. We will study a surface in $\boldsymbol{R}^3$ without any umbilical point such that the integral curves of some principal distribution are geodesics. In particular, the lines of curvature of such a surface will be characterized intrinsically and extrinsically: the semisurface structure of such a surface will be characterized in terms of local representation of the first fundamental form; the curvatures and the torsions of the lines of curvature as space curves will be characterized.

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1. Introduction

Let S be a surface in $\boldsymbol{R}^3$ and $\mathrm{Umb}(S)$ the set of umbilical points of S. Let $\mathcal{D}$ be a smooth one-dimensional distribution on $S \setminus \mathrm{Umb}(S)$ which gives a principal direction of $S \setminus \mathrm{Umb}(S)$ at each point. Such a distribution as $\mathcal{D}$ is called a *principal distribution* on S. Each integral curve of a principal distribution is called a *line of curvature* of S. Principal distributions are interesting objects of study from several viewpoints.

The behavior of principal distributions can be complicated around an isolated umbilical point. The *index* of an isolated umbilical point p_0 is a quantity in relation to the behavior of principal distributions around p_0 ([16, pp. 137]). It is known that an umbilical point of a surface with constant mean curvature which is not totally umbilical is isolated and has negative index ([16, pp. 139]). The same

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result holds on a special Weingarten surface ([15]). By Hopf-Poincaré's theorem, we see that if S is a compact, orientable surface of genus zero and with constant mean curvature, then S is a round sphere. The index of an isolated umbilical point on a Willmore surface is less than or equal to $1/2$ and this estimate is sharp ([11]). Thus, in contrast to a surface with constant mean curvature, a Willmore surface can have an isolated umbilical point with positive index. As is expected from such a difference between a surface with constant mean curvature and a Willmore surface, there exists a compact, orientable Willmore surface of genus zero different from any round sphere: any Willmore sphere is obtained from a complete minimal surface in $\boldsymbol{R}^3$ with finite total curvature such that each end is embedded and flat (see [14], [18]).

It is conjectured that the index of an isolated umbilical point on a surface in $\boldsymbol{R}^3$ is less than or equal to one. This is called the *index conjecture* or the *local Carathéodory's conjecture*. If the index conjecture is true, then by Hopf-Poincaré's theorem, we see that there exist at least two umbilical points on a compact, orientable surface of genus zero. Therefore if we can affirmatively solve the index conjecture, then we can also affirmatively solve Carathéodory's conjecture, which asserts that there exist at least two umbilical points on a compact, convex surface. The author discussed the index of an isolated umbilical point on the graph of a homogeneous polynomial of two variables in [1]–[4]. In addition, he discussed the index of an isolated umbilical point on a real-analytic or smooth surface in [6]–[8].

Let f be a smooth function on a domain D of $\boldsymbol{R}^2$ and set $\partial_{\bar{z}} := \big(\partial/\partial x + \sqrt{-1}\partial/\partial y\big)/2$. Then *Loewner's conjecture* for a positive integer $n \in \boldsymbol{N}$ asserts that if a vector field $\boldsymbol{V}_f^{(n)} := \mathrm{Re}(\partial_{\bar{z}}^n f)\partial/\partial x + \mathrm{Im}(\partial_{\bar{z}}^n f)\partial/\partial y$ has an isolated zero point, then its index with respect to $\boldsymbol{V}_f^{(n)}$ is less than or equal to n. Loewner's conjecture contains the index conjecture, i.e., Loewner's conjecture for $n = 2$ is equivalent to the index conjecture (see [20]). In [10], the author introduced and studied a conjecture which is different from Loewner's conjecture and contains the index conjecture.

In further study of the behavior of principal distributions, the author has interest in the relation between a pair of principal distributions and the first fundamental form of a surface. In [12], he studied the relation among the first fundamental form, principal distributions and principal curvatures. Let M be a smooth two-dimensional manifold and g a Riemannian metric on M. Let $\mathcal{D}_1$, $\mathcal{D}_2$ be two smooth one-dimensional distributions on M. A Riemannian manifold (M, g) equipped with $(\mathcal{D}_1, \mathcal{D}_2)$ is called a *semisurface* if $\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal to each other at any point of M with respect to g. If $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ is a semisurface, then a triplet $(g, \mathcal{D}_1, \mathcal{D}_2)$ is called a *semisurface structure* of M. For example, a surface S in $\boldsymbol{R}^3$ without any umbilical point is considered as a semisurface: the first fundamental form and two principal distributions on S form a semisurface structure of S. Let $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ be a semisurface. Suppose that M is oriented. For each point $p \in M$, there exist local coordinates (u, v) on a neighborhood U_p of p which satisfy $\partial/\partial u \in \mathcal{D}_1$ and $\partial/\partial v \in \mathcal{D}_2$, and give the orientation of M. Such coordinates are said to be *compatible with* $(\mathcal{D}_1, \mathcal{D}_2)$. The metric g and the curvature K are locally represented as

$$g = A^2du^2 + B^2dv^2, \qquad K = -\frac{1}{AB}\left\{\left(\frac{A_v}{B}\right)_v + \left(\frac{B_u}{A}\right)_u\right\},$$

respectively, where A and B are positive-valued. Suppose $K \neq 0$ on M. Then the *first Codazzi-Mainardi polynomial* P_{I} of $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ is defined by

$$P_{\mathrm{I}}(X_1, X_2) := c_{\mathrm{I}20}X_1^2 + c_{\mathrm{I}11}X_1X_2 + c_{\mathrm{I}02}X_2^2,$$

where

$$\begin{aligned} c_{\mathrm{I}20} &:= \frac{1}{AB}\{(\log|K|A^2)_v(\log B)_u - (\log B)_{uv}\}, \\ c_{\mathrm{I}11} &:= \frac{1}{AB}\{(\log|K|AB)_{uv} - 4(\log A)_v(\log B)_u\}, \\ c_{\mathrm{I}02} &:= \frac{1}{AB}\{(\log|K|B^2)_u(\log A)_v - (\log A)_{uv}\}. \end{aligned}$$

