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# Submersion of CR-Submanifolds of Locally Conformal Kaehler Manifold

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**Abstract.** In this paper, we discuss submersion of CR-submanifolds of locally conformal Kaehler manifold. We prove that if $\pi : \overline{M} \longrightarrow B_\circ$ is a submersion of CR-submanifold $M$ of a locally conformal Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold $B_\circ$, then $B_\circ$ is a locally conformal Kaehler manifold. Furthermore, we discuss totally umbilical CR-submanifold and cohomology of CR-submanifold of locally conformal Kaehler manifold under the submersion.

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## 1. Introduction

A Hermitian manifold $(\overline{M}, g)$ is called a *locally conformal Kaehler manifold (briefly l.c.K manifold)*, if every point of $\overline{M}$ has a neighborhood $U$ such that the restriction $g_U$ of $g$ to $U$ is conformal to a Kaehler metric $g'_U$ of $U : g_U = e^{\sigma_U} g'_U$ for some $c^\infty$ function $\sigma_U : U \longrightarrow \mathbb{R}$. $(\overline{M}, g)$ is a globally conformal Kaehler (g.c.K) manifold if one can choose $U = \overline{M}$; then $g'$ is a Kaehler metric on $\overline{M}$, and hence $(\overline{M}, g')$ is a Kaehler manifold.

Let $\Omega$ be a 2-form on $\overline{M}$. Then $\overline{M}$ is a l.c.K. manifold if and only if there is a global 1-form $\omega$ on $\overline{M}$ (the Lee form of $\overline{M}$) such that [15]

$$d\Omega = \omega \wedge \Omega, \qquad d\omega = 0, \tag{1.1}$$

and $\overline{M}$ is a g.c.K manifold if and only if $\omega$ is also exact.

Let $\overline{M}$ be an l.c.K. manifold. Then the vector field $B$ (the Lee field of $\overline{M}$) is defined by

$$g(X, B) = \omega(X). \tag{1.2}$$

The best known examples of l.c.K. manifolds which are not globally conformal Kaehler are the Hopf manifolds. Further examples and general properties of l.c.K. manifolds have been studied by I. Vaisman in a series of papers and by others (see [9] for detail). For most of the known examples of l.c.K manifolds, the Lee form $\omega$ turns out to be parallel with respect to the Levi-Civita connection.

Now, suppose that $\overline{\nabla}$ be the Levi-Civita connection of $g$. Then we have [14]

$$\tilde{\nabla}_X Y = \overline{\nabla}_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B. \tag{1.3}$$

$\tilde{\nabla}$ is a torsionless linear connection on $\overline{M}$ which is called the *Weyl connection of $g$.*

**Theorem 1.1.** [14] *The almost Hermitian manifold $\overline{M}$ is an l.c.K. manifold if and only if there is a closed 1-form $\omega$ on $\overline{M}$ such that the Weyl connection be almost complex i.e., $\tilde{\nabla} J = 0$.*

The notion of Cauchy-Riemann (CR-)submanifold was introduced by Bejancu [1] as a natural generalization of complex submanifolds and totally real submanifolds.

Now we have

**Definition.** *Let $M$ be an $m$-dimensional submanifold of an l.c.K manifold $\overline{M}$. If there exist two orthogonal complementary distributions $D$ and $D^{\perp}$ on $M$ satisfying $JD = D$ and $JD^{\perp} \subset \nu$, where $J$ is the almost complex structure on $M$ and $\nu$ is the normal bundle of $M$, then $M$ is called a CR-submanifold of $\overline{M}$.*

We call $D$ (resp. $D^{\perp}$) a *horizontal (resp. vertical) distribution.*

We denote by the same letter $g$ the induced metric on $M$. The Riemannian connection $\overline{\nabla}$ on $\overline{M}$ gives rise to the induced Riemannian connection $\nabla$ on $M$ and a connection $\nabla^{\perp}$ in the normal bundle $\nu$. Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.4}$$

$$\overline{\nabla}_X N = -\tilde{A}_N X + \nabla^{\perp}_X N, \tag{1.5}$$

for $X, Y \in T(M)$ and $N \in \nu$, where $h$ is the second fundamental form and $\tilde{A}_N$ is the Weingarten map and these are related by

$$g(\tilde{A}_N X, Y) = g(h(X, Y), N). \tag{1.6}$$

Now we state the following lemmas for later use.

