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Illumination of Direct Sums of Two Convex Figures

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Abstract. In this article we solve the problem of finding all possible values of the illumination number for the direct vector sum of two compact, convex planar figures which are not assumed to be smooth. It is proved that the possible values are only 7, 8, 9, 12, and 16. Keywords: direct vector sum, convex body, illumination number

Let $M \subset \mathbb{R}^n$ be a compact, convex body (i.e., a closed convex set with nonempty interior). A boundary point x of the body M is *illuminated* by a vector $a \neq 0$ if for any sufficiently small number $\lambda > 0$ the point $x + \lambda a$ belongs to the interior of M.

By $c(M)$ denote the *illumination number* of the body M, i.e., the least integer c for which there exist c nonzero vectors $a_1, a_2, \dots, a_c$ which illuminate the whole boundary of the body M. The problem of finding of the integer $c(M)$ was formulated in [1]. For more background information we refer to [5] and [8], see also [2] and [3].

For example, when $n = 2$, i.e., for a planar compact, convex figure M we have $c(M) = 4$ if M is a parallelogram, and $c(M) = 3$ for any other compact planar convex figure M, see [1] and [7].

It is known [1] that $c(M) = n + 1$ for an arbitrary *smooth* compact, convex body M (that is, for a body all boundary points of which are regular, i.e., through any boundary point of M passes only one support hyperplane), and moreover $c(M) = n + 1$ if M has no more than n non-regular boundary points. More general results are contained in [6] and [9].

In [4] the following theorem is proved.

Theorem 1. *Let $\mathbb{R}^n = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_k$ be a decomposition of the space $\mathbb{R}^n$ into the direct sum of its subspaces, and for every $i = 1, 2, \ldots, k$ in the subspace $\mathbb{L}_i$ a compact, convex body M_i is given. Then the inequality $c(M_1 \oplus M_2 \oplus \cdots \oplus M_k) \leq c(M_1) \cdot c(M_2) \cdot c(M_k)$ holds.*

It seems intuitively clear (in connection with a very simple proof of Theorem 1) that also the converse inequality holds, i.e., it seems that always the equality $c(M_1 \oplus M_2 \oplus \cdots \oplus M_k) = c(M_1) \cdots c(M_k)$ holds. Nevertheless, here our intuition is wrong. More detailed, in [4] the following theorem is proved.

Theorem 2. *For arbitrary 2-dimensional smooth compact, convex bodies M_1, M_2 the equality $c(M_1 \oplus M_2) = 7$ holds, i.e., $c(M_1 \oplus M_2) < c(M_1) \cdot c(M_2)$.*

In this article we consider the problem of finding the integer $c(M)$ when M is the direct sum of two compact planar convex figures which are not assumed to be smooth.

Lemma 1. *For arbitrary 2-dimensional compact, convex bodies M_1, M_2 the inequality $c(M_1 \oplus M_2) > 6$ holds.*

Proof. Let a_i, b_i be arbitrary unit vectors in $\mathbb{R}^2$, $i = 1, 2, \ldots, 6$. Consider six vectors $a_i + b_i \in \mathbb{R}^4$. Let x_1, x_2 be two boundary points of M_1 which are situated in two different support lines of the figure M_1 parallel to the vector a_1. We can suppose (changing the numeration of the points x_1, x_2 and the vectors a_j, $j = 2, 3, 4, 5, 6$, if necessary) that the vectors a_2, a_3, a_4 do not illuminate the point x_1. The vector a_1 does not illuminate the point x_1, too. Furthermore, let y_1 be a boundary point of the figure M_2 which is not illuminated by none of the vectors b_5, b_6. Then none of the vectors $a_j + b_j$, $j = 1, 2, \ldots, 6$, illuminates the boundary point $x_1 + y_1$ of the body $M_1 \oplus M_2$. Thus, any six vectors in $\mathbb{R}^4$ do not illuminate the whole boundary of the body $M_1 \oplus M_2$. □

Definition 1. *Let $M \subset \mathbb{R}^n$ be a compact, convex body. Boundary points x_1, x_2 of M are said to be antipodal if there are two distinct parallel support hyperplanes Γ_1 and Γ_2 of the body M such that $x_1 \in \Gamma_1, x_2 \in \Gamma_2$.*

It is clear that if x_1 and x_2 are antipodal boundary points of M, then no vector $a \neq 0$ illuminates both these points.

Definition 2. *Let $M \subset \mathbb{R}^n$ be a compact, convex body and $c = c(M)$ be its illumination number. We say that the body M is antipodal in the sense of illumination if there exist c boundary points of this body which are pairwise antipodal.*

For example, every parallelogram is a planar figure being antipodal in the sense of illumination. As another example of a planar figure that is antipodal in the sense of illumination we may indicate any *Reuleaux triangle*, i.e., the intersection of three circular disks of radius h centered at the vertices of an equilateral triangle with the side length h.

The following theorem describes a case when in Theorem 1 equality holds (a weaker result is contained in [4]).