Notice that P_{I} is determined by the semisurface structure $(g, \mathcal{D}_1, \mathcal{D}_2)$ of M and does not depend on the choice of (u, v). Let S be a surface in $\boldsymbol{R}^3$ without any umbilical point. Then S has a semisurface structure which consists of the first fundamental form and two principal distributions $\mathcal{D}_1$, $\mathcal{D}_2$. Suppose that the Gaussian curvature K of S is nowhere zero and that S is oriented. Let P_{I} be the first Codazzi-Mainardi polynomial determined by the above semisurface structure of S and k_1, k_2 the principal curvatures of S corresponding to $\mathcal{D}_1$, $\mathcal{D}_2$, respectively. Then $P_{\mathrm{I}}(k_1, k_2) = 0$ on S ([12]). Therefore noticing $K = k_1k_2$, we see that if $P_{\mathrm{I}} \not\equiv 0$, then k_1 and k_2 are represented by $c_{\mathrm{I}20}$, $c_{\mathrm{I}11}$, $c_{\mathrm{I}02}$ and K. In [12], the author showed that if $P_{\mathrm{I},q} \equiv 0$ for any point q of a domain U of S, then U is determined by the semisurface structure and a pair of principal curvatures at an arbitrarily chosen point of U and that if $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ is a semisurface with nowhere zero curvature satisfying $P_{\mathrm{I},q} \equiv 0$ for any $q \in M$, then (M, g) can be locally and isometrically immersed in $\boldsymbol{R}^3$ so that $\mathcal{D}_1$ and $\mathcal{D}_2$ give two principal distributions. Hence we see that the relation $P_{\mathrm{I}}(k_1, k_2) = 0$ between the semisurface structure and a pair of principal curvatures motivates us to consider a surface S as a semisurface such that the first fundamental form and a pair of two principal distributions $\mathcal{D}_1$, $\mathcal{D}_2$ are connected by some good relation.

When we consider a surface without any umbilical point as a semisurface, we can consider the equations of Codazzi-Mainardi as a general representation of the good relation between the first fundamental form and a pair of principal distributions. The author expects that according to each property which a surface can have, we can have a more concrete representation than the equations of Codazzi-Mainardi, by describing an intrinsic characterization of lines of curvature (for example, in terms of local representation of the first fundamental form). Then we can find, as a fundamental object of study, a surface S which has a family of geodesics of curvature, i.e., a surface S such that the integral curves of some principal distribution on S are geodesics. The first purpose of the present paper is to characterize the lines of curvature of S intrinsically: we will characterize the semisurface structure of S in terms of local representation of the first fundamental form.

Let $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ be a *partially geodesic semisurface*, that is, a semisurface such that the integral curves of just one of $\mathcal{D}_1$ and $\mathcal{D}_2$ are geodesics. Suppose that the integral curves of $\mathcal{D}_2$ are geodesics. Then we can suppose $B \equiv 1$ on U_p. Suppose that the curvature of (M, g) is identically equal to zero. Then noticing that the curvature K is locally represented as $K = -A_{vv}/A$, we obtain $A_{vv} \equiv 0$ and therefore we can suppose $A(u, v) = \alpha(u)v + 1$, where α is a smooth function of one variable. In addition, noticing the equations of Gauss and Codazzi-Mainardi, we see that for each $p \in M$, there exists an isometric immersion of a neighborhood U_p of p into $\boldsymbol{R}^3$ such that $\mathcal{D}_1$ and $\mathcal{D}_2$ give principal distributions and then we see that the principal curvature k_1 corresponding to $\mathcal{D}_1$ is locally represented as $k_1(u, v) = f(u)/A(u, v)$, where f is of one variable and that the principal curvature corresponding to $\mathcal{D}_2$ is identically equal to zero. Suppose that the curvature of (M, g) is nowhere zero. Then it is possible that there exists no isometric immersion of a neighborhood U_p of $p \in M$ into $\boldsymbol{R}^3$ such that $\mathcal{D}_1$ and $\mathcal{D}_2$ give principal distributions, even if we replace U_p with a small one. A parallel curved surface is an example of a surface with a family of geodesics of curvature. A surface S is said to be *parallel curved* if there exists a plane P in $\boldsymbol{R}^3$ such that at each point of S, at least one principal direction is parallel to P. If S is parallel curved, then such a plane as P is called a *base plane* of S. For example, a surface of revolution is parallel curved and any plane orthogonal to an axis of rotation is a base plane. For a parallel curved surface without any umbilical point, a line of curvature which is not contained in any base plane is a geodesic. A *canonical* parallel curved surface S is represented as a disjoint union of plane curves which are congruent in $\boldsymbol{R}^3$ with one another and tangent to principal directions of S. These curves are geodesics of S. In [12], the author studied the semisurface structure of a canonical parallel curved surface: if $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ is a partially geodesic semisurface with nowhere zero curvature such that the metric g is locally represented as

$$g = \{1 + A_1(u)A_2(v)\}^2 du^2 + dv^2, \tag{1}$$

where A_1 and A_2 are smooth functions of one variable, then there exists an isometric immersion of a neighborhood of each point of M into $\boldsymbol{R}^3$ satisfying

(i) the image is a canonical parallel curved surface;

(ii) $\mathcal{D}_1$, $\mathcal{D}_2$ give principal distributions, and in addition, the semisurface structure of a canonical parallel curved surface without any umbilical point and with nowhere zero Gaussian curvature is characterized by local representation of the first fundamental form as in (1).

