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Triangulations of Simplicial Polytopes

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Abstract. Various facts about triangulations of simplicial polytopes, particularly those pertaining to the equality case in the generalized lower bound conjecture, are collected together here. They include an apparently weaker restriction on the kind of triangulation which needs to be found, and an inductive argument which reduces the number of cases to be established.

1. Introduction

The lower bound conjecture for simplicial polytopes was proved in four and five dimensions by Walkup [22], and generally by Barnette [1, 2] just a few days before the upper bound conjecture was proved by the present author [14]. Since then, several different proofs of the lower bound conjecture have appeared; see [6, 8], and, for a general survey, [11].

At around the same time, the author and Walkup [18] formulated the generalized lower bound conjecture for simplicial polytopes (which we shall abbreviate to GLBC). This states that, if P is a simplicial d-polytope, then $g_r(P) \geqslant 0$ for each $r = 1, \ldots, \lfloor \frac{1}{2}d \rfloor$, where the g_r are certain linear combinations of the face numbers of P (we shall repeat the familiar definition in Section 2); moreover, if $g_r(P) = 0$, then P admits a triangulation with no interior $(d-r)$-faces (see Conjecture 3.1—the original lower bound conjecture is the case $r = 2$).

The proof of the g-theorem (Theorem 2.1) showed that, indeed, $g_r(P) \geqslant 0$ for each r; for the necessity of the conditions, see [20] or, for a proof using more elementary techniques, [16, 17]. (The sufficiency of the conditions of the g-theorem was established by Billera and Lee [4, 5]; this is less relevant to our present discussion, but see also [10] for an interesting stronger result about triangulations.) However, what is still lacking is a proof of the remaining part of the GLBC, namely, that characterizing the case of equality for $3 \leqslant r \leqslant \lfloor \frac{1}{2}d \rfloor$, though some ideas have been proposed in [12].

The idea behind this note is to collect together various facts about triangulations of simplicial polytopes, particularly insofar as they are relevant to the GLBC. One main result shown here is Theorem 3.5, which shows that Conjecture 3.1 holds if P has a triangulation with no interior $(r-1)$-faces. Another is Theorem 5.3, which says that, to prove the characterization of Conjecture 3.1, it is enough to establish the case $d = 2r$.

For the general background in polytope theory, the reader should consult [7, 23].

2. Triangulations

Let P be a simplicial d-polytope in d-dimensional euclidean space $\mathbb{E}^d$. We write $F \leqslant G$ to mean that F is a face of the polytope G; then $F < G$ means, in addition, that $F \neq G$. In the present context, $\emptyset$ will count as a face of every polytope P, whereas P itself generally will not.

Let $\mathcal{T}$ be a triangulation of P; we always suppose that the boundary complex of $\mathcal{T}$ coincides with that of P, and usually that the vertices of $\mathcal{T}$ are exactly those of P itself. We refer to the d-faces of $\mathcal{T}$ as *cells*. Those faces of $\mathcal{T}$ which are not faces of P itself, and so meet the interior $\operatorname{int} P$ of P, are called *interior* faces of $\mathcal{T}$; the set of these interior faces is denoted $\mathcal{T}_{\text{int}}$.

To some extent, we are interested in regular triangulations, which means that $\mathcal{T}$ will arise from the projection of the lower (or upper) surface of some simplicial $(d+1)$-polytope.

For $j = -1, \ldots, d-1$, let $f_j = f_j(P)$ be the number of j-faces of P, with the usual convention $f_{-1} = 1$ (corresponding to $\emptyset$). Although this builds in some redundancy, we find it convenient to define the *f-polynomial* of P to be

$$f(P, \sigma, \tau) := \sum_{j=0}^{d} f_{j-1} \sigma^{d-j} \tau^j, \tag{2.1}$$

a function of two indeterminates σ and τ. Observe that $f(P, \cdot, \cdot)$ encapsulates the purely numerical information about the boundary complex of P. The *h-polynomial* of P is then defined by

$$h(P, \sigma, \tau) = \sum_{r=0}^{d} h_r(P) \sigma^{d-r} \tau^r := f(P, \sigma - \tau, \tau). \tag{2.2}$$