**Lemma 1.2.** [13, P.5] *Let $\overline{M}$ be an l.c.K manifold and $M$ be a CR-submanifold of $\overline{M}$. If the horizontal distribution $D$ is integrable, then*

$$h(JX, Y) - h(X, JY) = g(X, JY)B^{\perp},$$

*for any $X, Y \in D$, where $B^{\perp}$ denotes the normal component of $B$.*

**Lemma 1.3** [9, P.167] *Let $\overline{M}$ be an l.c.K manifold and $M$ be a CR-submanifold of $\overline{M}$. Then vertical distribution $D^{\perp}$ is always integrable.*

The curvature tensor $R$ of the submanifold $M$ is related to the curvature tensor $\overline{R}$ of $\overline{M}$ by the following Gauss equation

$$\overline{R}(X,Y;Z,W) = R(X,Y;Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)), \tag{1.7}$$

for any $X, Y, Z, W \in T(M)$.

A CR-submanifold $M$ of a locally conformal Kaehler manifold $\overline{M}$ is called a *CR-product* if locally $M$ is a Riemannian product of an invariant submanifold and a totally real submanifold of $\overline{M}$ [9].

A CR-submanifold $M$ is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H,$$

where $H$ is the mean curvature vector.

The study of the Riemannian submersion $\pi : M \longrightarrow B_{\circ}$ of a Riemannian manifold $M$ onto a Riemannian manifold $B_{\circ}$ was initiated by O. Neill [12]. A submersion $\pi$ naturally gives rise to two distributions on $M$ called the horizontal and vertical distribution respectively, of which the vertical distribution is always integrable giving rise to the fibres of the submersion which are closed submanifold of $M$. Submersion of CR-submanifold of a Kaehler manifold was defined and studied by Kobayashi [10] and Deshmukh, Ali, Husain [8].

For the submersion of a CR-submanifold of an l.c.K. manifold onto an almost Hermitian manifold, we have

**Definition.** *Let $M$ be a CR-submanifold of a locally conformal Kaehler manifold $\overline{M}$. By a submersion $\pi : M \longrightarrow B_{\circ}$ of $M$ onto an almost Hermitian manifold $B_{\circ}$ we mean a Riemannian submersion $\pi : M \longrightarrow B_{\circ}$ together with the following conditions:*

(i) *$D^{\perp}$ is the kernel of $\pi_*$, i.e., $\pi_* D^{\perp} = \{0\}$,*

(ii) *$J$ interchanges $D^{\perp}$ and $\nu$, i.e., $JD^{\perp} = \nu$,*

(iii) *$\pi_* : D_p \longrightarrow D^*_{\pi(p)}$ is a complex isometry of the subspace $D_p$ onto $D^*_{\pi(p)}$ for every $p \in M$, where $D^*_{\pi(p)}$ denotes the tangent space of $B_{\circ}$ at $\pi(p)$.*

For a vector field $X$ on $M$, we set [10]

$$X = HX + VX, \tag{1.8}$$

where $H$ and $V$ denote the horizontal and vertical part of $X$.

We make the special choice of vector field in order to relate the geometry of $M$ with that of $B_{\circ}$ and call this as basic vector field.

**Definition.** *A vector field $X$ on $M$ is called basic if*

(i) *$X$ is horizontal, i.e., $X \in D$ and*

(ii) *$X$ is $\pi$-related to a vector field on $B_\circ$ i.e., there is a vector field $X_*$ on $B_\circ$ such that $(\pi_* X)_p = (X_*)_{\pi(p)}$ at all points $p \in M$.*

**Lemma 1.4.** [12] *Let $X$ and $Y$ be basic vector fields on $M$. Then*

(i) *$g(X,Y) = g_*(X_*, Y_*) \circ \pi$,*

(ii) *$H[X,Y]$ is basic and corresponds to $[X_*, Y_*]$,*

(iii) *$H(\bigtriangledown_X Y)$ is basic and corresponds to $\bigtriangledown^*_{X_*} Y_*$ where $\bigtriangledown^*$ is a Riemannian connection on $B_\circ$,*

