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Realizing Maps between Modules over Tate Cohomology Rings

Apostolos Beligiannis   Henning Krause

Department of Mathematics, University of the Aegean
83200 Karlovassi, Samos, Greece
e-mail: abeligia@mail.otenet.gr, abeligia@aegean.gr

Fakultät für Mathematik, Universität Bielefeld
D-33501 Bielefeld, Germany
e-mail: henning@mathematik.uni-bielefeld.de

Abstract. Let G be a finite group and k be a field. Given two representations A and B of G, we investigate when all homomorphisms $\widehat{H}^*(G,A) \to \widehat{H}^*(G,B)$ over the Tate cohomology ring $\widehat{H}^*(G,k)$ are of the form $\widehat{H}^*(G,\alpha)$ for some morphism $\alpha\colon A \to B$. We construct an extended Milnor sequence which computes the obstruction for homomorphisms $\widehat{H}^*(G,A) \to \widehat{H}^*(G,B)$ to be realizable.

Introduction

Let k be a field and G be a finite group. Given a graded module over the Tate cohomology ring $\widehat{H}^*(G,k)$, it is natural to ask when this module is of the form $\widehat{H}^*(G,A)$ for some module A over the group algebra kG. Recently, this question has been answered, at least for modules which are direct summands of modules of the form $\widehat{H}^*(G,A)$; see [5]. In this paper we continue this program and ask when a map $\widehat{H}^*(G,A) \to \widehat{H}^*(G,B)$ of $\widehat{H}^*(G,k)$-modules is of the form $\widehat{H}^*(G,\alpha)$ for some map $\alpha\colon A \to B$ of kG-modules. We do not give a complete answer to this question. However, we give a global answer for all maps $\widehat{H}^*(G,A) \to \widehat{H}^*(G,B)$ for a fixed kG-module A.

Theorem 1. *Let A be a kG-module. Then the following are equivalent:*

(i) *For every kG-module B, every map $\widehat{H}^*(G,A) \to \widehat{H}^*(G,B)$ of graded $\widehat{H}^*(G,k)$-modules is of the form $\widehat{H}^*(G,\alpha)$ for some map $\alpha\colon A \to B$ of kG-modules.*

(ii) *The projective dimension of $\widehat{H}^*(G,A)$ over $\widehat{H}^*(G,k)$ is at most 1 and there exists a decomposition $A = A' \oplus A''$ such that A' belongs to the localizing subcategory of the stable module category $\underline{\mathrm{Mod}}(kG)$ which is generated by k, and $\widehat{H}^*(G,A'') = 0$.*

For a p-group G this result is quite satisfactory because every kG-module belongs to the localizing subcategory generated by k. However, in general one cannot expect a simple homological condition on $\widehat{H}^*(G,A)$ which ensures that every map out of $\widehat{H}^*(G,A)$ is realizable by a map of kG-modules.

Theorem 1 is really a lot more general. We can replace the stable category of kG-modules by any triangulated category with arbitrary coproducts, and $\widehat{H}^*(G,-)$ can be replaced by any homology theory which is represented by a compact object. In this context we prove our result; see Theorem 3.3.

There is also a dual result which characterizes the fact that for a fixed kG-module B every map $\widehat{H}^*(G,A) \to \widehat{H}^*(G,B)$ is realizable by a map $A \to B$. This is stated in Theorem 4.2.

The proof of Theorem 1 and its generalization is based on a five-term exact sequence which is closely related to some generalized and extended variants of Milnor's sequence, appearing for example in [6, 4]. Our sequence describes the obstruction for the realizability of maps of the form $\widehat{H}^*(G,A) \to \widehat{H}^*(G,B)$. The obstruction is by definition the cokernel of the natural map

$$\underline{\mathrm{Hom}}_{kG}(A,B) \longrightarrow \mathrm{Hom}_{\widehat{H}^*(G,k)}\big(\widehat{H}^*(G,A), \widehat{H}^*(G,B)\big)$$

and we use for its description the ideal of maps $\alpha\colon A \to B$ satisfying $\widehat{H}^*(G,\alpha) = 0$. Such maps are known as phantom maps in algebraic topology and their analogue in group representation theory has been studied before by Benson and Gnacadja [3].

Next we discuss in this paper the realizability of extensions. It turns out that the long exact sequence which describes the realizability obstruction for maps $\widehat{H}^*(G,A) \to \widehat{H}^*(G,B)$, has a higher dimensional analogue which computes

$$\mathrm{Ext}^n_{\widehat{H}^*(G,k)}\big(\widehat{H}^*(G,A), \widehat{H}^*(G,B)\big)$$

for $n \geq 1$. We use this to describe the exact sequences $0 \to \widehat{H}^*(G,B) \to E \to \widehat{H}^*(G,A) \to 0$ which are of the form $\widehat{H}^*(G,\varepsilon)$ for some exact sequence $\varepsilon\colon 0 \to B \to C \to A \to 0$ of kG-modules.

The final part of this paper is devoted to various applications of our general theory. Most of these applications take place in the stable category of kG-modules.

Our motivating problem of realizing modules and maps over the Tate cohomology ring is closely related to the problem when Brown representability holds for homology theories. Recently, some progress has been made in [1, 8, 7], and we use some of the ideas which have been developed in these papers.

In contrast to [5], we do not exploit the A_∞-structure of Tate cohomology.

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1. Exact triangles and phantom maps

We begin this paper with a brief introduction into relative homological algebra for triangulated categories. Throughout we fix a triangulated category $\mathcal{T}$ containing all small coproducts. Given two objects A and B in $\mathcal{T}$, we denote by $\mathcal{T}(A,B)_*$ the graded Hom-group $\mathcal{T}(A,B)_* = \bigoplus_{i\in\mathbb{Z}} \mathcal{T}(\Sigma^i A, B)$. In particular, the graded Hom-group $\mathcal{T}(A,A)_*$ is a $\mathbb{Z}$-graded ring.

We fix a *homology theory* $\mathsf{H}\colon \mathcal{T} \to \mathcal{A}b$, that is, H is a functor which sends triangles to exact sequences and preserves all coproducts. We make the additional assumption that H is *represented* by an object $T \in \mathcal{T}$, that is, $\mathsf{H} = \mathcal{T}(T,-)_*$. Therefore the endomorphism ring $\Gamma = \mathcal{T}(T,T)_*$ acts in a natural way on $\mathsf{H}(A)$ for each object $A \in \mathcal{T}$. Thus we have a functor

$$\mathsf{H}\colon \mathcal{T} \longrightarrow \mathrm{Mod}(\Gamma), \quad A \mapsto \mathcal{T}(T,A)_*$$

into the category $\mathrm{Mod}(\Gamma)$ of $\mathbb{Z}$-graded Γ-modules.

There is an obvious way to generalize our setting. Instead of representing the homology theory H by a single object T, one could take a set of compact objects in $\mathcal{T}$ which is closed under the shift Σ. Recall that an object $A \in \mathcal{T}$ is *compact* if the functor $\mathcal{T}(A,-)$ preserves all coproducts. Viewing the full subcategory Γ of objects in such a set of compact objects as a $\mathbb{Z}$-graded ring with several objects, we obtain a homology theory $\mathsf{H}\colon \mathcal{T} \to \mathrm{Mod}(\Gamma)$ by sending an object A to $\mathcal{T}(-,A)|_\Gamma$.