In the present paper, we will prove

Theorem 1. *Let $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ be a partially geodesic semisurface with nowhere zero curvature. Suppose that on some neighborhood of each point of M, there exist local coordinates (u, v) compatible with $(\mathcal{D}_1, \mathcal{D}_2)$ such that g is locally represented as*

$$g = A^2 du^2 + dv^2, \qquad A(u, v) = 1 + A_1(u)A_2(u, v), \tag{2}$$

where A_2 is a smooth function of two variables u, v satisfying $(A_2)_v = \sin(\alpha_1(u)+\alpha_2(v))$, and A_1, α_1, α_2 are smooth functions of one variable satisfying $A_1 > 0$ and $\alpha_1(u) + \alpha_2(v) \in (-\pi/2, \pi/2)$. Then (M, g) can be locally and isometrically immersed in $\boldsymbol{R}^3$ so that the following hold:

(a) *$\mathcal{D}_1$ and $\mathcal{D}_2$ give principal distributions;*

(b) *for principal curvatures k_1, k_2 corresponding to $\mathcal{D}_1$, $\mathcal{D}_2$, respectively, a pair (k_1, k_2) is locally represented by*

$$(k_1, k_2) = \left(\frac{A_1(u)\cos(\alpha_1(u)+\alpha_2(v))}{A},\ -\alpha_2'(v)\right) \tag{3}$$

up to a sign.

Remark. Noticing (1), we see that in Theorem 1, the image of a neighborhood of each point of M is a canonical parallel curved surface if and only if g can be locally represented as in (2) so that α_1 is constant.

In addition, we will prove

Theorem 2. *Let S be a surface in $\boldsymbol{R}^3$ without any umbilical point and with nowhere zero Gaussian curvature. Then the integral curves of some principal distribution on S are geodesics if and only if on some neighborhood of each point of S, there exist local coordinates (u, v) compatible with principal distributions such that the first fundamental form is locally represented as in* (2).

Remark. Theorem 1 and Theorem 2 give an intrinsic characterization of the lines of curvature of a surface S in $\boldsymbol{R}^3$ without any umbilical point and with nowhere zero Gaussian curvature such that the integral curves of some principal distribution on S are geodesics: by these theorems, the semisurface structure of such a surface as S is characterized in terms of local representation of the first fundamental form.

Let S be a surface in $\boldsymbol{R}^3$ without any umbilical point and with nowhere zero Gaussian curvature such that the integral curves of some principal distribution on S are geodesics. The second purpose of the present paper is to characterize the lines of curvature of S extrinsically: we will characterize the curvatures and the torsions of the lines of curvature of S as space curves. Since S exists in the space, it is natural that such a characterization is expected. In [5] and [9], the author presented an extrinsic characterization of the lines of curvature of a canonical parallel curved surface. Referring to this characterization, we will prove two theorems introduced below.

Theorem 3. *Let S be a surface in $\boldsymbol{R}^3$ without any umbilical point and with nowhere zero Gaussian curvature such that the integral curves of some principal distribution on S are geodesics.*

(a) *For each point p of S, there exists a neighborhood U_p of p in S such that the integral curves of the above principal distribution on U_p are plane curves congruent in $\boldsymbol{R}^3$ with one another;*

(b) *U_p is a canonical parallel curved surface if and only if the integral curves of the other principal distribution on U_p are plane curves;*

(c) *The curvature k and the torsion τ of each integral curve of the principal distribution in* (b) *as a space curve are locally represented as*

$$k = \frac{A_1(u)}{A}, \qquad \tau = \frac{\alpha_1'(u)}{A},$$

respectively, where A, A_1 and α_1 are as in Theorem 1.

Remark. The curvature of a plane curve in (a) of Theorem 3 is given by k_2 as in Theorem 1.

Theorem 4. *Let C_b and C_g be simple curves in $\boldsymbol{R}^3$ with a unique intersection p_0 satisfying the following:*

(i) *the curvature of C_b as a space curve is nowhere zero;*

(ii) *C_g is a plane curve with nowhere zero curvature such that the plane which contains C_g is perpendicular to C_b at p_0;*

(iii) *the curvature vector of C_b at p_0 is not tangent to C_g;*

(iv) *if $\boldsymbol{n}_0$ is a unit vector normal to C_b and C_g at p_0, then the scalar product of $\boldsymbol{n}_0$ and the curvature vector of C_b at p_0 is not equal to the scalar product of $\boldsymbol{n}_0$ and the curvature vector of C_g at p_0.*

Then there exists a surface S in $\boldsymbol{R}^3$ without any umbilical point and with nowhere zero Gaussian curvature satisfying the following:

(a) *S contains a neighborhood of p_0 in $C_b \cup C_g$ so that $S \cap C_b$ and $S \cap C_g$ are lines of curvature of S*;

(b) *the integral curves of some principal distribution on S are geodesics and $S \cap C_g$ is an integral curve of this distribution.*

In addition, if S' is such a surface as S, then there exists a surface S_0 as S contained in S and S'.

Remark. Theorem 3 and Theorem 4 give an extrinsic characterization of the lines of curvature of a surface S as in Theorem 3: by these theorems, the curvatures and the torsions of the lines of curvature of S as space curves are characterized.