(iv) *$[X,Z] \in D^\perp$, for $Z \in D^\perp$.*

For a covariant differentiation operator $\bigtriangledown^*$ we define a corresponding operator $\tilde{\bigtriangledown}^*$ for basic vector fields of $M$ by

$$\tilde{\bigtriangledown}^*_X Y = H(\bigtriangledown_X Y). \tag{1.9}$$

Then $\tilde{\bigtriangledown}^*_X Y$ is a basic vector field, and we have

$$\pi_*(\tilde{\bigtriangledown}^*_X Y) = \bigtriangledown^*_{X_*} Y_*.$$

Next, we define a tensor field $C$ by

$$\bigtriangledown_X Y = H(\bigtriangledown_X Y) + C(X,Y),$$

for all $X, Y \in D$, where $C(X,Y)$ is the vertical part of $\bigtriangledown_X Y$. In particular, if $X$ and $Y$ are basic vector fields, then we have

$$\bigtriangledown_X Y = \tilde{\bigtriangledown}^*_X Y + C(X,Y). \tag{1.10}$$

It is known that $C$ is skew-symmetric, and if $X, Y \in D$, then

$$C(X,Y) = \frac{1}{2} V[X,Y]. \tag{1.11}$$

Next, for vertical vector fields $Z, W \in D^\perp$ we define $L$ by

$$\bigtriangledown_W Z = \hat{\bigtriangledown}_W Z + L(W,Z), \tag{1.12}$$

where $L(W,Z)$ is the horizontal part of $\bigtriangledown_W Z$. $L$ is a bilinear mapping $L : D^\perp \times D^\perp \longrightarrow D$ and it is symmetric tensor.

Since the fibres are closed submanifolds of $M$. So for vertical vector fields, $L$ defines the second fundamental form of the fibres in $M$. The fibres are totally geodesic if $L(W,Z) = 0$ for all $W, Z \in D^\perp$ [8].

For $X \in D$, $V \in D^\perp$ define an operator $A$ on $M$ by setting

$$\bigtriangledown_X V = A_X V + \nu(\bigtriangledown_X V), \tag{1.13}$$

where $A_X V$ (resp. $\nu(\bigtriangledown_X V)$) is the horizontal (resp. normal) component of $\bigtriangledown_X V$. Using (iv) of Lemma (1.4) we have

$$H(\bigtriangledown_X V) = H(\bigtriangledown_V X) = A_X V. \tag{1.14}$$

The operator $C$ and $A$ are related by

$$g(A_X V, Y) = -g(V, C(X,Y)), \tag{1.15}$$

for $X, Y \in D$, $V \in D^\perp$.

## 2. Submersion of CR-submanifold

It is known by a result of Chen [5] that the anti-invariant distribution $D^\perp$ of a CR-submanifold of a Kaehler manifold is always integrable. This is still true for CR-submanifold of locally conformal Kaehler manifold [9]. Now we have

**Proposition 2.1.** *Let $\pi : \overline{M} \longrightarrow B_\circ$ be a submersion of a CR-submanifold $M$ of a locally conformal Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold $B_\circ$. If the horizontal distribution $D$ is integrable and the vertical distribution $D^\perp$ is parallel, then $M$ is CR-product.*

*Proof.* Since the horizontal distribution $D$ is integrable. So for $X, Y \in D$, we have $[X, Y] \in D$. Therefore $V[X, Y] = 0$. Now using the equation (1.11) we get $C(X, Y) = 0$ for $X, Y \in D$. Putting the value of $C(X, Y)$ in (1.10) we have $\bigtriangledown_X Y = \tilde{\bigtriangledown}^*_X Y \in D$ which shows that $D$ is parallel. Since the horizontal distribution $D$ and vertical distribution $D^\perp$ are both parallel. So using de Rham's theorem , it follows that $M$ is the product $M_1 \times M_2$, where $M_1$ is invariant submanifold of $\overline{M}$ and $M_2$ is totally real submanifold of $\overline{M}$. Hence $M$ is a CR-product. □