The examples we have in mind include the following.

Example 1.1. 1. Fix a finite group G and a field k. Let $\mathcal{H}$ be the homotopy category of projective modules over the group algebra kG. Setting T to be a Tate resolution of the trivial representation k, we have that T is compact in $\mathcal{H}$ and $\Gamma = \mathcal{H}(T,T)_*$ is the Tate cohomology ring $\widehat{H}^*(G,k)$. We can use instead of $\mathcal{H}$ the full subcategory of $\mathcal{H}$ consisting of acyclic complexes of projective kG-modules, which is equivalent to the stable module category $\underline{\mathrm{Mod}}(kG)$ of the group algebra kG. The equivalence identifies T with k.

2. Let $\mathcal{D}(\mathrm{Mod}(\Lambda))$ be the derived category of unbounded complexes over a ring Λ. We fix a finitely presented Λ-module T of finite projective dimension and view T as a stalk complex concentrated in degree zero. Then T is a compact object in $\mathcal{D}(\mathrm{Mod}(\Lambda))$ and the graded endomorphism ring Γ of T equals the Yoneda algebra $\mathrm{Ext}^*_\Lambda(T,T)$. For instance, we can take T to be a tilting module, or simply $T = \Lambda$.

3. Let $\mathsf{Ho}(\mathcal{S})$ be the stable homotopy category of spectra, and let T be the sphere spectrum S^0. Then $\mathrm{Mod}(\Gamma)$ is the module category of the stable homotopy ring of spheres.

4. Let $\mathcal{T}$ be a compactly generated triangulated category and let $\mathcal{P}$ be the set of isoclasses of compact objects in $\mathcal{T}$. Viewing the full subcategory of objects in $\mathcal{P}$ as a ring Γ with several objects, we obtain a homology theory $\mathsf{H}\colon \mathcal{T} \to \mathrm{Mod}(\Gamma)$ which is the basis for the well-studied theory of purity in compactly generated triangulated categories.

We call a triangle $A \to B \to C \to \Sigma A$ in $\mathcal{T}$ H-*exact* if the induced sequence $0 \to \mathsf{H}(A) \to \mathsf{H}(B) \to \mathsf{H}(C) \to 0$ is exact in $\mathcal{A}b$. An object P in $\mathcal{T}$ is called H-*projective* if for any H-exact triangle as above, any morphism $P \to C$ factors through the map $B \to C$. An H-*projective* *presentation* of the object A is an H-exact triangle $K \to P \to A \to \Sigma A$ where P is H-projective. The concept of an H-*injective object* and an H-*injective presentation* are defined dually.

Closely related to the notion of H-exactness is the following concept. We call a map $f\colon A \to B$ H-*phantom* if the induced map $\mathsf{H}(f)$ is zero. Plainly the collection $\mathrm{Ph}_{\mathsf{H}}(A,B)$ of all H-phantom maps between A and B is a subgroup of $\mathcal{T}(A,B)$ and it is easy to see that in this way we obtain an ideal $\mathrm{Ph}_{\mathsf{H}}(-,-)$ of $\mathcal{T}$, i.e. an additive subfunctor of $\mathcal{T}(-,-)$, which is Σ-stable in the sense that $\Sigma^n f$ is H-phantom for all $n \in \mathbb{Z}$, provided that f is H-phantom. Obviously, a triangle $A \to B \to C \to \Sigma A$ in $\mathcal{T}$ is H-exact if and only if the third map $C \to \Sigma A$ is H-phantom.

Any object A of $\mathcal{T}$ admits an H-projective presentation

$$\Omega_{\mathsf{H}} A \xrightarrow{g} P(A) \xrightarrow{f} A \xrightarrow{h} \Sigma\Omega_{\mathsf{H}} A\,.$$

Indeed let $P(A) = \coprod_{i\in\mathbb{Z}} \coprod_{\Sigma^i T \to A} \Sigma^i T$ and let $f\colon P(A) \to A$ be the induced canonical morphism. Then the triangle in $\mathcal{T}$ with base f is plainly H-exact, hence it is an H-projective presentation of A since by construction $P(A)$ is H-projective. Observe that $h\colon A \to \Sigma\Omega_{\mathsf{H}} A$ is a weakly universal H-phantom map out of A in the sense that any other H-phantom map out of A factors through h. It follows from the above construction that the full subcategory of H-projective objects of $\mathcal{T}$ is identified with $\mathrm{Add}(\Sigma^* T)$, the full subcategory of $\mathcal{T}$ consisting of all direct summands of arbitrary coproducts of objects in $\{\Sigma^i T \mid i \in \mathbb{Z}\}$.

An H-*projective resolution* of an object A in $\mathcal{T}$ is a complex

$$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \longrightarrow A \longrightarrow 0 \tag{1.1}$$

in $\mathcal{T}$, where each P_n is H-projective and the induced complex

$$\cdots \longrightarrow \mathcal{T}(P,P_2) \longrightarrow \mathcal{T}(P,P_1) \longrightarrow \mathcal{T}(P,P_0) \longrightarrow \mathcal{T}(P,A) \longrightarrow 0$$

is exact for any H-projective P. Any object A admits an H-projective resolution. It is obtained by splicing together H-exact triangles

$$\Omega_{\mathsf{H}}^{n+1} A \xrightarrow{g_n} P_n \xrightarrow{f_n} \Omega^n(A) \xrightarrow{h_n} \Sigma\Omega_{\mathsf{H}}^{n+1} A$$

where each P_n is H-projective and $\Omega_{\mathsf{H}}^0(A) = A$. Defining $\delta_n = g_{n-1} \circ f_n$ for all $n \geq 1$ gives a resolution of the form (1.1). Actually any H-projective resolution is of this form. H-*injective resolutions* are defined dually.

Finally we define the H-*projective dimension* $\mathsf{H}\text{-pd}\, A$ of an object A in $\mathcal{T}$ to be the smallest integer $n \geq 0$ such that there exists an H-projective resolution (1.1) with $P_i = 0$ for all $i > n$. We put $\mathsf{H}\text{-pd}\, A = \infty$ if $\mathsf{H}\text{-pd}\, A \neq n$ for all $n \geq 0$.

2. A long exact sequence

Given two objects A and B in $\mathcal{T}$, we have a natural map $\mathsf{H}_{A,B}\colon \mathcal{T}(A,B) \to \mathrm{Hom}_{\Gamma}(\mathsf{H}(A),\mathsf{H}(B))$. In this section we construct a five-term exact sequence which describes the obstruction for this map to be surjective. We start with the following well-known result.

