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Crofton Measures in Polytopal Hilbert Geometries

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Abstract. The Hilbert geometry in an open bounded convex set in $\mathbb{R}^n$ is a classical example of a projective Finsler space. We construct explicitly a positive measure on the space of lines in a polytopal Hilbert geometry which yields an integral geometric representation of Crofton type for the Holmes-Thompson area of hypersurfaces.

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1. Introduction

There have recently been efforts to extend classical integral-geometric Crofton formulas to projective Finsler spaces. This was partly motivated by the integral-geometric approach to Hilbert's fourth problem, and by Busemann's generalization of the problem, asking for notions of areas in affine spaces for which flats are minimizing. Another impetus came from the work of Gelfand and Smirnov [10] on Crofton densities, aiming at bridging the gap between the integral geometries of Blaschke-Chern-Santaló on one side and that of Radon type transforms on the other.

A Crofton formula in $\mathbb{R}^n$, in a simple case, is of the form

$$\int_{A(n,n-k)} \operatorname{card}(E \cap M)\, \eta_{n-k}(\mathrm{d}E) = \operatorname{vol}_k(M). \tag{1}$$

Here $k \in \{1, \ldots, n-1\}$, $A(n, j)$ is the affine Grassmannian of j-dimensional flats (affine subspaces) in $\mathbb{R}^n$, and M is a k-dimensional convex set (we restrict ourselves to this special case, since the generality of the admissible subsets M is not an issue here). The classical version of formula (1) is the case where $\mathbb{R}^n$ is equipped with a Euclidean metric, vol_k denotes the k-dimensional Euclidean volume, and η_{n-k} is the rigid motion invariant measure on $A(n, n-k)$, with a suitable normalization. Recent investigations are concerned with the case where $\mathbb{R}^n$ or an open convex subset thereof is endowed with a projective Finsler metric and vol_k is the k-dimensional Holmes-Thompson area. For smooth projective Finsler spaces, it was shown by Álvarez and Fernandes [4], and with different approaches in [5, 7] (see also the surveys [6] and [8]), that a signed measure η_{n-k} exists so that (1) holds for the Holmes-Thompson area vol_k. Such a signed measure is called a Crofton measure for vol_k. The line measure η_1 is always positive, but generally not the measure η_j for $j = 2, \dots, n-1$. If the metric induced by the Finsler metric is a hypermetric, then all the measures η_j, $j = 1, \dots, n-1$, are positive. For the case of hypermetric Minkowski spaces (a Minkowski space is here a finite dimensional real normed space), the existence of positive Crofton measures for the Holmes-Thompson area was already proved in [15]. No smoothness assumption on the norm was made in that case. As a consequence, the obtained Crofton measures in general do not have densities (with respect to a Haar measure, say). For example, the line measure in a Minkowski space with a polytopal norm is concentrated on a lower-dimensional subset of $A(n, 1)$. The existence of positive Crofton measures for general hypermetric projective Finsler spaces, without smoothness assumptions, was proved in [12]. There it was further shown that a positive line measure, that is, a positive Crofton measure for vol_{n-1}, exists also without the hypermetric assumption. The employed approximation method ensures only the existence and does not yield an explicit description of the measures obtained.

In the present paper, we will explicitly construct the positive line measure for the Holmes-Thompson $(n-1)$-area in an n-dimensional Hilbert geometry in the interior of a convex polytope. Hilbert geometries are, besides Minkowski spaces, the second classical example of complete projective metrics, already mentioned (though not under this name) by Hilbert in the formulation of his fourth problem. The line measure in a polytopal Hilbert geometry shows similar features to that in a Minkowski space with a polytopal unit ball. For Hilbert geometries in planar polygons, the line measure was described briefly by Alexander [1] and more explicitly by Alexander, Berg and Foote [3]. The higher-dimensional case requires a different approach. Our constructed line measure is given by (16) together with (13); the details are explained below.

2. Preliminaries

We work in $\mathbb{R}^n$ ($n \geq 2$), where we choose a scalar product $\langle \cdot, \cdot \rangle$, with induced norm $|\cdot|$. The results will not depend on the choice of this auxiliary Euclidean structure, but its availability makes some calculations easier.