**Proposition 2.2.** *Let $\pi : \overline{M} \longrightarrow B_\circ$ be a submersion of a CR-submanifold $M$ of a locally conformal Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold $B_\circ$ such that $B \in D^\perp$, then*

$$H(\tilde{A}_{JY} X) = -JL(X, Y), \tag{2.1}$$

$$V(\tilde{A}_{JY} X) = -Jh(X, Y), \tag{2.2}$$

*for $X, Y \in D^\perp$.*

*Proof.* Since J is parallel with respect to the Weyl connection $\tilde{\bigtriangledown}$, so we have for $X, Y \in D^\perp$

$$\tilde{\bigtriangledown}_X JY = J\tilde{\bigtriangledown}_X Y.$$

Now using (1.3), we have

$$\begin{aligned}
\overline{\bigtriangledown}_X JY - {} & \frac{1}{2}\omega(X)JY - \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X, JY)B \\
= {} & J\big(\overline{\bigtriangledown}_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)B\big)
\end{aligned}$$

or,

$$-\tilde{A}_{JY}X + \bigtriangledown^{\perp}_{X}JY - \frac{1}{2}\omega(X)JY - \frac{1}{2}\omega(JY)X =$$
$$J\big(\bigtriangledown_X Y + h(X,Y) - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X,Y)B\big) \tag{2.3}$$

for $X, Y \in D^{\perp}$.

Since $J$ interchanges vertical and normal part i.e., $D^{\perp}$ and $\nu$. So using these facts in equation (2.3), we get

$$-H(\tilde{A}_{JY}X) - V(\tilde{A}_{JY}X) + \bigtriangledown^{\perp}_{X}JY - \frac{1}{2}\omega(JY)X = J\hat{\bigtriangledown}_X Y + JL(X,Y)$$
$$+ Jh(X,Y) - \frac{1}{2}\omega(Y)JX + \frac{1}{2}g(X,Y)JB. \tag{2.4}$$

As Lee vector field $B \in D^{\perp}$, then the above equation reduces to

$$-H(\tilde{A}_{JY}X) - V(\tilde{A}_{JY}X) + \bigtriangledown^{\perp}_{X}JY - \frac{1}{2}g(X,Y)JB - \frac{1}{2}\omega(JY)X = J\hat{\bigtriangledown}_X Y$$
$$+ JL(X,Y) + Jh(X,Y) - \frac{1}{2}\omega(Y)JX, \tag{2.5}$$

for $X, Y \in D^{\perp}$.

Equating horizontal and vertical components on both sides of equation (2.5), we get the result. □

Now we prove

**Theorem 2.3.** *Let $\pi : \overline{M} \longrightarrow B_{\circ}$ be a submersion of CR-submanifold $M$ of a locally conformal Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold $B_{\circ}$ such that Lee vector field $B \in D^{\perp}$. Then $B_{\circ}$ is a locally conformal Kaehler manifold.*

*Proof.* Let $X, Y \in D$ be the basis vectors. Using equation (1.10) in (1.4) we have

$$\overline{\bigtriangledown}_X Y = \tilde{\bigtriangledown}^{*}_{X} Y + h(X,Y) + C(X,Y) \quad \text{for} \quad X, Y \in D,$$

which by virtue of (1.3), we obtain

$$\tilde{\bigtriangledown}_X Y + \frac{1}{2}\omega(X)Y + \frac{1}{2}\omega(Y)X - \frac{1}{2}g(X,Y)B$$
$$= \tilde{\bigtriangledown}^{*}_{X} Y + h(X,Y) + C(X,Y). \tag{2.6}$$

Applying $J$ both the side of above equation, we have

$$J\tilde{\bigtriangledown}_X Y + \frac{1}{2}\omega(X)JY + \frac{1}{2}\omega(Y)JX - \frac{1}{2}g(X,Y)JB$$
$$= J\tilde{\bigtriangledown}^{*}_{X} Y + Jh(X,Y) + JC(X,Y). \tag{2.7}$$

Replacing $Y$ by $JY$ in equation (2.6), we get

$$\begin{aligned}\tilde{\nabla}_X JY + \frac{1}{2}\omega(X)JY + \frac{1}{2}\omega(JY)X - \frac{1}{2}g(X,JY)B \\ = \tilde{\nabla}^*_X JY + h(X,JY) + C(X,JY).\end{aligned} \tag{2.8}$$

Subtracting equations (2.7) and (2.8), we get

$$\begin{aligned}(\tilde{\nabla}_X J)(Y) + \frac{1}{2}\omega(JY)X - \frac{1}{2}g(X,JY)B - \frac{1}{2}\omega(Y)JX + \frac{1}{2}g(X,Y)JB \\ = (\tilde{\nabla}^*_X J)(Y) + h(X,JY) + C(X,JY) - Jh(X,Y) - JC(X,Y)\end{aligned} \tag{2.9}$$

for $X, Y \in D$.