Lemma 2.1. *The functor* $\mathsf{H}\colon \mathcal{T} \to \mathrm{Mod}(\Gamma)$ *induces an equivalence between* $\mathrm{Add}(\Sigma^* T)$ *and the category of projective* Γ*-modules. Moreover, for any* $X \in \mathrm{Add}(\Sigma^* T)$ *and any* $A \in \mathcal{T}$ *the canonical map* $\mathsf{H}_{X,A}\colon \mathcal{T}(X, A) \to \mathrm{Hom}_\Gamma(\mathsf{H}(T), \mathsf{H}(A))$ *is invertible.*

The lemma implies that H sends H-projective presentations in $\mathcal{T}$ to projective presentations in $\mathrm{Mod}(\Gamma)$. More precisely, if $\Omega_\mathsf{H} A \to P \to A \to \Sigma\Omega_\mathsf{H} A$ is an H-projective presentation of A in $\mathcal{T}$, then the sequence $0 \to \mathsf{H}(\Omega_\mathsf{H} A) \to \mathsf{H}(P) \to \mathsf{H}(A) \to 0$ is a projective presentation of the Γ-module $\mathsf{H}(A)$. In particular we have an isomorphism $\mathsf{H}(\Omega_\mathsf{H} A) \cong \Omega\mathsf{H}(A)$ up to projective summands.

The construction of the loop object $\Omega_\mathsf{H} A$ is not in general functorial; it depends on the choice of the map $P \to A$. But it is easy to see that if $\Omega'_\mathsf{H} A \to P' \to A \to \Sigma\Omega'_\mathsf{H} A$ is another projective presentation of A in $\mathcal{T}$, then the loop objects $\Omega_\mathsf{H} A$ and $\Omega'_\mathsf{H} A$ are isomorphic up to H-projective summands: we have an isomorphism $\Omega_\mathsf{H} A \cong \Omega'_\mathsf{H} A$ in the stable category $\mathcal{T}/\mathcal{P}$ of $\mathcal{T}$ modulo the ideal of maps in $\mathcal{T}$ factorizing through an H-projective object. Moreover the assignment $A \mapsto \Omega_\mathsf{H} A$ defines a functor $\Omega_\mathsf{H}\colon \mathcal{T}/\mathcal{P} \to \mathcal{T}/\mathcal{P}$.

Lemma 2.2. *The left ideal* $\mathrm{Ph}_\mathsf{H}(\Omega_\mathsf{H} A, -)$ *of* $\mathcal{T}(\Omega_\mathsf{H} A, -)$ *is independent of the choice of the loop object* $\Omega_\mathsf{H} A$.

Proof. If $\Omega_\mathsf{H} A \to P \to A \to \Sigma\Omega_\mathsf{H} A$ and $\Omega'_\mathsf{H} A \to P' \to A \to \Sigma\Omega'_\mathsf{H} A$ are H-projective presentations, then we have the following diagram of morphisms of triangles:

$$\begin{array}{ccccccc}
\Sigma^{-1}A & \xrightarrow{-\Sigma^{-1}h} & \Omega_\mathsf{H} A & \xrightarrow{g} & P & \xrightarrow{f} & A \\
\| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \| \\
\Sigma^{-1}A & \xrightarrow{-\Sigma^{-1}h'} & \Omega'_\mathsf{H} A & \xrightarrow{g'} & P' & \xrightarrow{f'} & A \\
\| & & \downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\alpha'} & & \| \\
\Sigma^{-1}A & \xrightarrow{-\Sigma^{-1}h} & \Omega_\mathsf{H} A & \xrightarrow{g} & P & \xrightarrow{f} & A
\end{array}$$

Observe that $(1_{\Omega_\mathsf{H} A} - \beta' \circ \beta) \circ \Sigma^{-1}h = 0$. Thus there exists a map $\rho\colon P \to \Omega_\mathsf{H} A$ such that $1_{\Omega_\mathsf{H} A} - \beta' \circ \beta = \rho \circ g$. Similarly there exists a map $\sigma\colon P' \to \Omega'_\mathsf{H} A$ such that $1_{\Omega'_\mathsf{H} A} - \beta \circ \beta' = \sigma \circ g'$. Next we define, for any object $B \in \mathcal{T}$, a map $\beta'_*\colon \mathrm{Ph}_\mathsf{H}(\Omega_\mathsf{H} A, B) \to \mathrm{Ph}_\mathsf{H}(\Omega'_\mathsf{H} A, B)$ by $\beta'_*(\phi) = \phi \circ \beta'$. Similarly we define a map $\beta_*\colon \mathrm{Ph}_\mathsf{H}(\Omega'_\mathsf{H} A, B) \to \mathrm{Ph}_\mathsf{H}(\Omega_\mathsf{H} A, B)$ by $\beta_*(\psi) = \psi \circ \beta$. We claim that β'_* is invertible with inverse the map β_*. Indeed for any H-phantom map $\phi\colon \Omega_\mathsf{H} A \to B$ we have $(\beta_* \circ \beta'_*)(\phi) = \phi \circ \beta' \circ \beta$. Since $1_{\Omega_\mathsf{H} A} - \beta' \circ \beta = \rho \circ g$, it follows that $(\beta_* \circ \beta'_*)(\phi) = \phi - \phi \circ \rho \circ g$. But ϕ, hence $\phi \circ \rho$, is an H-phantom map. Since $\phi \circ \rho$ starts from the H-projective object P, it follows that $\phi \circ \rho = 0$. Hence $(\beta_* \circ \beta'_*)(\phi) = \phi$ and we infer that $\beta_* \circ \beta'_* = 1_{\mathrm{Ph}_\mathsf{H}(\Omega_\mathsf{H} A, B)}$. In a dual way we have $\beta'_* \circ \beta_* = 1_{\mathrm{Ph}_\mathsf{H}(\Omega'_\mathsf{H} A, B)}$ which shows that β'_* is invertible with inverse the map β_*. □

Remark 2.3. If $\alpha\colon A' \to A$ is a morphism in $\mathcal{T}$, then we have a morphism of H-projective presentations

$$\begin{array}{ccccccc}
\Omega_\mathsf{H} A' & \xrightarrow{g'} & P' & \xrightarrow{f'} & A' & \xrightarrow{h'} & \Sigma\Omega_\mathsf{H} A' \\
\downarrow{\scriptstyle\Omega_\mathsf{H}\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\Sigma\Omega_\mathsf{H}\alpha} \\
\Omega_\mathsf{H} A & \xrightarrow{g} & P & \xrightarrow{f} & A & \xrightarrow{h} & \Sigma\Omega_\mathsf{H} A
\end{array}$$

We claim that for any object B in $\mathcal{T}$ the induced map $\mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}\alpha, B)\colon \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A, B) \to \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A', B)$ given by $\phi \mapsto \phi \circ \Omega_{\mathsf{H}}\alpha$, does not depend on the choices we made for the compatible maps $P' \to P$ and $\Omega_{\mathsf{H}}A' \to \Omega_{\mathsf{H}}A$. Indeed if

$$\begin{array}{ccccccc} \Omega A' & \xrightarrow{g'} & P' & \xrightarrow{f'} & A' & \xrightarrow{h'} & \Sigma\Omega_{\mathsf{H}}A' \\ {\scriptstyle \Omega'_{\mathsf{H}}\alpha}\downarrow & & {\scriptstyle \beta'}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \Sigma\Omega'_{\mathsf{H}}\alpha}\downarrow \\ \Omega_{\mathsf{H}}A & \xrightarrow{g} & P & \xrightarrow{f} & A & \xrightarrow{h} & \Sigma\Omega_{\mathsf{H}}A \end{array}$$

is another morphism of H-projective presentations induced by α, then there exists a morphism $\sigma\colon P' \to \Omega_{\mathsf{H}}A$ such that $\Omega_{\mathsf{H}}\alpha - \Omega'_{\mathsf{H}}\alpha = \sigma \circ g'$. Then for any H-phantom map $\phi\colon \Omega_{\mathsf{H}}A \to B$ we have $\phi \circ (\Omega_{\mathsf{H}}\alpha - \Omega'_{\mathsf{H}}\alpha) = \phi \circ \sigma \circ g'$. But $\phi \circ \sigma = 0$ as an H-phantom map out of the H-projective object P'. Hence $\phi \circ \Omega_{\mathsf{H}}\alpha = \phi \circ \Omega'_{\mathsf{H}}\alpha$.