An alternative form of (2.2) which we find useful is

$$h(P, \sigma, \tau) = \sum_{F < P} (\sigma - \tau)^{d-1-\dim F} \tau^{\dim F + 1}, \tag{2.3}$$

recalling the conventions introduced earlier. It is well known that the h-numbers $h_r = h_r(P)$ satisfy the *Dehn-Sommerville equations*

$$h_r = h_{d-r} \tag{2.4}$$

for $r = 0, \ldots, d$. Indeed, these relations hold more generally for triangulated $(d-1)$-spheres. We can write them in the convenient form

$$h(P, \sigma, \tau) = h(P, \tau, \sigma), \tag{2.5}$$

which already illustrates the utility of our convention. (Compare Theorem 2.3; in this case, all faces will be "interior".)

Of even more fundamental importance is the *g-polynomial* of P, which is given by

$$g(P, \sigma, \tau) = \sum_{r=0}^{d+1} g_r(P)\sigma^{d+1-r}\tau^r := (\sigma - \tau)h(P, \sigma, \tau). \tag{2.6}$$

In other words, with $g_r = g_r(P)$, we have

$$g_r = h_r - h_{r-1}$$

for each r, and, inversely,

$$h_r = \sum_{s \leqslant r} g_s.$$

It follows from (2.5) that the g-polynomial satisfies

$$g(P, \tau, \sigma) = -g(P, \sigma, \tau),$$

so that

$$g_r = -g_{d+1-r} \tag{2.7}$$

for $r = 0, \ldots, d+1$.

At this point, we can introduce the g-theorem. For our purposes, we define $(g_0, \ldots)$ to be an *M-sequence* if there exists some polynomial ring R generated by its elements of degree 1 (say a quotient of $\mathbb{R}[X_1, \ldots, X_k]$, for some k, by a homogeneous ideal—such a ring is often called a *standard graded algebra*), such that $g_r = \dim R_r$, the dimension of the rth graded subspace of R. Then we have

Theorem 2.1. *There exists a simplicial d-polytope P such that $g_r(P) = g_r$ for $r = 0, \ldots, \lfloor \frac{1}{2}d \rfloor$ if and only if $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ is an M-sequence.*

That part of the g-theorem which concerns us in these notes states that

$$g_r \geqslant 0 \qquad \text{for } r = 0, \ldots, \lfloor \tfrac{1}{2}d \rfloor; \tag{2.8}$$

since these g_r determine the h_r, and hence the f_j, we often think of just them as the g-numbers of P.

There are important relations between the numbers of faces of P and those of the faces of a triangulation $\mathcal{T}$ of P, or those of its interior faces; in the different notation adopted there, these comes from [18]. In leading up to them, we state some initial results in a more general context, that of triangulations of d-balls. Since $\mathcal{T}$ is a d-dimensional complex, the appropriate analogue for the f-polynomial of $\mathcal{T}$ is

$$f(\mathcal{T},\sigma,\tau) := \sum_{j=0}^{d} f_{j-1}(\mathcal{T})\sigma^{d+1-j}\tau^{j}; \tag{2.9}$$

that for the interior complex $\mathcal{T}_{\text{int}}$ is

$$f(\mathcal{T}_{\text{int}},\sigma,\tau) := \sum_{j=1}^{d} f_{j-1}(\mathcal{T}_{\text{int}})\sigma^{d+1-j}\tau^{j}, \tag{2.10}$$

where, exceptionally, we do not count the empty face (a justification for this convention is Lemma 2.2). The h-polynomials of $\mathcal{T}$ and $\mathcal{T}_{\text{int}}$ are then defined exactly as in (2.2).

There is an important relationship between $h(\mathcal{T},\cdot,\cdot)$ and $h(\mathcal{T}_{\text{int}},\cdot,\cdot)$. We begin with a subsidiary result.