As J interchanges vertical and normal part. So comparing horizontal, vertical and normal components of the above equation (2.9) for $B \in D^\perp$ we get

$$(\tilde{\nabla}^*_X J)(Y) = 0, \tag{2.10}$$

$$Jh(X,Y) = -C(X,JY) - \frac{1}{2}g(X,JY)B, \tag{2.11}$$

$$JC(X,Y) - h(X,JY) = \frac{1}{2}g(X,Y)JB, \tag{2.12}$$

for $X, Y \in D$, $B \in D^\perp$.

Hence equation (2.10) shows that the base manifold $B_\circ$ is a locally conformal Kaehler manifold, where $\tilde{\nabla}^*$ denotes the horizontal lift of the covariant derivative $\nabla^*$ of $B_\circ$. □

## 3. Totally umbilical CR-submanifold

In this section, we discuss submersion of totally umbilical CR-submanifold of l.c.K manifold. The equations of Gauss and Weingarten take the following forms:

$$\overline{\nabla}_X Y = \nabla_X Y + g(X,Y)H, \quad \overline{\nabla}_X N = -g(N,H)X + \nabla^\perp_X N, \tag{3.1}$$

for $X, Y \in TM$, $N \in \nu$ and $H$ is the mean curvature vector.

Now we have the following

**Proposition 3.1.** *Let $\pi : \overline{M} \longrightarrow B_\circ$ be a submersion of a totally umbilical CR-submanifold $M$ of a locally conformal Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold $B_\circ$ such that Lee vector field $B \in D^\perp$, $D$ is parallel and $g(JX, X) = 0$. Then $M$ is a totally geodesic submanifold.*

*Proof.* The mean curvature vector $H$ is normal and $J$ interchanges vertical and normal parts. Now we have

$$(\tilde{\nabla}_X J)H = 0$$

or,

$$\tilde{\nabla}_X JH = J\tilde{\nabla}_X H$$

or,

$$-\tilde{\nabla}_X H = J\tilde{\nabla}_X JH$$

$$\begin{aligned} &-\overline{\nabla}_X H + \frac{1}{2}\omega(X)H + \frac{1}{2}\omega(H)X - \frac{1}{2}g(X,H)B \\ &= J(\overline{\nabla} JH - \frac{1}{2}\omega(X)JH - \frac{1}{2}\omega(JH)X + \frac{1}{2}g(X,JH)B). \end{aligned} \tag{3.2}$$

But for $B \in D^{\perp}$ and $X \in D$ we have

$$\omega(H) = g(B,H) = 0, \quad g(X,H) = 0, \quad g(X,JH) = 0.$$

Thus using the above facts in (3.2) we get

$$-\overline{\nabla}_X H = J\overline{\nabla}_X JH - \frac{1}{2}\omega(JH)JX$$

or,

$$\tilde{A}_H X - \nabla^{\perp}_X H = J\nabla_X JH + Jh(X,JH) - \frac{1}{2}\omega(JH)JX.$$

Taking inner product with $X \in D$ in the above equation, we get

$$g(\tilde{A}_H X, X) = g(J\nabla_X JH, X)$$

or,

$$g(h(X,X),H) = -g(\nabla_X JH, JX) = g(\nabla_X JX, JH). \tag{3.3}$$

As $D$ is parallel, so $\nabla_X JX \in D$ for $X \in D$, giving $g(\nabla_X JX, JH) = 0$. So from equation (3.3), we get $||H|| = 0$ which gives $H = 0$. Hence $M$ is a totally geodesic submanifold. □