Remark. Let C_b and C_g be as in Theorem 4 and suppose that C_b is a plane curve. Then a pair (C_b, C_g) is called a *generating pair*; C_b and C_g are called a *base curve* and a *generating curve* of (C_b, C_g), respectively. For a generating pair (C_b, C_g), a surface S as in Theorem 4 can be a canonical parallel curved surface, and any canonical parallel curved surface is determined by some generating pair.

Remark. If condition (iii) in Theorem 4 is removed, then the Gaussian curvature of S at p_0 can be equal to zero; if condition (iv) in Theorem 4 is removed, then p_0 can be an umbilical point of S.

2. Semisurfaces

Let $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ be a semisurface with nowhere zero curvature K and P_{I} the first Codazzi-Mainardi polynomial of $(M, g, \mathcal{D}_1, \mathcal{D}_2)$. Suppose that $P_{\mathrm{I},q} \not\equiv 0$ for any $q \in M$ and that there exist smooth, real-valued functions k_1, k_2 on M satisfying $k_1 k_2 = K$ and $P_{\mathrm{I}}(k_1, k_2) = 0$. Then k_1 and k_2 are represented by $c_{\mathrm{I}20}$, $c_{\mathrm{I}11}$, $c_{\mathrm{I}02}$ and K. A Riemannian manifold (M, g) can be locally and isometrically immersed in $\boldsymbol{R}^3$ so that $\mathcal{D}_1$ and $\mathcal{D}_2$ give principal distributions if and only if k_1 and k_2 satisfy the equations of Codazzi-Mainardi:

$$(k_1)_v = -(\log A)_v (k_1 - k_2), \qquad (k_2)_u = (\log B)_u (k_1 - k_2). \tag{4}$$

If k_1 and k_2 satisfy (4), then k_1 and k_2 become principal curvatures corresponding to $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Suppose $P_{\mathrm{I},q} \equiv 0$ for any $q \in M$. Then for each point $p \in M$, there exists a neighborhood U_p of p which can be immersed in the above manner and in addition, for real numbers $k_1^{(0)}$, $k_2^{(0)}$ satisfying $k_1^{(0)} k_2^{(0)} = K(p)$, there exists a unique pair (k_1, k_2) of smooth functions on U_p such that k_1 and k_2 can be principal curvatures of some image of U_p corresponding to $\mathcal{D}_1$, $\mathcal{D}_2$, respectively, satisfying $k_i(p) = k_i^{(0)}$ for $i = 1, 2$ ([12]).

Let S be a surface in $\boldsymbol{R}^3$ without any umbilical point and with $K \neq 0$.

(i) A neighborhood of each point of S is a canonical parallel curved surface if and only if S satisfies $P_{\mathrm{I},q} \equiv 0$ for any $q \in S$ and the condition that the integral curves of some principal distribution are geodesics ([12]). These conditions hold if and only if the first fundamental form is locally represented as in (1) ([12]). Noticing the condition $P_{\mathrm{I}} \equiv 0$, we see that there exist plural canonical parallel curved surfaces which have the same semisurface structure. Kishimura presented a characterization of the relation between generating pairs of such two surfaces ([17]).

(ii) Suppose $P_{\mathrm{I},q} \not\equiv 0$ for any $q \in S$. Then S has constant mean curvature H_0 if and only if on a neighborhood of each point of S, there exist isothermal coordinates (u, v) compatible with principal distributions and a positive-valued function $\tilde{A}$ satisfying

$$g = \tilde{A}^2(du^2 + dv^2), \qquad \Delta \log \tilde{A} + H_0^2 - \frac{c_0^2}{\tilde{A}^4} = 0,$$

where $c_0 > 0$ (see [12]). If $P_{\mathrm{I},q} \equiv 0$ for any $q \in S$, then it is possible that the mean curvature of S is not constant, even though there exist (u, v) and $\tilde{A}$ as above ([12]).

Remark. Let S be a surface in $\boldsymbol{R}^3$ without any umbilical point. In [13], we studied another semisurface structure of S than the semisurface structure given by two principal distributions. For each $p \in S$, there exist just two one-dimensional subspaces L_1, L_2 of $T_p(S)$ such that the normal curvature of S at p with respect to L_i is equal to the mean curvature $H(p)$ of S at p. We call L_i an *H-direction* of S at p. There exist two smooth one-dimensional distributions $\mathcal{D}_1$, $\mathcal{D}_2$ which give the two H-directions of S at each point. We call $\mathcal{D}_i$ an *H-distribution* on S. We see that the first fundamental form g and two H-distributions $\mathcal{D}_1$, $\mathcal{D}_2$ on S form a semisurface structure of S. Suppose that S is oriented. Let (u, v) be local coordinates which are compatible with $(\mathcal{D}_1, \mathcal{D}_2)$ and give the orientation of S. We set

$$\boldsymbol{U}_1 := \frac{1}{A}\frac{\partial}{\partial u}, \qquad \boldsymbol{U}_2 := \frac{1}{B}\frac{\partial}{\partial v},$$

and

$$V_K := -\boldsymbol{U}_1(\log B)\boldsymbol{U}_1 - \boldsymbol{U}_2(\log A)\boldsymbol{U}_2.$$