In the following, $P \subset \mathbb{R}^n$ is a fixed convex polytope, with nonempty interior denoted by C. For $x, y \in C$, $x \neq y$, let a, b be the points where the line through x and y intersects the boundary of C, such that x is between a and y. Then

$$d(x, y) := \ln \frac{|x-b||y-a|}{|y-b||x-a|} \tag{2}$$

defines a metric d on C. It is projective, or linearly additive, which means that $d(x,z) = d(x,y) + d(y,z)$ if x, y, z are on a line and y is between x and z. The pair (C, d) is the *Hilbert geometry* in C.

There is a Finsler metric inducing the metric d. A (generalized) Finsler metric on C is a continuous function $F : C \times \mathbb{R}^n \to [0, \infty)$ such that $F(x, \cdot) =: \| \cdot \|_x$ is a norm on $\mathbb{R}^n$ for each $x \in C$. The Finsler metric inducing d (which means that $d(x, y)$ is the infimum of the Finsler lengths of all piecewise C^1 curves connecting x and y) is given by

$$F(x,u) := h((P-x)^o, u) + h((P-x)^o, -u), \tag{3}$$

where h denotes the support function and $(P-x)^o$ is the polar body of $P-x$, the translate of P by the vector x. (The easy calculation can be found, e.g., in the survey article [14].) For each $x \in C$, the norm $F(x, \cdot) = \| \cdot \|_x$ has the unit ball

$$B_x := \{u \in \mathbb{R}^n : \|u\|_x \le 1\},$$

and its polar is

$$B_x^o := \{\xi \in \mathbb{R}^n : \langle \xi, u \rangle \le 1 \, \forall \, u \in B_x\}.$$

Writing $(P-x)^o =: P^x$, we see from (3) and the general relation $\| \cdot \|_x = h(B_x^o, \cdot)$ that

$$B_x^o = P^x - P^x = \mathrm{D}P^x, \tag{4}$$

where D denotes the difference body operator.

For an introduction to the Holmes-Thompson area in Minkowski spaces, we refer to [16], and for the Holmes-Thompson area in Finsler spaces to the survey [8]; see also [13]. Holmes-Thompson areas appearing below always refer to the projective Finsler space (C, F).

In the following, we denote by $\mathcal{M}$ the set of all $(n-1)$-dimensional relatively open bounded convex sets contained in C. This is a sufficiently rich class of submanifolds for studying Crofton measures for the Holmes-Thompson $(n-1)$-area. For $M \in \mathcal{M}$, we denote by TM the linear subspace parallel to the affine hull of M and by u_M one of the two unit normal vectors of TM.

For $M \in \mathcal{M}$, the Holmes-Thompson $(n-1)$-area of M can be represented in the form

$$\mathrm{vol}_{n-1}(M) = \frac{1}{\kappa_{n-1}} \int_M \lambda_{n-1}(B_x^o|TM) \, \lambda_{n-1}(\mathrm{d}x), \tag{5}$$

where λ_{n-1} is the $(n-1)$-dimensional Lebesgue measure and $|TM$ denotes orthogonal projection to TM. The constant κ_{n-1} is the $(n-1)$-volume of the $(n-1)$-dimensional Euclidean unit ball. The representation (5) involves the auxiliary Euclidean structure in several ways, but $\mathrm{vol}_{n-1}(M)$ is independent of its choice.

A *Crofton measure* for vol_{n-1} is a signed measure η_1 on the Borel sets of the space $A(n, 1)$ of lines which is locally finite (that is, finite on compact sets) and satisfies

$$\int_{A(n,1)} \mathrm{card}(L \cap M) \, \eta_1(\mathrm{d}L) = \mathrm{vol}_{n-1}(M) \tag{6}$$

for all $M \in \mathcal{M}$. There is at most one such signed measure (see [12]). In the next section, we construct a positive Crofton measure for vol_{n-1}.