## 4. Curvature properties

First we prove

**Theorem 4.1.** *Let $\pi : \overline{M} \longrightarrow B_\circ$ be a submersion of a CR-submanifold $M$ of a locally conformal Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold $B_\circ$ such that $B \in D^{\perp}$. Then the sectional curvature of locally conformal Kaehler manifold $\overline{M}$ and the fibres is given by*

$$\overline{K}(X \wedge Y) = \hat{K}(X \wedge Y) - g(\tilde{A}_{JY}X, \tilde{A}_{JX}Y) + g(\tilde{A}_{JX}X, \tilde{A}_{JY}Y) - ||\tilde{A}_{JY}X||^2,$$

*for all vertical vector fields $X, Y \in D^{\perp}$.*

*Proof.* From the definition of curvature tensor $R$, we have

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{4.1}$$

Using (1.12) in (4.1) we have

$$R(X,Y) = \nabla_X(\hat{\nabla}_Y Z + L(Y,Z)) - \nabla_Y(\hat{\nabla}_X Z + L(X,Z)) - \hat{\nabla}_{[X,Y]}Z - L([X,Y],Z),$$

for $X, Y, Z \in D^\perp$.
Since $L(X,Y)$ is the horizontal part of $\nabla_X Y$ for $X, Y \in D^\perp$ i.e., $L(X,Y) = H(\nabla_X Y)$, we get

$$\begin{aligned} R(X,Y)Z &= \nabla_X \hat{\nabla}_Y Z + \nabla_X L(Y,Z) \\ &\quad - \nabla_Y \hat{\nabla}_X Z - \nabla_Y L(X,Z) - \hat{\nabla}_{[X,Y]}Z - H(\nabla_{[X,Y]}Z), \end{aligned}$$

for $X, Y \in D^\perp$.

Taking inner product with a vertical vector field $W \in D^\perp$ in the above equation, we get

$$\begin{aligned} g(R(X,Y)Z,W) &= g(\nabla_X \hat{\nabla}_Y Z, W) + g(\nabla_X L(Y,Z), W) \\ &\quad - g(\nabla_Y \hat{\nabla}_X Z, W) - g(\nabla_Y L(X,Z), W) - g(\hat{\nabla}_{[X,Y]}Z, W) \\ &= g(\nabla_X \hat{\nabla}_Y Z, W) - g(\nabla_Y \hat{\nabla}_X Z, W) - g(\hat{\nabla}_{[X,Y]}Z, W) \\ &\quad - g(L(Y,Z), \nabla_X W) + g\big(L(X,Z), \nabla_Y W\big), \end{aligned}$$

for $X, Y, W \in D^\perp$.

Therefore

$$R(X,Y,Z,W) = \hat{R}(X,Y,Z,W) - g\big(L(Y,Z), L(X,W)\big) + g\big(L(X,Z), L(Y,W)\big).$$

Now, from Gauss equation, the above equation reduces to

$$\begin{aligned} \overline{R}(X,Y,Z,W) &= \hat{R}(X,Y,Z,W) - g\big(L(Y,Z), L(X,W)\big) \\ &\quad + g\big(L(X,Z), L(Y,W)\big) \\ &\quad + g\big(h(X,Z), h(Y,W)\big) - g\big(h(Y,Z), h(X,W)\big). \end{aligned}$$

Thus, putting $Y = X, Z = X, W = Y$ in the above equation, we have

$$\begin{aligned} \overline{R}(X,Y;X,Y) &= \hat{R}(X,Y;X,Y) - g\big(L(Y,X), L(X,Y)\big) + g\big(L(X,X), L(Y,Y)\big) \\ &\quad + g\big(h(X,X), h(Y,Y)\big) - g\big(h(Y,X), h(X,Y)\big), \end{aligned}$$

which implies that

$$\begin{aligned} \overline{K}(X \wedge Y) &= \hat{K}(X \wedge Y) - g\big(L(X,Y), L(Y,X)\big) + g\big(L(X,X), L(Y,Y)\big) \\ &\quad + g\big(h(X,X), h(Y,Y)\big) - g\big(h(X,Y), h(X,Y)\big) \end{aligned} \tag{4.2}$$

for $X, Y \in D^\perp$.