Let A be an object in $\mathcal{T}$ and choose an H-projective presentation $\Omega_{\mathsf{H}}A \xrightarrow{g} P \xrightarrow{f} A \xrightarrow{h} \Sigma\Omega_{\mathsf{H}}A$ of A in $\mathcal{T}$. Then for any object B in $\mathcal{T}$, we define a map

$$\zeta_{A,B}\colon \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A, B) \longrightarrow \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B)$$

by sending an H-phantom $\phi\colon \Omega_{\mathsf{H}}A \to B$ to $\zeta_{A,B}(\phi) = \phi \circ \Sigma^{-1}h$. Note that $\zeta_{A,B}(\phi)$ lies in $\mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B)$ since $\Sigma^{-1}h$ is an H-phantom map.

Lemma 2.4. *The natural map* $\zeta_{A,B}\colon \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A, B) \longrightarrow \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B)$ *is independent of the choice of loops.*

Proof. Let $\zeta'_{A,B}\colon \mathrm{Ph}_{\mathsf{H}}(\Omega'_{\mathsf{H}}A, B) \to \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B)$ be the map resulting from another H-projective presentation $\Omega'_{\mathsf{H}}A \to P' \to A \xrightarrow{h'} \Sigma\Omega'_{\mathsf{H}}A$ of A in $\mathcal{T}$, that is $\zeta'_{A,B}(\psi) = \psi \circ \Sigma^{-1}h'$. We claim that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A, B) & \xrightarrow{\zeta_{A,B}} & \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B) \\ {\scriptstyle \beta'_*}\downarrow{\scriptstyle \cong} & & \| \\ \mathrm{Ph}_{\mathsf{H}}(\Omega'_{\mathsf{H}}A, B) & \xrightarrow{\zeta'_{A,B}} & \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B) \end{array}$$

where β'_* is the isomorphism constructed in Lemma 2.2. Indeed for any H-phantom map $\phi\colon \Omega_{\mathsf{H}}A \to B$, we have $(\zeta'_{A,B} \circ \beta'_*)(\phi) = \zeta'_{A,B}(\phi \circ \beta') = \phi \circ \beta' \circ \Sigma^{-1}h' = \phi \circ \Sigma^{-1}h = \zeta_{A,B}(\phi)$. Hence the above diagram commutes and this shows that the map $\zeta_{A,B}$ does not depend on the choices of the loop objects. □

Remark 2.5. As in Remark 2.3 one can prove that any morphism $\alpha\colon A' \to A$ in $\mathcal{T}$ induces a commutative square

$$\begin{array}{ccc} \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A, B) & \xrightarrow{\zeta_{A,B}} & \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B) \\ {\scriptstyle \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}\alpha, B)}\downarrow & & {\scriptstyle \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}\alpha, B)}\downarrow \\ \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A', B) & \xrightarrow{\zeta_{A',B}} & \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A', B) \end{array}$$

The main result of this section is the following.

Proposition 2.6. *For any objects A, B in $\mathcal{T}$, there exists a natural map*

$$\vartheta_{A,B}\colon \operatorname{Hom}_\Gamma\big(\mathsf{H}(A), \mathsf{H}(B)\big) \longrightarrow \operatorname{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A, B)$$

which induces a functorial exact sequence

$$0 \longrightarrow \operatorname{Ph}_{\mathsf{H}}(A,B) \longrightarrow \mathcal{T}(A,B) \xrightarrow{\mathsf{H}_{A,B}} \operatorname{Hom}_\Gamma\big(\mathsf{H}(A), \mathsf{H}(B)\big) \xrightarrow{\vartheta_{A,B}} \operatorname{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A, B) \xrightarrow{\zeta_{A,B}} \operatorname{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B) \longrightarrow 0\ . \tag{2.1}$$

Proof. First we need to define the map $\vartheta_{A,B}$. To this end choose an H-projective presentation $\Omega_{\mathsf{H}}A \xrightarrow{g} P \xrightarrow{f} A \xrightarrow{h} \Sigma\Omega_{\mathsf{H}}A$ of A. Then we have a short exact sequence $0 \rightarrow \mathsf{H}(\Omega_{\mathsf{H}}A) \xrightarrow{\mathsf{H}(g)} \mathsf{H}(P) \xrightarrow{\mathsf{H}(f)} \mathsf{H}(A) \rightarrow 0$ in $\operatorname{Mod}(\Gamma)$. For any morphism $\widetilde{\alpha}\colon \mathsf{H}(A) \rightarrow \mathsf{H}(B)$ in $\operatorname{Mod}(\Gamma)$, there exists a unique morphism $\alpha'\colon P \rightarrow B$ such that $\mathsf{H}(\alpha') = \widetilde{\alpha}\circ\mathsf{H}(f)$. Then the morphism $\alpha'\circ g\colon \Omega_{\mathsf{H}}A \rightarrow B$ is H-phantom, since $\mathsf{H}(\alpha'\circ g) = \widetilde{\alpha}\circ\mathsf{H}(f)\circ\mathsf{H}(g) = 0$. We define $\vartheta_{A,B}(\widetilde{\alpha}) = \alpha'\circ g$, and it is easy to see that this gives a well-defined homomorphism $\vartheta_{A,B}\colon \operatorname{Hom}_\Gamma\big(\mathsf{H}(A), \mathsf{H}(B)\big) \rightarrow \operatorname{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A, B)$.

Let $\widetilde{\alpha}\colon \mathsf{H}(A) \rightarrow \mathsf{H}(B)$ be in $\operatorname{Ker}\vartheta_{A,B}$, i.e. $\alpha'\circ g = 0$. Then there exists a morphism $\rho\colon A \rightarrow B$ such that $\alpha' = \rho\circ f$. Then $\mathsf{H}(\alpha') = \mathsf{H}(\rho)\circ\mathsf{H}(f) = \widetilde{\alpha}\circ\mathsf{H}(f)$, hence $\widetilde{\alpha} = \mathsf{H}(\rho)$. The morphism ρ is uniquely determined up to H-phantom maps. Indeed if $\alpha' = \sigma\circ f$, then the morphism $\rho - \sigma$ factors through the H-phantom map h, hence it is H-phantom. It follows that the map $\mathcal{T}(A,B)/\operatorname{Ph}_{\mathsf{H}}(A,B) \rightarrow \operatorname{Hom}_\Gamma\big(\mathsf{H}(A), \mathsf{H}(B)\big)$ induced by $\mathsf{H}_{A,B}$ is the kernel of $\vartheta_{A,B}$, and this proves the exactness of the first part of the sequence (2.1).