Lemma 2.2. *Let $\mathcal{T}$ be a triangulation of a d-ball, and let $F \in \mathcal{T}$. Then*

$$\sum_{G \geqslant F} (-1)^{d-\dim G} = \begin{cases} 1, & \text{if } F \in \mathcal{T}_{\text{int}}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This is a concealed form of the Euler relation. If $F \in \mathcal{T}_{\text{int}}$, then the link of F in $\mathcal{T}$ is a sphere (of the appropriate dimension); otherwise, the link is a ball (in particular, this holds for $F = \emptyset$). □

Theorem 2.3. *Let $\mathcal{T}$ be a triangulation of a d-ball. Then*

$$h(\mathcal{T}_{\text{int}},\sigma,\tau) = h(\mathcal{T},\tau,\sigma).$$

Proof. We perform the calculation directly. In analogy with (2.3), and using Lemma 2.2 and the fact that each face G of $\mathcal{T}$ is a simplex, we have

$$\begin{aligned} h(\mathcal{T}_{\text{int}},\sigma,\tau) &= \sum_{F \in \mathcal{T}_{\text{int}}} (\sigma-\tau)^{d-\dim F}\tau^{\dim F+1} \\ &= \sum_{F \in \mathcal{T}} \Big(\sum_{G \geqslant F} (-1)^{d-\dim G}\Big)(\sigma-\tau)^{d-\dim F}\tau^{\dim F+1} \\ &= \sum_{G \in \mathcal{T}} (\tau-\sigma)^{d-\dim G}\Big(\sum_{F \leqslant G} (\sigma-\tau)^{\dim G-\dim F}\tau^{\dim F+1}\Big) \\ &= \sum_{G \in \mathcal{T}} (\tau-\sigma)^{d-\dim G}\sigma^{\dim G+1} \\ &= h(\mathcal{T},\tau,\sigma), \end{aligned}$$

as claimed. □

For future reference, we collect together some important facts about the h-vector $(h_0,\ldots,h_{d+1})$ (with $h_r := h_r(\mathcal{T})$) of a triangulation $\mathcal{T}$.

Proposition 2.4. *Let $\mathcal{T}$ be a triangulation of a simplicial d-polytope P. Then*
(a) $h_0 = 1$, *and* $h_r \geqslant 0$ *for* $r \geqslant 1$;
(b) *if* $h_r = 0$ *for some* r, *then* $h_s = 0$ *for* $s \geqslant r$;
(c) $h_r = 0$ *if and only if* $\mathcal{T}$ *has no interior* $(d-r)$*-faces.*

Proof. Part (a) follows from the fact that $(h_0, \ldots, h_{d+1})$ is an M-sequence; see [19, 21]. In particular, if $h_r = 0$ for some r, then $h_s = 0$ for all $s \geqslant r$, which is part (b). For part (c), we reverse the relationship between the f- and h-polynomials, to get

$$f(\mathcal{T}_{\mathrm{int}}, \sigma, \tau) = h(\mathcal{T}_{\mathrm{int}}, \sigma + \tau, \tau) = h(\mathcal{T}, \tau, \sigma + \tau),$$

by Lemma 2.3. Hence $f_{d-r}(\mathcal{T}_{\mathrm{int}})$ is the coefficient of $\sigma^r \tau^{d+1-r}$ in $h(\mathcal{T}, \tau, \sigma + \tau)$, so that

$$f_{d-r}(\mathcal{T}_{\mathrm{int}}) = \sum_{s \geqslant r} \binom{s}{r} h_s \geqslant 0,$$

with equality if and only if $h_s = 0$ for each $s \geqslant r$, and thus, in view of part (b), if and only if $h_r = 0$. □

We also need the relationship between the g-polynomial of P and the h-polynomials of $\mathcal{T}$ and $\mathcal{T}_{\mathrm{int}}$.

Theorem 2.5. *Let P be a simplicial d-polytope, and let $\mathcal{T}$ be a triangulation of P. Then*

$$\begin{aligned} g(P, \sigma, \tau) &= h(\mathcal{T}, \sigma, \tau) - h(\mathcal{T}, \tau, \sigma) \\ &= h(\mathcal{T}_{\mathrm{int}}, \tau, \sigma) - h(\mathcal{T}_{\mathrm{int}}, \sigma, \tau). \end{aligned}$$

Proof. Of course, in view of Theorem 2.3, these two equations are equivalent to each other, and also to

$$g(P, \sigma, \tau) = h(\mathcal{T}, \sigma, \tau) - h(\mathcal{T}_{\mathrm{int}}, \sigma, \tau).$$

If we observe that

$$f(\mathcal{T}, \sigma, \tau) - f(\mathcal{T}_{\mathrm{int}}, \sigma, \tau) = \sigma f(P, \sigma, \tau),$$

we deduce at once that

$$h(\mathcal{T}, \sigma, \tau) - h(\mathcal{T}_{\mathrm{int}}, \sigma, \tau) = (\sigma - \tau) h(P, \sigma, \tau) = g(P, \sigma, \tau),$$

exactly as was wanted. □

We wish to employ Theorem 2.5 in its numerical form.