We will need conjugate faces of polar polytopes. Let $Q \subset \mathbb{R}^n$ be a polytope with 0 in its interior, so that the polar polytope Q^o is defined. Let G be a face of Q^o and u a unit vector such that $G = F(Q^o, u)$ (as usual, $F(K, u)$ denotes the support set of the convex body K with outer normal vector u; there is no danger of confusion with the Finsler metric). The ray $\{\lambda u : \lambda \geq 0\}$ meets the boundary of Q in a point which is in the relative interior of a unique face F of Q (depending only on G, for fixed Q); we denote this face by $R(Q, u)$. It is independent of the choice of the vector u with $G = F(Q^o, u)$. Since F is the face of Q of smallest dimension containing a point λu with $\lambda > 0$, the faces F and G are conjugate to each other under the duality of Q and Q^o; we denote this fact by $F^* = G$, thus

$$R(Q, u)^* = F(Q^o, u). \tag{7}$$

Let $v_1, \dots, v_k$ be the outer unit normal vectors of the facets of Q containing F. Then

$$F^* = \mathrm{conv}\left\{\frac{v_1}{h(Q, v_1)}, \dots, \frac{v_k}{h(Q, v_k)}\right\}, \tag{8}$$

where h denotes the support function (this follows from [11, p. 99]).

3. Construction of the line measure

Our starting point is the representation (5) of the Holmes-Thompson $(n-1)$-area. Let $M \in \mathcal{M}$ be given. Let $x \in C$. The polar unit ball $B_x^o = \mathrm{D}P^x$ is a polytope with 0 as centre of symmetry. Let $S_1, \dots, S_k$ be the facets of $\mathrm{D}P^x$. Then

$$\lambda_{n-1}(B_x^o|TM) = \frac{1}{2}\sum_{i=1}^{k} \lambda_{n-1}(S_i|TM). \tag{9}$$

Let $i \in \{1, \dots, k\}$, and let u_i be the outer unit normal vector of $\mathrm{D}P^x$ at its facet S_i. We have

$$S_i = F(\mathrm{D}P^x, u_i) = F(P^x, u_i) + F(-P^x, u_i) = F((P-x)^o, u_i) - F((P-x)^o, -u_i).$$

According to (7), it follows that

$$S_i = R(P-x, u_i)^* - R(P-x, -u_i)^*.$$

Writing $R(P-x, u_i) =: F - x$ and $R(P-x, -u_i) =: G - x$, so that F and G are faces of P, we have $\dim F + \dim G \leq n-1$, since $\dim[(F-x)^* - (G-x)^*] = n-1$. There are numbers $\lambda, \mu > 0$ with $x + \lambda u_i \in F$ and $x - \mu u_i \in G$, hence the point x lies on a segment with one endpoint in F and the other in G. We set

$$\Delta(F, G) := C \cap \mathrm{conv}(F \cup G)$$

and denote by $\mathcal{T}$ the set of all triples (F, G, y) consisting of two distinct faces F, G of P with $\dim F + \dim G \leq n-1$ and a point $y \in \Delta(F, G)$. If the pair of faces

F, G is *complementary*, which means that the dimensions of F and G add up to $n-1$ and together they affinely span $\mathbb{R}^n$, then every point $y \in \Delta(F,G)$ lies on a unique segment with one endpoint in F and the other in G.

For $(F,G,x) \in \mathcal{T}$, let

$$Q_x^{F,G} := (F-x)^* - (G-x)^*$$

where $(F-x)^*$ denotes the face of the polar polytope $(P-x)^o$ that is conjugate to $F-x$. Thus, each facet S_i of $\mathrm{D}P^x$ is of the form $S_i = Q_x^{F,G}$ with $(F,G,x) \in \mathcal{T}$. Conversely, if $(F,G,x) \in \mathcal{T}$, then $Q_x^{F,G}$ is a face of $\mathrm{D}P^x$, but $\lambda_{n-1}(Q_x^{F,G}) = 0$ if it is not a facet. From (5) and (9) it now follows that