We now show that $\zeta_{A,B}$ is surjective. So let β be in $\operatorname{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B)$. Then there exists a factorization $\beta = \beta_2\circ\beta_1\colon \Sigma^{-1}A \xrightarrow{\beta_1} X \xrightarrow{\beta_2} B$ where the β_i are H-phantoms. Since β_1 is H-phantom, the composition $\beta_1\circ\Sigma^{-1}f$ is zero. Hence there exists a morphism $\gamma\colon \Omega_{\mathsf{H}}A \rightarrow X$ such that $\beta_1 = \gamma\circ\Sigma^{-1}h$. Consider the H-phantom map $\delta = \beta_2\circ\gamma\colon \Omega_{\mathsf{H}}A \rightarrow B$. Then by construction we have $\zeta_{A,B}(\delta) = \delta\circ\Sigma^{-1}h = \beta$. We infer that $\zeta_{A,B}$ is surjective.

It remains to prove that $\operatorname{Ker}\zeta_{A,B} = \operatorname{Im}\vartheta_{A,B}$. First let $\alpha\colon \Omega_{\mathsf{H}}A \rightarrow B$ be an H-phantom map such that $\zeta_{A,B}(\alpha) = \alpha\circ\Sigma^{-1}h = 0$. Then there exists a morphism $\beta\colon P \rightarrow B$ such that $\alpha = \beta\circ g$. Since α is H-phantom we have $0 = \mathsf{H}(\alpha) = \mathsf{H}(\beta)\circ\mathsf{H}(g)$. Hence there exists a unique morphism $\gamma\colon \mathsf{H}(A) \rightarrow \mathsf{H}(B)$ such that $\gamma\circ\mathsf{H}(f) = \mathsf{H}(\beta)$. Then by construction $\vartheta_{A,B}(\gamma) = \beta\circ g = \alpha$. Hence α lies in $\operatorname{Im}\vartheta_{A,B}$. Finally let $\alpha\colon \Omega_{\mathsf{H}}A \rightarrow B$ be in the image of $\vartheta_{A,B}$, that is $\alpha = \alpha'\circ g$ where $\alpha'\colon P \rightarrow B$ is the unique morphism such that $\mathsf{H}(\alpha') = \widetilde{\alpha}\circ\mathsf{H}(f)$ for some morphism $\widetilde{\alpha}\colon \mathsf{H}(A) \rightarrow \mathsf{H}(B)$. Then $\zeta_{A,B}(\alpha) = \alpha\circ\Sigma^{-1}h = \alpha'\circ g\circ\Sigma^{-1}h = 0$. Hence $\vartheta_{A,B}(\alpha) \in \operatorname{Ker}\zeta_{A,B}$. We infer that $\operatorname{Ker}\zeta_{A,B} = \operatorname{Im}\vartheta_{A,B}$, hence the sequence (2.1) is exact. The naturality of (2.1) follows from the preceding discussion in this section. □

The above result suggests the following definition.

Definition 2.7. *Let A, B be objects of $\mathcal{T}$. The obstruction group $\mathcal{O}_{A,B}$ of A and B is the cokernel of the map $\mathcal{T}(A,B) \xrightarrow{\mathsf{H}_{A,B}} \operatorname{Hom}_\Gamma\big(\mathsf{H}(A), \mathsf{H}(B)\big)$.*

Thus we have an isomorphism $\mathcal{O}_{A,B} \cong \operatorname{Ker} \zeta_{A,B} = \operatorname{Im} \vartheta_{A,B}$. Next we construct a sequence which computes $\operatorname{Ext}^{n+1}_{\Gamma}\big(\mathsf{H}(A), \mathsf{H}(B)\big)$.

Proposition 2.8. *For any objects A, B in $\mathcal{T}$ and $n \geq 0$, there exist a natural map*

$$\varphi_{\Omega^n_{\mathsf{H}} A,B} \colon \operatorname{Ph}_{\mathsf{H}}(\Sigma^{-1}\Omega^n_{\mathsf{H}} A, B) \longrightarrow \operatorname{Ext}^{n+1}_{\Gamma}\big(\mathsf{H}(A), \mathsf{H}(B)\big)$$

which induces a functorial exact sequence

$$\begin{array}{l} 0 \longrightarrow \operatorname{Ph}^2_{\mathsf{H}}(\Sigma^{-1}\Omega^n_{\mathsf{H}} A, B) \longrightarrow \operatorname{Ph}_{\mathsf{H}}(\Sigma^{-1}\Omega^n_{\mathsf{H}} A, B) \xrightarrow{\varphi_{\Omega^n_{\mathsf{H}} A,B}} \\ \qquad \operatorname{Ext}^{n+1}_{\Gamma}(\mathsf{H}(A), \mathsf{H}(B)) \longrightarrow \mathcal{O}_{\Omega^{n+1}_{\mathsf{H}} A,B} \longrightarrow 0\ . \end{array} \tag{2.2}$$

Proof. The H-exact triangle $\Omega_{\mathsf{H}} A \xrightarrow{g} P \xrightarrow{f} A \xrightarrow{h} \Sigma\Omega_{\mathsf{H}} A$ induces the following exact commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathcal{T}(A,B) & \xrightarrow{\mathcal{T}(f,B)} & \mathcal{T}(P,B) & \xrightarrow{\mathcal{T}(g,B)} \\ & \downarrow{\scriptstyle \mathsf{H}_{A,B}} & & \downarrow{\scriptstyle \mathsf{H}_{P,B}}\ \cong & \\ 0 \longrightarrow & \operatorname{Hom}_{\Gamma}\big(\mathsf{H}(A), \mathsf{H}(B)\big) & \xrightarrow{(\mathsf{H}(f),\mathsf{H}(B))} & \operatorname{Hom}_{\Gamma}\big(\mathsf{H}(P), \mathsf{H}(B)\big) & \xrightarrow{(\mathsf{H}(g),\mathsf{H}(B))} \end{array}$$

$$\begin{array}{ccccc} \mathcal{T}(\Omega_{\mathsf{H}} A, B) & \xrightarrow{\varepsilon} & \operatorname{Im}\mathcal{T}(\Sigma^{-1}h, B) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathsf{H}_{\Omega_{\mathsf{H}} A,B}} & & \downarrow{\scriptstyle \varphi} & & \\ \operatorname{Hom}_{\Gamma}\big(\mathsf{H}(\Omega_{\mathsf{H}} A), \mathsf{H}(B)\big) & \xrightarrow{\delta} & \operatorname{Ext}^1_{\Gamma}\big(\mathsf{H}(A), \mathsf{H}(B)\big) & \longrightarrow & 0 \end{array}$$

in which the morphism $\mathsf{H}_{P,B}$ is invertible. Therefore we have an exact sequence $0 \to \mathcal{O}_{A,B} \to \operatorname{Ker}\varepsilon \to \operatorname{Ker}\delta \to 0$. Next observe that $\operatorname{Im}\mathcal{T}(\Sigma^{-1}h, B) = \operatorname{Ph}_{\mathsf{H}}(\Sigma^{-1}A, B)$. Using the property of $\zeta_{A,B}$ from Proposition 2.6, the right hand square induces the following exact commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{A,B} & \longrightarrow & \operatorname{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}} A, B) & \xrightarrow{\zeta_{A,B}} & \operatorname{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}\varepsilon & \longrightarrow & \mathcal{T}(\Omega_{\mathsf{H}} A, B) & \longrightarrow & \operatorname{Ph}_{\mathsf{H}}(\Sigma^{-1}A, B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi_{A,B}} & & \\ 0 & \longrightarrow & \operatorname{Ker}\delta & \longrightarrow & \operatorname{Hom}_{\Gamma}\big(\mathsf{H}(\Omega_{\mathsf{H}} A), \mathsf{H}(B)\big) & \longrightarrow & \operatorname{Ext}^1_{\Gamma}(\mathsf{H}(A), \mathsf{H}(B)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & \mathcal{O}_{\Omega_{\mathsf{H}} A,B} & \xrightarrow{=} & \mathcal{O}_{\Omega_{\mathsf{H}} A,B} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