Theorem 3. *Let $\mathbb{R}^n = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_k$ be a decomposition of the space $\mathbb{R}^n$ into the direct sum of its subspaces, and for every $i = 1, 2, \ldots, k$ in the subspace $\mathbb{L}_i$ a compact, convex body M_i is given. If the bodies $M_1, \ldots, M_{k-1}$ are antipodal in the sense of illumination, then $c(M_1 \oplus M_2 \oplus \cdots \oplus M_k) = c(M_1) \cdot c(M_2) \cdots c(M_k)$.*

Proof. Denote by $c_1, c_2, \ldots, c_{k-1}$ the illumination numbers of the bodies $M_1, M_2, \ldots, M_{k-1}$, and let $x_1^{(i)}, x_2^{(i)}, \ldots, x_{c_i}^{(i)}$ be pairwise antipodal points of the body M_i, $i = 1, 2, \ldots, k-1$. Then $x_{i_1}^{(1)} + x_{i_2}^{(2)} + \cdots + x_{i_{k-1}}^{(k-1)}$ are pairwise antipodal points of the body $M_1 \oplus M_2 \oplus \cdots \oplus M_{k-1}$, and the number of these points is equal to $c(M_1) \cdot c(M_2) \cdots c(M_{k-1})$. Thus, using induction over k, we obtain that the body $M_1 \oplus M_2 \oplus \cdots \oplus M_{k-1}$ is antipodal in the sense of illumination. Hence it is sufficient to consider in Theorem 3 only the case $k = 2$.

Thus, consider compact, convex bodies M_1 and M_2, the first of which is antipodal in the sense of illumination. Let $c(M_i) = c_i$, $i = 1, 2$, and $x_1, x_2, \ldots, x_{c_1}$ be pairwise antipodal boundary points of the body M_1. Assume $c(M_1 \oplus M_2) < c(M_1) \cdot c(M_2)$, and let $g_1, g_2, \ldots, g_q$ be vectors illuminating the whole boundary of the body $M_1 \oplus M_2$, where $q < c(M_1) \cdot c(M_2)$. Then for an index $i \in \{1, 2, \ldots, c(M_1)\}$ we have among $g_1, g_2, \ldots, g_q$ less than $c(M_2)$ vectors illuminating the boundary points of the set $x_i \oplus M_2$, and hence there is a point of this set that is not illuminated by any of the vectors $g_1, g_2, \ldots, g_q$, what is contradictory. Thus $c(M_1 \oplus M_2) \geq c(M_1) \cdot c(M_2)$, and consequently, by Theorem 1, equality holds. □

We can now prove the main result on the illumination number of direct vector sums of two planar compact convex figures (which are not assumed to be smooth).

Theorem 4. *For arbitrary 2-dimensional compact, convex figures M_1, M_2 the number $c(M_1 \oplus M_2)$ can take only the values* 7, 8, 9, 12, 16.

Proof. If the figure M_1 is a parallelogram, then, by Theorem 3, we have $c(M_1 \oplus M_2) = 4 \cdot c(M_2)$, i.e., $c(M_1 \oplus M_2)$ is equal to 12 or 16. The same holds if M_2 is a parallelogram.

Let now none of the figures M_1, M_2 be a parallelogram. Then, by Lemma 1 and Theorem 1, the inequality $7 \leq c(M_1 \oplus M_2) \leq 9$ holds. It remains to prove that all values 7, 8, 9 are possible. Theorem 2 establishes the possibility of the equality $c(M_1 \oplus M_2) = 7$. The possibility of the equalities $c(M_1 \oplus M_2) = 8$ and $c(M_1 \oplus M_2) = 9$ is proved in the following two lemmas. □

Lemma 2. *Assume that $M_1 = M_2$ is a Reuleaux triangle. Then the equality $c(M_1 \oplus M_2) = 9$ holds.*

Proof. The Reuleaux triangle $M_1 = M_2$ is a planar figure that is antipodal in the sense of illumination. Applying Theorem 3, we obtain $c(M_1 \oplus M_2) = c(M_1) \cdot c(M_2) = 9$. □

Lemma 3. *Assume that $M_1 = M_2$ is a regular pentagon. Then the equality $c(M_1 \oplus M_2) = 8$ holds.*

Proof. To illuminate the boundary of the polytope $M_1 \oplus M_2$ it is sufficient to illuminate all its vertices. Assume that $c(M_1 \oplus M_2) = 7$, and let $a_i + b_i$ be vectors which illuminate all vertices of the polytope $M_1 \oplus M_2$, where a_i, b_i are some unit vectors in $\mathbb{R}^2$, $i = 1, 2, \dots, 7$. Since any of the vectors $a_1, a_2, \dots, a_7$ illuminates no more than two vertices of the polygon M_1, then, by the inequality $2 \cdot 7 < 3 \cdot 5$, there is a vertex x of the polygon M_1 that is illuminated by no more than two of the vectors $a_1, a_2, \dots, a_7$. Assume that the vertex x is illuminated only by the vectors a_1 and a_2. Each vector b_1, b_2 illuminates no more than two vertices of the polygon M_2, and hence there is a vertex $y \in M_2$ that is not illuminated by any of the vectors b_1, b_2, i.e., the vertex $x + y$ of the polytope $M_1 \oplus M_2$ is not illuminated by any of the vectors $a_i + b_i$, $i = 1, 2, \dots, 7$. This contradiction shows that any seven vectors in $\mathbb{R}^4$ do not illuminate all vertices of the polytope $M_1 \oplus M_2$.