$$\begin{aligned} \mathrm{vol}_{n-1}(M) &= \frac{1}{\kappa_{n-1}} \int_M \frac{1}{2} \sum_{(F,G,x)\in\mathcal{T}} \lambda_{n-1}(Q_x^{F,G}|TM)\mathbf{1}_{\Delta(F,G)}(x)\,\lambda_{n-1}(\mathrm{d}x) \\ &= \frac{1}{2} \sum_{(F,G)} \frac{1}{\kappa_{n-1}} \int_M \lambda_{n-1}(Q_x^{F,G}|TM)\mathbf{1}_{\Delta(F,G)}(x)\,\lambda_{n-1}(\mathrm{d}x), \end{aligned}$$

where the last sum extends over all pairs (F,G) of distinct faces of P with $\dim F + \dim G \le n-1$. Here we need only sum over the complementary pairs (F,G). In fact, suppose that F and G are faces of P for which $S := \mathrm{aff}(F \cup G) \neq \mathbb{R}^n$. If $M \not\subset S$, then $\dim M \cap S < n-1$, hence

$$\int_M \mathbf{1}_{\Delta(F,G)}(x)\,\lambda_{n-1}(\mathrm{d}x) = 0.$$

If $M \subset S$ and $x \in M$, then the normal cones of $\mathrm{aff}(F-x)^*$ and $\mathrm{aff}(G-x)^*$, and hence that of $\mathrm{aff}[(F-x)^* - (G-x)^*]$, are subspaces of $S-x$. It follows that $\dim Q_x^{F,G}|TM < n-1$ and hence $\lambda_{n-1}(Q_x^{F,G}|TM) = 0$.

For every ordered pair (F,G) of complementary faces, we define a *partial area* $\mathrm{vol}_{F,G}$ by

$$\mathrm{vol}_{F,G}(M) := \frac{1}{\kappa_{n-1}} \int_M \lambda_{n-1}(Q_x^{F,G}|TM)\mathbf{1}_{\Delta(F,G)}(x)\,\lambda_{n-1}(\mathrm{d}x), \tag{10}$$

for $M \in \mathcal{M}$. Thus, we have shown that

$$\mathrm{vol}_{n-1}(M) = \frac{1}{2} \sum_{(F,G)} \mathrm{vol}_{F,G}(M), \tag{11}$$

where the sum extends over all ordered pairs (F,G) of complementary faces of P.

For a pair (F,G) of complementary faces of P we denote by $A(F,G) \subset A(n,1)$ the set of all lines that meet F and G.

We will show that there exists a positive measure $\eta_{F,G}$ on $A(n,1)$ which is concentrated on $A(F,G)$ and satisfies

$$\mathrm{vol}_{F,G}(M) = \int_{A(n,1)} \mathrm{card}(L \cap M)\,\eta_{F,G}(\mathrm{d}L) \tag{12}$$

for $M \in \mathcal{M}$. We fix a complementary pair (F,G) and explain how the measure $\eta_{F,G}$ is constructed.

Let $x \in \Delta(F,G)$. By Carathéodory's theorem, there is a representation

$$x = \sum_{i=0}^{m} \alpha_i a_i \quad \text{with } \alpha_i \geq 0,\ a_i \in F \cup G \text{ for } i = 0,\dots,m,\ \sum_{i=0}^{m} \alpha_i = 1,$$

where $a_0,\dots,a_m$ are affinely independent; without loss of generality, $a_0,\dots,a_j \in F$ and $a_{j+1},\dots,a_m \in G$, with $j \in \{0,\dots,m-1\}$. The number $\mu := \sum_{i=j+1}^{m} \alpha_i$ satisfies $\mu \notin \{0,1\}$, since $x \notin F \cup G$. With

$$y := \sum_{i=0}^{m} \frac{\alpha_i}{1-\mu} a_i, \qquad z := \sum_{i=j+1}^{m} \frac{\alpha_i}{\mu} a_i$$

we have $y \in F$, $z \in G$ and $x = (1-\mu)y + \mu z$. The representation $x = (1-\mu)y + \mu z$ with $y \in \operatorname{aff} F$, $z \in \operatorname{aff} G$ and $\mu \in (0,1)$ is unique. Thus, every point $x \in \Delta(F,G)$ lies on a unique line $L(x) \in A(F,G)$, meeting F in a point $y = y_x$ and G in a point $z = z_x$.