The right hand vertical exact sequence is the desired sequence for $n = 0$. The general case is obtained by replacing A with $\Omega^n_{\mathsf{H}} A$ for $n \geq 1$. □

Remark 2.9. The map $\varphi_{A,B}\colon \mathrm{Ph}_{\mathsf{H}}(\Sigma^{-1}A, B) \to \mathrm{Ext}^1_\Gamma\big(\mathsf{H}(A), \mathsf{H}(B)\big)$ has an explicit description which we include for later reference. Let $\alpha\colon \Sigma^{-1}A \to B$ be an H-phantom map. We complete this to a triangle $\Sigma^{-1}A \to B \to C \to A$ and apply H to obtain an exact sequence $\varepsilon\colon 0 \to \mathsf{H}(B) \to \mathsf{H}(C) \to \mathsf{H}(A) \to 0$. We have $\varphi_{A,B}(\alpha) = \varepsilon$.

Combining the exact sequences (2.1) and (2.2) we have the following.

Corollary 2.10. *For any objects A, B in $\mathcal{T}$ and any $n \geq 1$, there exists an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}\Omega^{n-1}_{\mathsf{H}}A, B) \longrightarrow &\mathrm{Ph}_{\mathsf{H}}(\Sigma^{-1}\Omega^{n-1}_{\mathsf{H}}A, B) \xrightarrow{\varphi_{\Omega^{n-1}_{\mathsf{H}}A,B}} \\ \mathrm{Ext}^n_\Gamma(\mathsf{H}(A), \mathsf{H}(B)) \longrightarrow &\mathrm{Ph}_{\mathsf{H}}(\Omega^{n+1}_{\mathsf{H}}A, B) \xrightarrow{\zeta_{\Omega^n_{\mathsf{H}}A,B}} \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}\Omega^n_{\mathsf{H}}A, B) \longrightarrow 0\ . \end{aligned}$$

We end this section with a remark which is useful for the computation of the ideal $\mathrm{Ph}_{\mathsf{H}}(-,-)$.

Remark 2.11. Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$ be an H-exact triangle. Then for any $X \in \mathcal{T}$, the following sequences of H-phantom ideals are exact

$$\begin{aligned} \mathrm{Ph}_{\mathsf{H}}(C, X) \xrightarrow{\beta_*} \mathrm{Ph}_{\mathsf{H}}(B, X) \xrightarrow{\alpha_*} \mathrm{Ph}_{\mathsf{H}}(A, X) \\ \mathrm{Ph}_{\mathsf{H}}(X, A) \xrightarrow{\alpha^*} \mathrm{Ph}_{\mathsf{H}}(X, B) \xrightarrow{\beta^*} \mathrm{Ph}_{\mathsf{H}}(X, C)\ . \end{aligned}$$

Indeed, let $\rho\colon B \to X$ be an H-phantom map such that $\alpha_*(\rho) = \rho \circ \alpha = 0$. Then there exists a morphism $\sigma\colon C \to X$ such that $\sigma \circ \beta = \rho$. Hence $\beta_*(\sigma) = \rho$ and it suffices to show that σ is H-phantom. But $\mathsf{H}(\sigma) \circ \mathsf{H}(\beta) = \mathsf{H}(\rho) = 0$. Hence $\mathsf{H}(\sigma) = 0$ since $\mathsf{H}(\beta)$ is an epimorphism. It follows that σ is H-phantom. The exactness of the second sequence is proved similarly.

3. Realizability of morphisms

We say that a morphism $\widetilde{\alpha}\colon \mathsf{H}(A) \to \mathsf{H}(B)$ in $\mathrm{Mod}(\Gamma)$ is *realizable* if there exists a morphism $\alpha\colon A \to B$ in $\mathcal{T}$ such that $\widetilde{\alpha} = \mathsf{H}(\alpha)$. The exact sequence

$$\mathcal{T}(A, B) \xrightarrow{\mathsf{H}_{A,B}} \mathrm{Hom}_\Gamma\big(\mathsf{H}(A), \mathsf{H}(B)\big) \longrightarrow \mathcal{O}_{A,B} \longrightarrow 0$$

shows that the map $\widetilde{\alpha}\colon \mathsf{H}(A) \to \mathsf{H}(B)$ in $\mathrm{Mod}(\Gamma)$ is realizable if and only if its image in the obstruction group $\mathcal{O}_{A,B}$ vanishes. Hence all maps between $\mathsf{H}(A)$ and $\mathsf{H}(B)$ are realizable if and only if $\mathcal{O}_{A,B} = 0$. We denote by $\mathcal{O}_{A,-}\colon \mathcal{T} \to \mathcal{A}b$ the functor defined by $\mathcal{O}_{A,-}(B) = \mathcal{O}_{A,B}$. The functor $\mathcal{O}_{-,B}$ is defined dually. Note that $\mathcal{O}_{A,-}$ preserves all products, whereas $\mathcal{O}_{-,B}$ preserves all coproducts.

Our aim in this section is a characterization of the objects A satisfying $\mathcal{O}_{A,-} = 0$. We start with some preparations. The Jacobson radical of a ring Λ is denoted by $\mathrm{Rad}\,\Lambda$.

Lemma 3.1. *Let A be an object in $\mathcal{T}$. Then the following are equivalent.*

(i) $\mathrm{Ph}^2_{\mathsf{H}}(A, A) = 0$ *and* $\mathcal{O}_{A,-} = 0$.

(ii) $\mathrm{Ph}_{\mathsf{H}}(A,A) \subseteq \mathrm{Rad}\,\mathcal{T}(A,A)$ *and* $\mathcal{O}_{A,-} = 0$.
(iii) $\mathsf{H}\text{-pd}\,A \leq 1$.

Proof. (i) $\Rightarrow$ (ii): Clear, since $\mathrm{Rad}\,\mathcal{T}(A,A)$ contains any nilpotent ideal of $\mathcal{T}(A,A)$.