We see that V_K does not depend on the choice of (u, v). Therefore V_K is a vector field well-defined on S and determined by the semisurface structure $(g, \mathcal{D}_1, \mathcal{D}_2)$. Then the equations of Codazzi-Mainardi are represented as follows ([13]):

$$2K\mathrm{grad}(H) = \mathrm{W}\big(\mathrm{grad}(K) + 4\big(H^2 - K\big)V_K\big), \tag{5}$$

where W is the Weingarten map (shape operator). We see that V_K satisfies

$$\mathrm{div}(V_K) = K, \qquad \mathrm{rot}(V_K) = \frac{1}{AB}(\log(B/A))_{uv},$$

where $\mathrm{div}(V_K)$ and $\mathrm{rot}(V_K)$ are the divergence and the rotation of V_K, respectively. In particular, we see that (u, v) can be isothermal coordinates if and only if $\mathrm{rot}(V_K) = 0$ holds and that if we denote by Ω the area element of S, then $\mathrm{rot}(V_K)\Omega$ is determined by $\mathcal{D}_1$, $\mathcal{D}_2$ and the conformal class of g. Computing the rotations of the both sides of (5), we see that if $K \neq 0$, then $P_{\mathrm{II}}\big(H, \sqrt{H^2 - K}\big) = 0$, where P_{II} is defined by

$$\begin{aligned}
P_{\mathrm{II}}(Y_1, Y_2) &:= c_{\mathrm{II}20}Y_1^2 + c_{\mathrm{II}11}Y_1Y_2 + c_{\mathrm{II}02}Y_2^2, \\
c_{\mathrm{II}20} :=&\ -\frac{1}{2}\{\boldsymbol{U}_1\boldsymbol{U}_1(\log|K|) - \boldsymbol{U}_2\boldsymbol{U}_2(\log|K|)\} \\
&\ -\frac{3}{2}\{\boldsymbol{U}_1(\log|K|)\boldsymbol{U}_1(\log B) - \boldsymbol{U}_2(\log|K|)\boldsymbol{U}_2(\log A)\}, \\
c_{\mathrm{II}11} :=&\ -2\mathrm{rot}(V_K) - 2\{\boldsymbol{U}_1(\log|K|)\boldsymbol{U}_2(\log A) - \boldsymbol{U}_2(\log|K|)\boldsymbol{U}_1(\log B)\}, \\
c_{\mathrm{II}02} :=&\ \frac{1}{2}\big\{\boldsymbol{U}_1\boldsymbol{U}_1\big(\log|K|B^4\big) - \boldsymbol{U}_2\boldsymbol{U}_2\big(\log|K|A^4\big) \\
&\ -\boldsymbol{U}_1\big(\log|K|B^4\big)\boldsymbol{U}_1(\log B) + \boldsymbol{U}_2\big(\log|K|A^4\big)\boldsymbol{U}_2(\log A)\big\}
\end{aligned}$$

and determined by the semisurface structure of S given by H-distributions $\mathcal{D}_1$, $\mathcal{D}_2$ ([13]). We call P_{II} the *second Codazzi-Mainardi polynomial*. Let (M, g) be an oriented two-dimensional Riemannian manifold with $K \neq 0$. Let $(g, \mathcal{D}_1^+, \mathcal{D}_2^+)$, $(g, \mathcal{D}_1^\times, \mathcal{D}_2^\times)$ be two semisurface structures of M such that the angle between $\mathcal{D}_i^\times$ and $\mathcal{D}_i^+$ is equal to $\pi/4$ at any point of M. Let P_{I}^+ (respectively, $P_{\mathrm{II}}^\times$) be the first (respectively, second) Codazzi-Mainardi polynomial of $(M, g, \mathcal{D}_1^+, \mathcal{D}_2^+)$ (respectively, $(M, g, \mathcal{D}_1^\times, \mathcal{D}_2^\times)$). Then $P_{\mathrm{I}}^+(X_1, X_2) = P_{\mathrm{II}}^\times(Y_1, Y_2)$ holds, where X_i, $Y_i \in \boldsymbol{R}$ satisfy $X_1 = Y_1 + Y_2$ and $X_2 = Y_1 - Y_2$ ([13]). In particular, if M is an oriented surface S in $\boldsymbol{R}^3$ without any umbilical point and with $K \neq 0$ and if

$\mathcal{D}_i^+$ and $\mathcal{D}_i^\times$ are a principal distribution and an H-distribution on S, respectively, then $P_{\mathrm{I}}^+(k_1, k_2) = P_{\mathrm{II}}^\times\big(H, \sqrt{H^2 - K}\big)(= 0)$, where k_i is a principal curvature of S corresponding to $\mathcal{D}_i^+$ $(i = 1, 2)$.

3. Partially geodesic semisurfaces

Let $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ be a partially geodesic semisurface. Then we can suppose that for each $p \in M$, there exist local coordinates (u, v) on a neighborhood U_p of p compatible with $(\mathcal{D}_1, \mathcal{D}_2)$ satisfying $g = A^2du^2 + dv^2$ on U_p. The integral curves of $\mathcal{D}_2$ are geodesics. The curvature K is locally represented as $K = -A_{vv}/A$. Suppose $K \neq 0$ on M. Then $c_{\mathrm{I}20}$, $c_{\mathrm{I}11}$ and $c_{\mathrm{I}02}$ are represented as follows:

$$c_{\mathrm{I}20} = 0, \quad c_{\mathrm{I}11} = \frac{1}{A}(\log|K|A)_{uv}, \quad c_{\mathrm{I}02} = \frac{1}{A}\{(\log A)_v(\log|K|)_u - (\log A)_{uv}\}. \tag{6}$$