With x,y,z as before, let t_x be the unit vector which is a positive multiple of $y - z$. The local polar unit ball B_x^o has a facet $Q_x^{F,G}$ with outer normal vector t_x, and this facet is given by

$$Q_x^{F,G} = R(P-x,t_x)^* - R(P-x,-t_x)^*,$$

where $R(P-x,t_x) = F - x$ and $R(P-x,-t_x) = G - x$.

Let H be the hyperplane parallel to $\operatorname{aff} F + \operatorname{aff} G$ and at the same distance from F and G. Then every point $x \in H \cap \Delta(F,G)$ has the unique representation $x = (y+z)/2$ with $y \in L(x) \cap F$, $z \in L(x) \cap G$. For $L \in A(F,G)$, let $\pi(L)$ be the point in $L \cap H$. The map $\pi : A(F,G) \to H$ is injective.

For a Borel set $\mathcal{B} \subset A(n,1)$ of lines, we define

$$\eta_{F,G}(\mathcal{B}) := \frac{1}{\kappa_{n-1}} \int_H \lambda_{n-1}(Q_x^{F,G}|H) \mathbf{1}_{\pi(\mathcal{B}\cap A(F,G))}(x)\, \lambda_{n-1}(\mathrm{d}x). \tag{13}$$

Then $\eta_{F,G}$ is a positive measure, concentrated on $A(F,G)$. We assert that it satisfies (12).

To prove this, let $M \in \mathcal{M}$ and $E := \operatorname{aff} M$. For a line L, the intersection $L \cap M$ is either empty or contains one or infinitely many points; in the latter case, $L \subset E$. Set

$$\mathcal{L}(M) := \{L \in A(F,G) : \operatorname{card}(L \cap M) = \infty\}.$$

If $x \in \pi(\mathcal{L}(M))$, then the line $L(x)$ is contained in E, hence $x \in H \cap E$. It follows that $\lambda_{n-1}(\pi(\mathcal{L}(M))) = 0$ and hence, by (13), that $\eta_{F,G}(\mathcal{L}(M)) = 0$.

We write

$$M' := (M \cap \Delta(F,G)) \setminus \{x \in \Delta(F,G) : L(x) \in \mathcal{L}(M)\}$$

and

$$M_H := \{x \in H : L(x) \cap M' \neq \emptyset\}.$$

There is a bijective mapping

$$\alpha : M_H \to M'$$

such that $\alpha(x)$ is the intersection point of the line $L(x)$ with E.

Denoting the right-hand side of (12) by $I(M)$, we have, by definition (13),

$$\begin{aligned} I(M) &= \int_{A(F,G)\setminus\mathcal{L}(M)} \operatorname{card}(L\cap M)\,\eta_{F,G}(\mathrm{d}L) \\ &= \frac{1}{\kappa_{n-1}}\int_{M_H} \lambda_{n-1}(Q_x^{F,G}|H)\,\lambda_{n-1}(\mathrm{d}x). \end{aligned} \tag{14}$$

We assert that also

$$I(M) = \frac{1}{\kappa_{n-1}}\int_{M'} \lambda_{n-1}(Q_w^{F,G}|E)\,\lambda_{n-1}(\mathrm{d}w). \tag{15}$$

By (10), the right-hand side is equal to $\mathrm{vol}_{F,G}(M)$. In fact, if $x \in (M\cap\Delta(F,G))\setminus M'$, then $L(x)\subset E$ and, hence, $\lambda_{n-1}(Q_x^{F,G}|E) = 0$. Thus, if (15) is established, then (12) follows.

To prove (15), we use the map α to transform (14) in an integral over M'. For this, we first relate $\lambda_{n-1}(Q_x^{F,G})$ to $\lambda_{n-1}(Q_w^{F,G})$ if $L(x) = L(w)$.