(ii) $\Rightarrow$ (iii): Let $\Omega_{\mathsf{H}}A \xrightarrow{g_0} P_0 \xrightarrow{f_0} A \xrightarrow{h_0} \Sigma\Omega_{\mathsf{H}}A$ and $\Omega^2_{\mathsf{H}}A \xrightarrow{g_1} P_1 \xrightarrow{f_1} \Omega_{\mathsf{H}}A \xrightarrow{h_1} \Sigma\Omega^2_{\mathsf{H}}A$ be H-projective presentations of A and $\Omega_{\mathsf{H}}A$ respectively. Then the composition $\delta_1 := g_0 \circ f_1 : P_1 \longrightarrow P_0$ induces the following octahedral diagram:

$$\begin{array}{ccccccc}
P_1 & \xrightarrow{f_1} & \Omega_{\mathsf{H}}A & \xrightarrow{h_1} & \Sigma\Omega^2_{\mathsf{H}}A & \xrightarrow{-\Sigma g_1} & \Sigma P_1\\
\| & & {\scriptstyle g_0}\downarrow & & {\scriptstyle \gamma}\downarrow & & \|\\
P_1 & \xrightarrow{\delta_1} & P_0 & \xrightarrow{\kappa} & B & \xrightarrow{\lambda} & \Sigma P_1\\
{\scriptstyle f_1}\downarrow & & \| & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \Sigma f_1}\downarrow\\
\Omega_{\mathsf{H}}A & \xrightarrow{g_0} & P_0 & \xrightarrow{f_0} & A & \xrightarrow{h_0} & \Sigma\Omega_{\mathsf{H}}A\\
 & & & & {\scriptstyle \Sigma h_1 \circ h_0}\downarrow & & \\
 & & & & \Sigma^2\Omega^2_{\mathsf{H}}A & &
\end{array}$$

where the third vertical triangle is H-exact. Since h_0 is H-phantom the same is true for $h_0 \circ \beta = \Sigma f_1 \circ \lambda$. Hence $\mathsf{H}(\Sigma f_1 \circ \lambda) = 0$ and consequently there exists a unique map $\varepsilon\colon \mathsf{H}(B) \longrightarrow \mathsf{H}(\Sigma\Omega^2_{\mathsf{H}}A)$ such that $\mathsf{H}(\Sigma g_1) \circ \varepsilon = \mathsf{H}(\lambda)$. Then $\mathsf{H}(\Sigma g_1) \circ \varepsilon \circ \mathsf{H}(\gamma) = \mathsf{H}(\lambda) \circ \mathsf{H}(\gamma) = -\mathsf{H}(\Sigma g_1)$. Since $\mathsf{H}(\Sigma g_1)$ is monic we have $(-\varepsilon) \circ \mathsf{H}(\gamma) = 1_{\mathsf{H}(\Sigma\Omega^2_{\mathsf{H}}A)}$, hence $\mathsf{H}(\gamma)$ is split monic. Then from the short exact sequence $0 \longrightarrow \mathsf{H}(\Sigma\Omega^2_{\mathsf{H}}A) \xrightarrow{\mathsf{H}(\gamma)} \mathsf{H}(B) \xrightarrow{\mathsf{H}(\beta)} \mathsf{H}(A) \longrightarrow 0$ we infer that $\mathsf{H}(\beta)$ is split epic. Hence there exists a map $\widetilde{\alpha}\colon \mathsf{H}(A) \longrightarrow \mathsf{H}(B)$ such that $\mathsf{H}(\beta) \circ \widetilde{\alpha} = 1_{\mathsf{H}(A)}$. Since $\mathcal{O}_{A,-} = 0$ the map $\widetilde{\alpha}$ is realizable, hence there exists a map $\alpha\colon A \longrightarrow B$ such that $\mathsf{H}(\alpha) = \widetilde{\alpha}$. Then $1_A - \beta \circ \alpha$ belongs to $\mathrm{Ph}_{\mathsf{H}}(A,A)$. Using the hypothesis $\mathrm{Ph}_{\mathsf{H}}(A,A) \subseteq \mathrm{Rad}\,\mathcal{T}(A,A)$ we infer that $\beta \circ \alpha$ is invertible. Hence β is split epic and consequently $\Sigma h_1 \circ h_0 = 0$. Since $\mathcal{O}_{A,-} = 0$, by Proposition 2.6 the map $\zeta_{A,-}\colon \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A,-) \longrightarrow \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A,-)$ is invertible. Since $\zeta_{A,\Sigma\Omega^2_{\mathsf{H}}A}(h_1) = h_1 \circ \Sigma^{-1}h_0 = 0$, we infer that $h_1 = 0$ and consequently f_1 is split epic. Hence $\Omega_{\mathsf{H}}A$ is H-projective and therefore $\mathsf{H}\text{-pd}\,A \leq 1$.

(iii) $\Rightarrow$ (i): Since $\Omega_{\mathsf{H}}A$ is H-projective, the exact sequence (2.1) shows that $\mathcal{O}_{A,-} = 0$ and $\mathrm{Ph}^2_{\mathsf{H}}(A,-) = \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A,-) = 0$. □

We denote by $\mathcal{T}'$ the localizing subcategory of $\mathcal{T}$ which is generated by the compact object T representing H. That is, $\mathcal{T}'$ is the smallest full triangulated subcategory containing T which is closed under taking arbitrary coproducts. This category is compactly generated, and therefore the inclusion functor $\mathcal{T}' \to \mathcal{T}$ has a right adjoint $\mathsf{R}\colon \mathcal{T} \to \mathcal{T}'$; see [9]. For each object $A \in \mathcal{T}$, we denote by $\varepsilon_A\colon \mathsf{R}(A) \to A$ the corresponding counit.

Lemma 3.2. *Let A be an object in $\mathcal{T}'$.*
(i) $\mathrm{Ph}_{\mathsf{H}}(A,A) \subseteq \mathrm{Rad}\,\mathcal{T}(A,A)$.
(ii) $\mathsf{H}\text{-pd}\,A = \mathrm{pd}_{\Gamma}\,\mathsf{H}(A)$.

Proof. (i) This follows from the fact that H reflects isomorphisms between objects in $\mathcal{T}'$.

(ii) We have always $\mathsf{H}\text{-pd}\, A \geq \mathrm{pd}_\Gamma \mathsf{H}(A)$. Now suppose A belongs to $\mathcal{T}'$. It is not difficult to see that A is H-projective if and only $\mathsf{H}(A)$ is projective. The equality $\mathsf{H}\text{-pd}\, A = \mathrm{pd}_\Gamma \mathsf{H}(A)$ then follows by induction. □

The following result characterizes the realizability of morphisms, using homological conditions in the module category $\mathrm{Mod}(\Gamma)$.

Theorem 3.3. *Let A be an object of $\mathcal{T}$. Then the following are equivalent.*

(i) *For any object B in $\mathcal{T}$, all maps $\mathsf{H}(A) \longrightarrow \mathsf{H}(B)$ are realizable.*

(ii) $\mathcal{O}_{A,-} = 0$.

(iii) $\mathrm{pd}_\Gamma \mathsf{H}(A) \leq 1$ *and the counit $\varepsilon_A \colon \mathsf{R}(A) \to A$ is a section.*

(iv) $\mathrm{pd}_\Gamma \mathsf{H}(A) \leq 1$ *and there is a decomposition $A = A' \oplus A''$ such that A' belongs to the localizing subcategory which is generated by the object representing H, and $\mathsf{H}(A'') = 0$.*

Proof. (i) ⇔ (ii): This follows from the definitions.