Corollary 2.6. *Let P be a simplicial d-polytope, and let $\mathcal{T}$ be a triangulation of P. Then, for each $r = 0, \ldots, d+1$,*

$$g_r(P) = h_r(\mathcal{T}) - h_{d+1-r}(\mathcal{T}).$$

3. Small faces

As before, let $\mathcal{T}$ be a triangulation of a simplicial d-polytope P. A face F of $\mathcal{T}_{\text{int}}$ is called *small* if $\dim F \leqslant \lfloor \frac{1}{2}d \rfloor$. We call $\mathcal{T}$ *small-face-free*, abbreviated *sff*, if $\mathcal{T}$ has no small interior faces. (Compare here the slightly different concept of *shallow* triangulations in [3], which were, however, employed for another purpose.)

Before we discuss sff triangulations in general, let us restate the open part of the GLBC.

Conjecture 3.1. *Let P be a simplicial d-polytope, and let $1 \leqslant r \leqslant \lfloor \frac{1}{2}d \rfloor$. If $g_r(P) = 0$, then P admits a triangulation with no interior $(d-r)$-faces.*

In [18], such a polytope was called $(r-1)$-stacked (the indices there differed from those here by 1—there is a mismatch of these indices in [9]). Of course, the conjecture is known to hold if $r = 1, 2$.

Remark 3.2. In fact, we believe that Conjecture 3.1 holds in a stronger form, in that the triangulation is actually regular. Our feeling is that it will probably be necessary to establish the strong form anyway, if we are to be able to prove the conjecture at all.

Our first result helps to narrow the range we have to consider. Throughout the next part of the section, P will be a fixed simplicial d-polytope.

Theorem 3.3. *A simplicial polytope P has at most one sff triangulation.*

Proof. We consider two different triangulations $\mathcal{T}$ and $\mathcal{T}'$ of P, and show that at most one is sff. Our blanket assumption is that their boundary sub-complexes coincide with the boundary complex of P. If we work in from $\operatorname{bd} P$, it is clear that there exist two cells $F \in \mathcal{T}$ and $F' \in \mathcal{T}'$ which lie on the same side of a common $(d-1)$-face G, say, such that $F \neq F'$. If $\operatorname{vert} G = \{b_1, \ldots, b_d\}$, and the two remaining vertices involved are $b \in F$ and $b' \in F'$, then Radon's Theorem yields a partition of $\{b_1, \ldots, b_d\}$ into two disjoint subsets B and B', say, such that $\operatorname{conv}(B \cup \{b\}) \cap \operatorname{conv}(B' \cup \{b'\}) \neq \emptyset$. Indeed, there is a unique point in this intersection, and it must lie in $\operatorname{int} P$. But one at least of $\operatorname{conv}(B \cup \{b\}) \subseteq F$ and $\operatorname{conv}(B' \cup \{b'\}) \subseteq F'$ is a face of dimension at most $\lfloor \frac{1}{2}d \rfloor$, which proves our claim. □

Remark 3.4. Note that the triangulation which is supposed to exist in Conjecture 3.1 would be sff, and hence unique. We were told of this latter fact by Carl Lee (without proof, however); the foregoing proof of the more general result is our own.

Proposition 2.4 has an extremely useful implication for Conjecture 3.1.