Using equations (2.1) and (2.2) in (4.2) we get

$$\begin{aligned} \overline{K}(X \wedge Y) &= \hat{K}(X \wedge Y) - g(H\tilde{A}_{JY}X, H\tilde{A}_{JX}Y) + g(H\tilde{A}_{JX}X, H\tilde{A}_{JY}Y) \\ &\quad + g(V(\tilde{A}_{JX}X), V(\tilde{A}_{JY}Y)) - g(V(\tilde{A}_{JY}X), V(\tilde{A}_{JY}X)) \\ &= \hat{K}(X \wedge Y) - g(\tilde{A}_{JY}X, \tilde{A}_{JX}Y) \\ &\quad + g(\tilde{A}_{JX}X, \tilde{A}_{JY}Y) - g(\tilde{A}_{JY}X, \tilde{A}_{JY}X) \end{aligned}$$

for $X, Y \in D^{\perp}$, which completes the proof. □

Next, we discuss the holomorphic sectional curvature of l.c.K manifold $\overline{M}$ and $B_\circ$ respectively. Using equation (1.13) and (1.15), the curvature tensor $R$ on $M$ is given by

$$R(X,Y,Z,W) = R^*(X_*, Y_*, Z_*, W_*) + g(C(X,Z), C(Y,W)) \\ - g(C(Y,Z), C(X,W)) + 2g(C(X,Y), C(Z,W)), \tag{4.3}$$

where $\pi_* X = X_*,\ \pi_* Y = Y_*,\ \pi_* Z = Z_*,\ \pi_* W = W_* \in B_\circ$ for $X, Y, Z, W \in D$.

Thus for $X, Y \in D$ making use of (4.3) in (1.7) and the fact that $C$ is skew-symmetric, we get

$$\overline{H}(X) = \overline{R}(X, JX, JX, X) = H^*(X_*) + ||h(X, JX)||^2 \\ - g(h(JX, JX), h(X,X)) - 3||C(X, JX)||^2, \tag{4.4}$$

where $\overline{H}(X)$ and $H^*(X_*)$ are the holomorphic sectional curvature tensors of $\overline{M}$ and $B_\circ$ respectively. Thus we have

**Proposition 4.2.** *Let $\pi : \overline{M} \longrightarrow B_\circ$ be a submersion of a CR-submanifold $M$ of a locally conformal Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold $B_\circ$. Then the holomorphic sectional curvature of $\overline{M}$ and $B_\circ$ is related by*

$$\overline{H}(X) = \overline{R}(X, JX, JX, X) = H^*(X_*) + ||h(X, JX)||^2 \\ - g(h(JX, JX), h(X,X)) - 3||C(X, JX)||^2,$$

*for $X, Y \in D$.*

**Corollary 4.3.** *Let $\overline{M}$ be an l.c.K manifold and $M$ be a CR-submanifold of $\overline{M}$ with integrable distribution $D$. Let $B_\circ$ be an almost Hermitian manifold and $\pi : \overline{M} \longrightarrow B_\circ$ be a submersion. Then the holomorphic sectional curvatures $\overline{H}$ and $H^*$ of $\overline{M}$ and $B_\circ$ respectively satisfy*

$$\overline{H}(X) \geq H^*(X_*) \qquad \text{for } X \in D,\ g(X, JX) = 0.$$

*Proof.* Since $D$ is integrable. So from Lemma 1.2 using $g(X, JX) = 0$, we have

$$h(JX, JX) = -h(X, X).$$

Also, from (2.11) by taking $Y = X$ we obtain $C(X, JX) = 0$. Thus from (4.4) we have

$$\overline{H}(X) = H^*(X_*) + ||h(X, JX)||^2 + ||h(X,X)||^2, \quad X \in D,$$

which proves that $\overline{H}(X) \geq H^*(X_*)$. □

## 5. Cohomology of CR-submanifolds

The cohomology class was originally introduced by Chen in his article [6] for CR-submanifold in Kaehler manifolds. Later this was extended to CR-submanifold in locally conformal Kaehler manifold by Chen and Piccinni [7]. In fact, the cohomology class has also been studied by several geometers [9]. In this section, we discuss how the submersion $\pi : M \longrightarrow B_\circ$ of a CR-submanifold $M$ with integrable $D$ effects the topology of $M$. Let $M$ be a CR-submanifold of a locally conformal Kaehler manifold $\overline{M}$ and $\pi : \overline{M} \longrightarrow B_\circ$ be a submersion of $\overline{M}$ onto an almost Hermitian manifold $B_\circ$.