Therefore the first Codazzi-Mainardi polynomial of $(M, g, \mathcal{D}_1, \mathcal{D}_2)$ is represented as

$$\begin{aligned} P_{\mathrm{I}}(X_1, X_2) = {} & \frac{1}{A}(\log|A_{vv}|)_{uv}X_1X_2 \\ & + \frac{1}{A}\{(\log A)_v(\log|A_{vv}|)_u - (\log A)_u(\log A)_v - (\log A)_{uv}\}X_2^2. \end{aligned} \tag{7}$$

Proof of Theorem 2. Let S be a surface in $\boldsymbol{R}^3$ without any umbilical point and with nowhere zero Gaussian curvature K. Let $\mathcal{D}_1$, $\mathcal{D}_2$ be two principal distributions on S orthogonal to each other at any point. Suppose that the integral curves of $\mathcal{D}_2$ are geodesics. Then on a neighborhood of each point of S, there exist local coordinates (u, v) compatible with $(\mathcal{D}_1, \mathcal{D}_2)$ such that the first fundamental form g is locally represented as $g = A^2du^2 + dv^2$. Let k_1, k_2 be principal curvatures of S corresponding to $\mathcal{D}_1$, $\mathcal{D}_2$, respectively. Then k_1 and k_2 satisfy the equations of Gauss and Codazzi-Mainardi:

$$k_1k_2 = K = -\frac{A_{vv}}{A}, \qquad (k_1)_v = -(\log A)_v(k_1 - k_2), \qquad (k_2)_u = 0. \tag{8}$$

From the third relation, we see that k_2 is of one variable v. Noticing $K \neq 0$, by the first and the second relations in (8), we obtain

$$A_{vv}k_2' = A_vk_2^3 + A_{vvv}k_2.$$

This implies

$$\left\{A_v^2 + \left(\frac{A_{vv}}{k_2}\right)^2\right\}_v = 0.$$

Therefore we see that there exists a smooth, positive-valued function A_1 of one variable u satisfying

$$A_v^2 + \left(\frac{A_{vv}}{k_2}\right)^2 = A_1(u)^2.$$

This implies

$$k_2(v)^2 = \frac{A_{vv}^2}{A_1(u)^2 - A_v^2}. \tag{9}$$

There exists a smooth function θ of two variables u, v satisfying $\theta \in (-\pi/2, \pi/2)$ and $A_v = A_1(u)\sin\theta$. Then $A_{vv} = A_1(u)(\cos\theta)\theta_v$ holds. Therefore by (9), we obtain $\theta_v^2 = k_2(v)^2$. We suppose $\theta_v k_2(v) < 0$. Then $\theta_v = -k_2(v)$ holds. By this together with $\theta = \arcsin(A_v/A_1(u))$, we obtain

$$\arcsin\left(\frac{A_v}{A_1(u)}\right) = \alpha_1(u) + \alpha_2(v),$$

where α_1 and α_2 are smooth functions of one variable u, v, respectively, satisfying $\alpha_2' = -k_2$. Therefore we obtain

$$A_v = A_1(u)\sin(\alpha_1(u) + \alpha_2(v)).$$

This implies

$$A(u, v) = A_1(u)A_2(u, v) + f(u),$$

where A_2 is a smooth function of two variables u, v satisfying $(A_2)_v = \sin(\alpha_1(u) + \alpha_2(v))$ and f is a smooth function of one variable u. Since $A_1 \neq 0$, A can be represented as

$$A = 1 + A_1(u)\{A_2(u, v) + (f(u) - 1)/A_1(u)\}.$$

Since

$$\{A_2(u, v) + (f(u) - 1)/A_1(u)\}_v = \sin(\alpha_1(u) + \alpha_2(v)),$$

we see that g is locally represented as in (2). It is clear that if on a neighborhood of each point of S, there exist local coordinates (u, v) compatible with $(\mathcal{D}_1, \mathcal{D}_2)$ such that g is locally represented as in (2), then the integral curves of $\mathcal{D}_2$ are geodesics. Hence we obtain Theorem 2. □

Remark. In the proof of Theorem 2, by (9) together with $k_1k_2 = -A_{vv}/A$, we obtain

$$k_1^2 = \frac{A_1(u)^2 - A_v^2}{A^2}.$$

Therefore we see that a pair (k_1, k_2) is locally represented by

$$(k_1, k_2) = \left(\frac{\sqrt{A_1(u)^2 - A_v^2}}{A}, \; -\frac{A_{vv}}{\sqrt{A_1(u)^2 - A_v^2}}\right) \tag{10}$$

up to a sign. If A is as in (2), then from (10), we see that (k_1, k_2) is locally represented as in (3).

Proof of Theorem 1. Noticing the above remark, we set (k_1, k_2) as in (3). Then we see that k_1 and k_2 satisfy (8). Therefore by the fundamental theorem of the theory of surfaces, we obtain Theorem 1. □

Remark. Let (u, v) be local coordinates compatible with $(\mathcal{D}_1, \mathcal{D}_2)$ such that g is locally represented as in (2). Then from (6), we obtain

$$c_{\mathrm{I}11} = -\frac{\alpha_1'(u)\alpha_2'(v)}{A\cos^2(\alpha_1(u) + \alpha_2(v))}, \qquad c_{\mathrm{I}02} = -\frac{A_1(u)\alpha_1'(u)}{A^2\cos(\alpha_1(u) + \alpha_2(v))}.$$

Therefore we can rewrite (7) into

$$P_{\mathrm{I}}(X_1, X_2) \\ = -\frac{\alpha_1'(u)X_2}{A^2\cos^2(\alpha_1(u)+\alpha_2(v))}\{A\alpha_2'(v)X_1 + A_1(u)\cos(\alpha_1(u)+\alpha_2(v))X_2\}. \tag{11}$$

In particular, we see that if $P_{\mathrm{I}} \not\equiv 0$, i.e., if $\alpha_1' \neq 0$, then (k_1, k_2) satisfying $k_1k_2 = K$ and $P_{\mathrm{I}}(k_1, k_2) = 0$ is uniquely determined by the semisurface structure $(g, \mathcal{D}_1, \mathcal{D}_2)$ up to a sign and locally represented as in (3).