Theorem 3.5. *In order to prove Conjecture* 3.1*, it suffices to show that $g_r(P) = 0$ implies that P has a triangulation with no interior $(r-1)$-faces.*

Proof. If we could find such a triangulation $\mathcal{T}$, then $h_{d+1-r}(\mathcal{T}) = 0$ by Proposition 2.4(c). Corollary 2.6 and the fact that $g_r(P) = 0$ now yield $h_r(\mathcal{T}) = 0$, and Proposition 2.4(c) again then implies that $\mathcal{T}$ has no interior $(d-r)$-faces, as required. □

Remark 3.6. Obviously, the condition of Theorem 3.5 extends to $\mathcal{T}$ having no interior k-faces for some $r-1 \leqslant k \leqslant d-r$, and so, in particular, to $\mathcal{T}$ being sff.

4. Combinatorial triangulations

A (pure) simplicial d-complex whose boundary complex is isomorphic to that of a given simplicial d-polytope P is called a *combinatorial triangulation* of P. In contrast, we refer to an actual triangulation of P as *geometric*. We extend the concept of sff to combinatorial triangulations in the natural way.

Theorem 4.1. *A sff combinatorial triangulation $\mathcal{T}$ of a simplicial d-polytope P is geometric.*

Proof. The proof is very similar to that of Theorem 3.3. When we try to realize a combinatorial triangulation geometrically, a cell of the geometric triangulation of P must, of course, just be the convex hull of the vertices of P corresponding to those of the combinatorial triangulation. We may thus run into two kinds of problem. What could go wrong is that either such a cell collapses into a hyperplane, or two adjacent cells fold over at their common $(d-1)$-face. In each case, Radon's Theorem leads to the existence of an interior face of dimension at most $\lfloor \frac{1}{2}d \rfloor$ (or $\lfloor \frac{1}{2}(d-1) \rfloor$ in the former case), contrary to $\mathcal{T}$ being sff. $\square$

The implication of Theorem 4.1 for the equality case Conjecture 3.1 of the GLBC is obvious—all we need is an appropriate *combinatorial* triangulation of our polytope P. However, there are no corresponding pointers to finding such a triangulation.

5. Inductive arguments

In this section, we work with the weaker form of the equality case in the GLBC, namely, Conjecture 3.1; that is, if P is a simplicial d-polytope, then $g_r(P) = 0$ implies that P has some triangulation, not necessarily regular, with no interior $(d-r)$-faces. We refer to this property as $F(d, r)$.

If $v \in \operatorname{vert} P$, then we write P/v for the vertex-figure of P at v. We first generalize a result from [14].

Proposition 5.1. *For each $r = 0, \ldots, d-1$,*

$$\sum_{v \in \operatorname{vert} P} g_r(P/v) = (d+1-r)g_r(P) + (r+1)g_{r+1}(P).$$

Proof. This result holds more generally for links of vertices of triangulated spheres, but there is a nice proof for a simplicial polytope P, which we give here. We first establish the actual result of [14]. We recall that the boundary complex of P is shellable (indeed, our proof thus extends to shellable spheres), and (following [14]) we obtain a contribution 1 to $h_r(P)$ each time when, in adjoining the new facet F in the shelling, we add a new $(r-1)$-face $G \leqslant F$ and, for each $s > r$, all the $(s-1)$-faces of F which contain G. In terms of the h-polynomial, the change $\Delta h(P, \sigma, \tau)$ in $h(P, \sigma, \tau)$ is given by

$$\Delta h(P, \sigma, \tau) = \Delta f(P, \sigma - \tau, \tau) = \sum_{s \geqslant r} \binom{d-r}{d-s} (\sigma - \tau)^{d-s} \tau^s = \sigma^{d-r} \tau^r.$$

At this stage, we repeat the observation of [14], that reversing the shelling order (which interchanges the rôles of r and $d-r$) leads directly to the Dehn-Sommerville equations (2.4).