Let dim $M = m$ and dim $D = 2p$. We choose a local orthonormal frame $\{e_1, \dots, e_p, Je_1, \dots, Je_p, e_{2p+1}, \dots, e_m\}$ on CR-submanifold $M$ such that $\{e_1, \dots, e_p, Je_1, \dots, Je_p\}$ is a local orthonormal frame of $D$ and $\{e_{2p+1}, \dots, e_m\}$ is a local orthonormal frame of $D^\perp$. Let $\{\phi^1, \dots, \phi^{2p}, \phi^{2p+1}, \dots, \phi^m\}$ be the dual frame of 1-forms to the above local orthonormal frame. Let us define a $2p$-forms $\phi$ on CR-submanifold $M$ by

$$\phi = \{\phi^1 \wedge \cdots \wedge \phi^{2p}\}, \tag{5.1}$$

where $\{\phi^1, \dots, \phi^{2p}\}$ be the dual frame of an orthonormal frame $\{e_1, \dots, e_p, Je_1, \dots, Je_p\}$ of $D$. From definition (5.1) of $\phi$, we have

$$d\phi = \sum_{i=1}^{2p} (-1)^{i-1} \phi^1 \wedge \cdots \wedge d\phi^i \wedge \cdots \wedge d\phi^{2p}.$$

From above equation, it follows that $d\phi = 0$ if and only if

$$d\phi(Z_1, Z_2, X_1, \dots, X_{2p-1}) = 0, \tag{5.2}$$

and

$$d\phi(Z_1, X_1, \dots, X_{2p}) = 0, \tag{5.3}$$

where $Z_1, Z_2 \in D^\perp$ and $X_1, \dots, X_{2p} \in D$.
Choosing the vectors $X_1, \dots, X_{2p} \in D$ as a local orthonormal frame $\{e_1, \dots, e_p, Je_1, \dots, Je_p\}$ of $D$ we see that equation (5.2) holds if and only if $D^\perp$ is integrable and equation (5.3) holds if and only if $D$ is minimal. However, it is known that totally real distribution $D^\perp$ of CR-submanifold of a locally conformal Kaehler manifold is integrable [7]. The hypothesis of the theorem gives that $D$ is integrable and this together with the proof of Proposition 2.1 gives that $D$ is minimal. Thus it follows that $d\phi = 0$ on CR-submanifold $M$ which means that $\phi$ is closed. Hence it defines a de Rham cohomology class $[\phi]$ in $H^{2p}(M, R)$.

Next, suppose that $D^\perp$ is minimal and we show that $\phi$ is harmonic. For this, define an $(m-2p)$-form $\phi^\perp$ on CR-submanifold $M$ by

$$\phi^\perp = \{\phi^{2p+1}, \dots, \phi^m\} \tag{5.4}$$

where $\{\phi^{2p+1}, \dots, \phi^m\}$ is dual frame to the orthonormal frame $\{e_{2p+1}, \dots, e_m\}$ of $D^\perp$. Then using the similar argument as above we see that $\phi^\perp$ is closed, i.e.,

$d\phi^{\perp} = 0$ if $D$ is integrable and $D^{\perp}$ is minimal. Hence $\phi$ is closed and co-closed too and $M$ is closed CR-submanifold which means that $\phi$ is harmonic, i.e., $\Delta\phi = 0$. Since $\phi$ is a nonzero form. Therefore $c(M) = [\phi] \neq 0$. Thus we have

**Theorem 5.1.** *Let $\pi : \overline{M} \longrightarrow B_\circ$ be a submersion of a closed CR-submanifold $M$ of a locally conformal Kaehler manifold $\overline{M}$ onto an almost Hermitian manifold $B_\circ$ with integrable horizontal distribution $D$. Then the $2p$-form $\phi$ is closed and defines de Rham cohomology $[\phi]$ in $H^{2p}(M, R)$. Moreover, the cohomology group $H^{2p}(M, R)$ is non-trivial if the vertical distribution $D^{\perp}$ is minimal.*

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