Proof of Theorem 3. Suppose that the integral curves of a principal distribution $\mathcal{D}_2$ on S are geodesics. Then we see that the absolute value of the principal curvature k_2 corresponding to $\mathcal{D}_2$ is the curvature of each integral curve as a space curve. From the Weingarten formula, we see that if we denote by $\boldsymbol{n}$ a unit normal vector field, then $\boldsymbol{n}_v = -k_2\partial/\partial v$ holds. Comparing this with the Frenet-Serret formula, we see that the torsion of any integral curve of $\mathcal{D}_2$ as a space curve is identically equal to zero. This implies that the integral curves of $\mathcal{D}_2$ are plane curves. As we saw in the proof of Theorem 2, k_2 is of one variable v. Therefore we obtain (a) of Theorem 3. If S is a canonical parallel curved surface, then the integral curves of another principal distribution $\mathcal{D}_1$ than $\mathcal{D}_2$ are plane curves. Suppose that the integral curves of $\mathcal{D}_1$ are plane curves. Then we can suppose that S is locally represented as the image by an immersion Φ of an open rectangular set of $\boldsymbol{R}^2$ into $\boldsymbol{R}^3$ in the following form:

$$\Phi(u,v) := \begin{pmatrix} \phi_1(u) \\ \phi_2(u) \\ 0 \end{pmatrix} + \phi_3(v)\boldsymbol{x}(u) + \phi_4(v)\boldsymbol{y}(u),$$

where

$$\boldsymbol{x}(u) := (\cos\theta(u))\begin{pmatrix} \phi_2'(u) \\ -\phi_1'(u) \\ 0 \end{pmatrix} + (\sin\theta(u))\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\boldsymbol{y}(u) := -(\sin\theta(u))\begin{pmatrix} \phi_2'(u) \\ -\phi_1'(u) \\ 0 \end{pmatrix} + (\cos\theta(u))\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and θ, ϕ_1, ϕ_2, ϕ_3, ϕ_4 are smooth functions of one variable satisfying

(i) $(\phi_1')^2 + (\phi_2')^2 \equiv 1$, $(\phi_3')^2 + (\phi_4')^2 \equiv 1$;

(ii) (u, v) are compatible with $(\mathcal{D}_1, \mathcal{D}_2)$.

Then the following holds:

$$0 = \Phi_u(u,v)\cdot\Phi_v(u,v) = \theta'(u)\{\phi_3(v)\phi_4'(v) - \phi_3'(v)\phi_4(v)\}.$$

If $\phi_3\phi_4' - \phi_3'\phi_4 \equiv 0$, then k_2 is identically equal to zero. This implies $K = 0$, which causes a contradiction. Therefore $\theta' \equiv 0$ holds and θ is constant. Then we see that the image by Φ is a canonical parallel curved surface. Hence we obtain (b) of Theorem 3. In the following, we suppose that S is the image by an immersion Φ of a domain D of $\boldsymbol{R}^2$ into $\boldsymbol{R}^3$ which is not necessarily in the above form and that coordinates (u,v) on $\boldsymbol{R}^2$ are compatible with $(\mathcal{D}_1,\mathcal{D}_2)$. We set $\boldsymbol{n} := \Phi_u \times \Phi_v/|\Phi_u \times \Phi_v|$. Let g be the metric on D induced by Φ. Then we can suppose $g = A^2du^2 + dv^2$ on D, where $A > 0$. We denote by $\boldsymbol{e}_1$ the vector $(1/A)\Phi_u$ in $\boldsymbol{R}^3$ or the tangent vector $(1/A)\partial/\partial u$ to D. Let l_1 be the geodesic curvature of each integral curve of $\mathcal{D}_1$ in (D,g). Then the following holds:

$$l_1\frac{\partial}{\partial v} = \nabla_{\boldsymbol{e}_1}\boldsymbol{e}_1 = -(\log A)_v\frac{\partial}{\partial v},$$

where ∇ is the covariant differentiation with respect to the Levi-Civita connection of (D,g). Therefore we obtain $l_1 = -(\log A)_v$. Let k be the curvature of each integral curve of $\mathcal{D}_1$ as a space curve. Then noticing $k^2 = k_1^2 + l_1^2$ and (3), we obtain $k = A_1(u)/A$. We set

$$\boldsymbol{e}_2 := \frac{1}{k}(l_1\Phi_v + k_1\boldsymbol{n}), \qquad \boldsymbol{e}_3 := \boldsymbol{e}_1 \times \boldsymbol{e}_2.$$

Let τ be the torsion of each integral curve of $\mathcal{D}_1$ as a space curve. Then noticing the Frenet-Serret formula, we obtain

$$-k\boldsymbol{e}_1 + \tau\boldsymbol{e}_3 = \frac{1}{A}(\boldsymbol{e}_2)_u = \frac{1}{A}(-A_1(u)\boldsymbol{e}_1 + \alpha_1'(u)\boldsymbol{e}_3).$$

Therefore we obtain $\tau = \alpha_1'(u)/A$. Hence we obtain (c) of Theorem 3. $\square$

Remark. Noticing (11) and $\tau = \alpha_1'(u)/A$, we see that $P_{\mathrm{I},q} \equiv 0$ is equivalent to $\tau(q) = 0$.