When we look at the corresponding contributions to

$$\varphi(P,\sigma,\tau) := \sum_{v\in \operatorname{vert} P} h(P/v,\sigma,\tau),$$

these come from the d vertices of F. For the r vertices $v \in G$, we are adding a new $(r-2)$-face (namely, the facet of G opposite v), while for the other $d-r$ vertices $v \notin G$ we are adding a new $(r-1)$-face (namely, G itself). The analysis analogous to that for P yields

$$\Delta\varphi(P,\sigma,\tau) = r\sigma^{d-r}\tau^{r-1} + (d-r)\sigma^{d-r-1}\tau^{r} = \Big(\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}\Big)\sigma^{d-r}\tau^{r}.$$

Summing over $v \in \operatorname{vert} P$, we obtain the attractive formula

$$\sum_{v\in \operatorname{vert} P} h(P/v,\sigma,\tau) = \Big(\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}\Big)h(P,\sigma,\tau). \tag{5.1}$$

Since $g(P,\sigma,\tau) = (\sigma-\tau)h(P,\sigma,\tau)$, an easy calculation yields exactly the same formula for the g-polynomials, namely,

$$\sum_{v\in \operatorname{vert} P} g(P/v,\sigma,\tau) = \Big(\frac{\partial}{\partial\sigma} + \frac{\partial}{\partial\tau}\Big)g(P,\sigma,\tau). \tag{5.2}$$

Comparing the coefficients of $\sigma^{d+1-r}\tau^{r}$ in the two sides of (5.2) gives the result. □

Remark 5.2. The simple expressions for the relations of (5.1) and (5.2) provide another justification for using two indeterminates to write the h- and g-polynomials.

Our inductive argument is the following.

Theorem 5.3. *If $r \leqslant \lfloor \frac{1}{2}(d-1)\rfloor$, then $F(d-1,r)$ implies $F(d,r)$.*

Proof. First, recall that $(g_0(P),\ldots,g_{\lfloor d/2\rfloor})$ is an M-sequence (this is Theorem 2.1). Hence $g_r(P)=0$ implies that $g_{r+1}(P)=0$ also, because $r+1 \leqslant \lfloor \frac{1}{2}(d+1)\rfloor$. Since $g_r(P/v) \geqslant 0$ for each $v \in \operatorname{vert} P$, we have

$$0 \leqslant \sum_{v\in \operatorname{vert} P} g_r(P/v) = (d+1-r)g_r(P) + (r+1)g_{r+1}(P) = 0 \tag{5.3}$$

from Proposition 5.1, and so it follows that $g_r(P/v) = 0$ for each v. By the inductive assumption $F(d-1,r)$, we deduce that each vertex-figure P/v has a triangulation $\mathcal{U}_v$, say, with no interior $(d-r-1)$-faces. This induces a "local" triangulation $\mathcal{T}_v$ at v, with no interior $(d-r)$-faces containing v; initially, at least, we make no assumption that $\mathcal{T}_v$ extends to a triangulation of P itself. However, if $E = \operatorname{conv}\{v, w\}$ is an edge of P, then $\mathcal{U}_v$ induces a triangulation $\mathcal{V}_v$ of the quotient P/E of P at E which has no interior $(d-r-2)$-faces. Now

$$r \leqslant \lfloor \tfrac{1}{2}(d-1) \rfloor \implies d-r-2 \geqslant \lfloor \tfrac{1}{2}d \rfloor - 1 = \lfloor \tfrac{1}{2}(d-2) \rfloor,$$

so that $\mathcal{V}_v$ is sff. Similarly, the triangulation $\mathcal{V}_w$ of P/E induced by $\mathcal{U}_w$ is sff, and from Theorem 3.3 we conclude that $\mathcal{V}_v = \mathcal{V}_w$. It follows at once that the local triangulations $\mathcal{T}_v$ are indeed compatible, and so they fit together to form a triangulation $\mathcal{T}$ of P with no interior $(d-r)$-faces, as claimed. □

A particular case of this is the lower bound theorem for a simplicial d-polytope P, namely, $g_2(P) \geqslant 0$, with equality implying that P has a triangulation with no interior $(d-2)$-faces. Putting the original proof by Walkup [22] for $d = 4$ together with the g-theorem, we see that we have the proof (with equality) for all $d \geqslant 4$. Moreover, in this case, it is easy to see that the resulting triangulation is actually regular.

Remark 5.4. The cases $r \leqslant \lfloor \frac{1}{2}(d-3) \rfloor$ of Theorem 5.3 were (in effect) shown in [9], again using (5.3); the extension to $r \leqslant \lfloor \frac{1}{2}(d-1) \rfloor$ uses Theorems 3.3 and 4.1.

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