Let $x \in \Delta(F,G)$, let $y \in L(x)\cap F$, $z \in L(x)\cap G$. Let

$$H^-_{u_i,\langle y,u_i\rangle} := \{p\in\mathbb{R}^n : \langle p,u_i\rangle \le \langle y,u_i\rangle\}, \qquad i=1,\dots,m$$

be the supporting halfspaces of P that contain F in their boundary, and let

$$H^-_{v_i,\langle z,v_i\rangle}, \qquad i=1,\dots,l$$

be the supporting halfspaces of P that contain G in their boundary. By (8), the face of $(P-x)^o$ that is conjugate to the face $F-x$ of $P-x$ is given by

$$(F-x)^* = \operatorname{conv}\left\{\frac{u_1}{\langle y-x,u_1\rangle},\dots,\frac{u_m}{\langle y-x,u_m\rangle}\right\}.$$

Similarly,

$$(G-x)^* = \operatorname{conv}\left\{\frac{v_1}{\langle z-x,v_1\rangle},\dots,\frac{v_l}{\langle z-x,v_l\rangle}\right\}.$$

We write $q := y-z$, then $y-x = \mu q$ with $\mu\in(0,1)$ and $x-z = (1-\mu)q$. Since

$$Q_x^{F,G} = (F-x)^* - (G-x)^*,$$

we obtain, with $\dim F =: j$ and $\dim G = n-1-j$,

$$\lambda_{n-1}(Q_x^{F,G}) = \lambda_{n-1-j}((F-x)^*)\lambda_j((G-x)^*)s(F,G) = \frac{1}{\mu^{n-1-j}}\frac{1}{(1-\mu)^j}\cdot V(x)$$

where $s(F,G)$ depends only on the relative position of aff F and aff G, and where

$$V(x) := \\ s(F,G)\lambda_{n-1-j}\left(\operatorname{conv}\left\{\frac{u_i}{\langle q,u_i\rangle} : i=1,\dots,m\right\}\right)\lambda_j\left(\operatorname{conv}\left\{\frac{v_i}{\langle q,v_i\rangle} : i=1,\dots,l\right\}\right).$$

Let $w \in L(x)$, $y - w = \nu q$, $w - z = (1-\nu)q$. Then it follows that

$$\frac{\lambda_{n-1}(Q_x^{F,G})}{\lambda_{n-1}(Q_w^{F,G})} = \left(\frac{\nu}{\mu}\right)^{n-1-j} \left(\frac{1-\nu}{1-\mu}\right)^j.$$

Now suppose that the hyperplane H has unit normal vector u and the hyperplane E has unit normal vector v. Let $x \in M_H$ and $w = \alpha(x)$. Then we obtain

$$\frac{\lambda_{n-1}(Q_x^{F,G}|H)}{\lambda_{n-1}(Q_w^{F,G}|E)} = \frac{\lambda_{n-1}(Q_x^{F,G})|\langle t,u\rangle|}{\lambda_{n-1}(Q_w^{F,G})|\langle t,v\rangle|} = \left(\frac{\nu}{\mu}\right)^{n-1-j} \left(\frac{1-\nu}{1-\mu}\right)^j \frac{|\langle t,u\rangle|}{|\langle t,v\rangle|},$$

where $t = t_x = t_w := (y-z)/|y-z|$ and $\mu = 1/2$.

The map $\alpha : M_H \to M'$ is a diffeomorphism. We have to determine the factor $D(\alpha, x)$ by which it distorts the Euclidean $(n-1)$-volume at a point x (i.e., the absolute determinant of the differential of α at x, with respect to the Euclidean metrics in the tangent spaces). For this, we fix a point $x_0 \in M_H$, with image $\alpha(x_0) = w_0$ and corresponding parameter ν_0. We represent α as the composition of two differentiable maps φ and ψ. The map φ is defined by

$$\varphi(x) := (1-\nu_0)y + \nu_0 z,$$

where $x = (1-\mu)y + \mu z$ with $y \in F$, $z \in G$ and $\mu = 1/2$. Thus, $\varphi(M_H)$ lies in the hyperplane H_0 through w_0 parallel to H. Instead of x, we may equivalently use y, z as independent variables. Writing $\varphi(x) =: \overline{x}$, we have, in a self-explanatory notation,