We now show that there are eight vectors in $\mathbb{R}^4$ which illuminate the whole boundary of $M_1 \oplus M_2$. Denote by $x_1, x_2, \dots, x_5$ the successive vertices of the pentagon M_1 and by $y_1, y_2, \dots, y_5$ the successive vertices of M_2. By a_{ij} denote a vector which illuminates the neighboring vertices x_i and x_j of the pentagon M_1, and by b_{pq} a vector which illuminates the neighboring vertices y_p and y_q of the pentagon M_2. Then it is easily shown that the eight vectors

$$a_{15} + b_{12}, a_{15} + b_{45}, a_{12} + b_{23}, a_{45} + b_{34},$$

$$a_{23} + b_{15}, a_{23} + b_{34}, a_{34} + b_{23}, a_{34} + b_{15}$$

illuminate all vertices (hence the whole boundary) of the polytope $M_1 \oplus M_2$. □

Lemma 3 considers a particular case of the problem on the illumination number of the 4-dimensional polytope $M_k \oplus M_k$, where M_k is a regular polygon. An analogous reasoning shows that the following more general result holds.

Theorem 5. *Let M_k be a regular polygon with k vertices. Then, depending on k, the 4-dimensional polytope $M_k \oplus M_k$ has the following illumination numbers:*

$c(M_k \oplus M_k) = 9$ *for* $k = 3$ *or* 6;
$c(M_k \oplus M_k) = 16$ *for* $k = 4$;
$c(M_k \oplus M_k) = 8$ *for* $k = 5, 8, 10$ *or* 12;
$c(M_k \oplus M_k) = 7$ *for* $k = 7, 9, 11$ *and for all* $k \geq 13$.

Proof. The proof is analogous to the previous one. Let, for example, $M_1 = M_2$ be a regular polygon with 12 vertices. To illuminate $\mathrm{bd}(M_1 \oplus M_2)$ it is sufficient to illuminate the vertices. Assume that $c(M_1 \oplus M_2) = 7$, and let $a_i + b_i$ be vectors which illuminate all vertices of the polytope $M_1 \oplus M_2$, where $a_i, b_i \in \mathbb{R}^2$,

$i = 1, 2, \dots, 7$. Every vector $a_1, a_2, \dots, a_7$ illuminates no more than five vertices of M_1, and hence, by the inequality $5 \cdot 7 < 3 \cdot 12$, there is a vertex $x \in M_1$ that is illuminated by no more than two vectors $a_1, a_2, \dots, a_7$. Therefore, as above, the vectors $a_i + b_i$, $i = 1, 2, \dots, 7$, do not illuminate all vertices of the polytope $M_1 \oplus M_2$. □

We note that if we change a small part of the boundary of one of the figures M_1, M_2, then the number $c(M_1 \oplus M_2)$ may be changed. For example, let M_1 be a regular hexagon and M_2 be a smooth planar figure. Then $c(M_1 \oplus M_2) = 9$. If even we change one of the sides of the hexagon M_1 by a suitable circular arc (of any radius), then the obtained figure M_1' satisfies the equality $c(M_1' \oplus M_2) = 8$. Moreover, if all angles of M_1 will be changed by inscribed circular arcs, then the obtained figure M_1'' satisfies the equality $c(M_1'' \oplus M_2) = 7$.

In conclusion we formulate some problems.

Problem 1. It is proved in [4] that if $M_1, M_2, \dots, M_k$ are smooth 2-dimensional compact, convex bodies, then the equality

$$c(M_1 \oplus M_2 \oplus \cdots \oplus M_k) = 2^{k+1} - 1$$

holds, i.e.,

$$c(M_1 \oplus M_2 \oplus \cdots \oplus M_k) < 2 \cdot \left(\frac{2}{3}\right)^k \cdot c(M_1) \cdot c(M_2) \cdots c(M_k).$$

Is it true that for smooth n-dimensional compact, convex bodies $M_1, M_2, \dots, M_k$ the inequality

$$c(M_1 \oplus M_2 \oplus \cdots \oplus M_k) < q\,\lambda^k \cdot c(M_1) \cdot c(M_2) \cdots c(M_k)$$

holds, where q and $\lambda < 1$ are positive numbers?

Problem 2. What are possible values of the number $c(M_1 \oplus M_2 \oplus \cdots \oplus M_k)$ for 2-dimensional compact, convex bodies $M_1, M_2, \dots, M_k$ which are not assumed to be smooth?

Problem 3. Find the number $c(M_1 \oplus M_2 \oplus \cdots \oplus M_k)$ for arbitrary smooth n-dimensional compact, convex bodies $M_1, M_2, \dots, M_k$.

Problem 4. Find the number $c(M_1 \oplus M_2 \oplus \cdots \oplus M_k)$ for arbitrary smooth compact, convex bodies $M_1, M_2, \dots, M_k$ of given dimensions $n_1, n_2, \dots, n_k$.

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