Remark. By the definition of $\boldsymbol{e}_2$ in the proof of Theorem 3, we see that the angle between $\boldsymbol{e}_2$ and $\boldsymbol{n}$ is equal to $|\alpha_1(u) + \alpha_2(v)|$ or $\pi - |\alpha_1(u) + \alpha_2(v)|$.

Proof of Theorem 4. Let γ_b be a smooth map of an open interval I_b into $\boldsymbol{R}^3$ satisfying $\gamma_b(I_b) = C_b$ and $|\gamma_b'(u)| = 1$ for any $u \in I_b$. Let γ_g be a smooth map of an open interval I_g into $\boldsymbol{R}^3$ satisfying $\gamma_g(I_g) = C_g$ and $|\gamma_g'(v)| = 1$ for any $v \in I_g$. We suppose that both I_b and I_g contain 0 and that γ_b and γ_g satisfy $\gamma_b(0) = \gamma_g(0) = p_0$. We set $\boldsymbol{n}_0 := \gamma_b'(0)\times\gamma_g'(0)$. Let $\theta_0 \in [0,\pi]\setminus\{\pi/2\}$ be the angle between $\gamma_b''(0)$ and $\boldsymbol{n}_0$. We can suppose $\theta_0 \in [0,\pi/2)$. Let A_1, τ be the curvature and the torsion of C_b, respectively. Let k_2 be the curvature of C_g. Then there exists a unique smooth function α_1 on I_b satisfying $\alpha_1(0) = \theta_0$ and $\alpha_1'(u) = \tau(u)$ for any $u \in I_b$, and there exists a unique smooth function α_2 on I_g satisfying $\alpha_2(0) = 0$ and $\alpha_2'(v) = -k_2(v)$ for any $v \in I_g$. In addition, there exists a unique smooth function A_2 on $I_b \times I_g$ satisfying $A_2(u,0) = 0$ and $(A_2)_v(u,v) = \sin(\alpha_1(u) + \alpha_2(v))$ for any $u \in I_b$ and any $v \in I_g$. We set $A(u,v) := 1 + A_1(u)A_2(u,v)$. Let $(M,g,\mathcal{D}_1,\mathcal{D}_2)$ be a semisurface defined by

$$M = I_b \times I_g, \qquad g = A^2du^2 + dv^2, \qquad \frac{\partial}{\partial u} \in \mathcal{D}_1, \qquad \frac{\partial}{\partial v} \in \mathcal{D}_2.$$

Then noticing Theorem 1, Theorem 3 and the fundamental theorem of the theory of curves, we see that there exist a neighborhood U_0 of $(0,0)$ in M and an isometric immersion Φ of U_0 into $\boldsymbol{R}^3$ such that $S := \Phi(U_0)$ is as in Theorem 4. Noticing Theorem 2 and the fundamental theorem of the theory of surfaces, we see that if S' is such a surface as S, then there exists a surface S_0 as S contained in S and S'. Hence we have proved Theorem 4. □

Remark. Let S be a surface in $\boldsymbol{R}^3$ without any umbilical point and with everywhere zero Gaussian curvature such that the integral curves of a principal distribution $\mathcal{D}_2$ on S are geodesics. Then we can suppose that the integral curves of $\mathcal{D}_2$ are straight lines in $\boldsymbol{R}^3$. The first fundamental form g of S is locally represented as $g = (1+\alpha(u)v)^2du^2 + dv^2$. The principal curvature k_1 of S corresponding to another principal distribution $\mathcal{D}_1$ than $\mathcal{D}_2$ is locally represented as $k_1 = f(u)/(1+\alpha(u)v)$. Then referring to the proof of (c) of Theorem 3, we see that the curvature and the torsion of each integral curve of $\mathcal{D}_1$ as a space curve are represented by α and f. Suppose that the integral curves of $\mathcal{D}_1$ are plane curves. Then referring to the proof of (b) of Theorem 3, we can suppose that S is locally represented as the image by an immersion Φ of an open rectangular set of $\boldsymbol{R}^2$ into $\boldsymbol{R}^3$ in the following form:

$$\Phi(u,v) := \begin{pmatrix} \phi_1(u) \\ \phi_2(u) \\ 0 \end{pmatrix} + v\left\{(\cos\theta(u))\begin{pmatrix} \phi_2'(u) \\ -\phi_1'(u) \\ 0 \end{pmatrix} + (\sin\theta(u))\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\},$$

where θ, ϕ_1, ϕ_2 are smooth functions of one variable satisfying

(i) $(\phi_1')^2 + (\phi_2')^2 \equiv 1$;

(ii) (u,v) are compatible with $(\mathcal{D}_1, \mathcal{D}_2)$.

Then $\Phi_u \cdot \Phi_v = 0$ always holds. In addition, noticing that (u,v) are compatible with principal distributions, we see that $\Phi_{uv} \cdot (\Phi_u \times \Phi_v) = 0$ must hold and this condition is equivalent to $\theta' \equiv 0$. Therefore we see that a neighborhood of each point of S is a canonical parallel curved surface. Hence we obtain an analogue of Theorem 3 for a surface without any umbilical point and with everywhere zero Gaussian curvature such that the integral curves of some principal distribution are geodesics. Referring to the proof of Theorem 4, we can obtain an analogue of Theorem 4 for such a surface.

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