$$\lambda_{n-1}(\mathrm{d}\overline{x}) = (1-\nu_0)^j \nu_0^{n-1-j} s(F,G) \lambda_j(\mathrm{d}y) \lambda_{n-1-j}(\mathrm{d}z),$$

where $s(F,G)$ depends only on the relative position of F and G, and

$$\lambda_{n-1}(\mathrm{d}x) = (1-\mu)^j \mu^{n-1-j} s(F,G) \lambda_j(\mathrm{d}y) \lambda_{n-1-j}(\mathrm{d}z),$$

hence

$$D(\varphi, x) = \left(\frac{\nu_0}{\mu}\right)^{n-1-j} \left(\frac{1-\nu_0}{1-\mu}\right)^j.$$

The map $\psi : \varphi(M_H) \to M'$ is defined by letting $\psi(\overline{x})$ be the intersection point of the line $L(\overline{x}) = L(x)$ with E. One finds that

$$D(\psi, w_0) = \frac{|\langle t,u\rangle|}{|\langle t,v\rangle|}$$

with $t := t_{x_0} = t_{w_0}$. For this, it is convenient to choose in the tangent space to H_0 at w_0 an orthonormal basis with one vector orthogonal to the ($(n-2)$-dimensional) direction of the intersection of H_0 and E. The distortion factor of the length of this vector under the differential of ψ at w_0 is then easily determined using the sine rule; the lengths of the other basis vectors remain unchanged. Altogether we get

$$D(\alpha, x_0) = D(\psi, w_0) D(\varphi, x_0) = \left(\frac{\nu_0}{\mu}\right)^{n-1-j} \left(\frac{1-\nu_0}{1-\mu}\right)^j \frac{|\langle t,u\rangle|}{|\langle t,v\rangle|} = \frac{\lambda_{n-1}(Q_{x_0}^{F,G}|H)}{\lambda_{n-1}(Q_{w_0}^{F,G}|E)}.$$

Since $x_0 \in M_H$ was arbitrary, this shows that (15) holds.

Remark. For $x \in \Delta(F,G)$ and vectors $\xi_1,\dots,\xi_{n-1} \in \mathbb{R}^n$, we may define

$$\gamma(x,\xi_1,\dots,\xi_{n-1}) := \lambda_{n-1}(Q_x^{F,G})|\det(t_x,\xi_1,\dots,\xi_{n-1})|.$$

Since the right-hand side depends only on the simple $(n-1)$-vector $\xi_1 \wedge \cdots \wedge \xi_{n-1}$, this defines a (smooth) $(n-1)$-density γ on the n-manifold $\Delta(F,G)$ (we identify every tangent space $T_x\mathbb{R}^n$ with $\mathbb{R}^n$). For $M \in \mathcal{M}$, with unit normal vector u_M, we see from the preceding result that

$$\begin{aligned}\int_M \gamma &= \int_M \lambda_{n-1}(Q_x^{F,G})|\langle t_x,u_M\rangle|\,\lambda_{n-1}(\mathrm{d}x)\\ &= \int_M \lambda_{n-1}(Q_x^{F,G}|TM)|\,\lambda_{n-1}(\mathrm{d}x)\\ &= \mathrm{vol}_{F,G}(M).\end{aligned}$$

Finally, we define a measure η_1 on $A(n,1)$ by

$$\eta_1 := \frac{1}{2}\sum_{(F,G)} \eta_{F,G)}, \tag{16}$$

where the sum extends over all ordered pairs (F,G) of complementary faces of P. Then (11) and (12) together show that

$$\mathrm{vol}_{n-1}(M) = \int_{A(n,1)} \mathrm{card}(L \cap M)\,\eta_1(\mathrm{d}L)$$

for $M \in \mathcal{M}$. Thus, η_1 is a (positive) Crofton measure for the Holmes-Thompson area vol_{n-1}. As shown in [12], it is uniquely determined.

We observe that the line measure in a polytopal Hilbert geometry has a similar singularity property as the line measure in a polytopal Minkowski space: the measure η_1 is concentrated on a subset of $A(n,1)$ of dimension $n-1$, whereas $A(n,1)$ itself has dimension $2(n-1)$.

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