(ii) ⇒ (iii): The counit $\varepsilon_A \colon \mathsf{R}(A) \to A$ induces an isomorphism $\mathsf{H}(\mathsf{R}(A)) \xrightarrow{\cong} \mathsf{H}(A)$. Therefore $\mathcal{O}_{\mathsf{R}(A),-} = 0$. Combining Lemma 3.1 and 3.2 we obtain

$$\mathrm{pd}_\Gamma \mathsf{H}(A) = \mathrm{pd}_\Gamma \mathsf{H}(\mathsf{R}(A)) = \mathsf{H}\text{-pd}\, \mathsf{R}(A) \leq 1.$$

The inverse for the counit $\varepsilon_A \colon \mathsf{R}(A) \to A$ is obtained by realizing the inverse of the isomorphism $\mathsf{H}(\varepsilon_{\mathsf{R}(A)})$.

(iii) ⇒ (iv): Take $A' = \mathsf{R}(A)$ and let A'' be the cofibre of the counit ε_A. This gives a decomposition $A = A' \oplus A''$.

(iv) ⇒ (ii): We apply again Lemma 3.1 and 3.2. We have $\mathrm{pd}_\Gamma \mathsf{H}(A') = \mathrm{pd}_\Gamma \mathsf{H}(A) \leq 1$ and therefore $\mathcal{O}_{A',-} = 0$. Using the canonical map $A \to A'$, we see that $\mathcal{O}_{A,-} = 0$. □

The following example shows that we cannot expect any homological criterion on $\mathsf{H}(A)$ which decides the realizability of maps $\mathsf{H}(A) \to \mathsf{H}(B)$.

Example 3.4. Let Λ be the ring of upper triangular 2×2 matrices over a field k, and let $\mathcal{T} = \mathcal{D}(\mathrm{Mod}(\Lambda))$ be the derived category of unbounded complexes over Λ. We number the indecomposable projective Λ-modules P_1 and P_2 such that $\mathrm{Hom}_\Lambda(P_1, P_2) \cong k$ and view them as complexes concentrated in degree 0. Let $\mathsf{H} = \mathcal{T}(P_1, -)_*$. Then $\Gamma = \mathcal{T}(P_1, P_1)_* \cong k$ and the non-zero map $\alpha \colon P_1 \to P_2$ induces an isomorphism $\mathsf{H}(P_1) \xrightarrow{\cong} \mathsf{H}(P_2)$. Clearly, $\mathsf{H}(P_2) \cong k$ is a projective Γ-module. However, the map $\mathsf{H}(\alpha)^{-1}$ is not realizable since $\mathcal{T}(P_2, P_1) = 0$.

4. A dual result

In this section we discuss the dual statement of Theorem 3.3. Thus we characterize the fact that for a fixed object B in $\mathcal{T}$ all maps $\mathsf{H}(A) \to \mathsf{H}(B)$ are realizable. This requires an extra assumption on the triangulated category $\mathcal{T}$. Throughout this section we assume that $\mathcal{T}$ satisfies *Brown representability for cohomology theories*, that is, for every exact contravariant functor $\mathsf{F} \colon \mathcal{T} \to \mathcal{A}b$ which sends coproducts in $\mathcal{T}$ to products, there exists an object X in $\mathcal{T}$

such that $\mathsf{F} \cong \mathcal{T}(-, X)$. For example, a compactly generated triangulated category satisfies Brown representability for cohomology theories [9]. In particular, all examples from Section 1 have this property.

An immediate consequence of our assumption on $\mathcal{T}$ is the fact that every object in $\mathcal{T}$ has an H-injective presentation. In addition, we have the following lemma.

Lemma 4.1. *There exists a localization functor $\mathsf{L}\colon \mathcal{T} \to \mathcal{T}$ such that $\mathsf{L}(A) = 0$ if and only if $\mathsf{H}(A) = 0$ for every object A in $\mathcal{T}$.*

Proof. Recall from Section 3 that the inclusion $\mathcal{T}' \to \mathcal{T}$ for the localizing subcategory $\mathcal{T}'$ generated by T has a right adjoint $\mathsf{R}\colon \mathcal{T} \to \mathcal{T}'$. This right adjoint preserves coproducts. Thus R has a right adjoint $\mathsf{R}'\colon \mathcal{T}' \to \mathcal{T}$ since $\mathcal{T}$ satisfies Brown representability. Now put $\mathsf{L} = \mathsf{R}' \circ \mathsf{R}$. □

We are now in a position to state the analogue of Theorem 3.3. To this end we say that an object A in $\mathcal{T}$ is H-*local* if the natural map $A \to \mathsf{L}(A)$ is an isomorphism.

Theorem 4.2. *Let B be an object of $\mathcal{T}$. Then the following are equivalent.*

(i) For any object A in $\mathcal{T}$, all maps $\mathsf{H}(A) \to \mathsf{H}(B)$ are realizable.

(i) $\mathcal{O}_{-,B} = 0$.

(ii) $\operatorname{id}_\Gamma \mathsf{H}(B) \leq 1$ *and the natural map $B \to \mathsf{L}(B)$ is a retraction.*

(iii) $\operatorname{id}_\Gamma \mathsf{H}(B) \leq 1$ *and there is a decomposition $B = B' \oplus B''$ such that B' is H-local and $\mathsf{H}(B'') = 0$.*

The proof of this result is similar to that of Theorem 3.3. It is therefore omitted.

5. Realizing extensions

Now we turn our attention to the problem of realizing short exact sequences. We call an exact sequence $0 \to \mathsf{H}(B) \xrightarrow{\tilde{\beta}} E \xrightarrow{\tilde{\alpha}} \mathsf{H}(A) \to 0$ in $\operatorname{Mod}(\Gamma)$ *realizable* if there exists a triangle $B \xrightarrow{\beta} C \xrightarrow{\alpha} A \to \Sigma B$ in $\mathcal{T}$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{H}(B) & \xrightarrow{\mathsf{H}(\beta)} & \mathsf{H}(C) & \xrightarrow{\mathsf{H}(\alpha)} & \mathsf{H}(A) & \longrightarrow & 0 \\ & & \| & & \cong\downarrow & & \| & & \\ 0 & \longrightarrow & \mathsf{H}(B) & \xrightarrow{\tilde{\beta}} & E & \xrightarrow{\tilde{\alpha}} & \mathsf{H}(A) & \longrightarrow & 0 \end{array}$$

Our analysis of realizable sequences is based on the long exact sequence

$$0 \longrightarrow \operatorname{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A, B) \longrightarrow \operatorname{Ph}_{\mathsf{H}}(\Sigma^{-1}A, B) \xrightarrow{\varphi_{A,B}} \operatorname{Ext}^1_\Gamma(\mathsf{H}(A), \mathsf{H}(B)) \longrightarrow \mathcal{O}_{\Omega_{\mathsf{H}}A,B} \longrightarrow 0 \tag{5.1}$$

which is obtained from the sequence (2.2). Using the description of the map $\varphi_{A,B}$, we get the following lemma.

Lemma 5.1. *A short exact sequence* $\varepsilon\colon 0 \to \mathsf{H}(B) \to E \to \mathsf{H}(A) \to 0$ *is realizable if and only if* ε *belongs to the image of the map* $\varphi_{A,B}$. *Moreover, the map* $A \to \Sigma B$ *in a triangle* $B \to C \to A \to \Sigma B$ *realizing* ε *is unique up to a map in* $\mathrm{Ph}^2_{\mathsf{H}}(A, \Sigma B)$.

Proof. Combine the description of $\varphi_{A,B}$ from Remark 2.9 and the long exact sequence (5.1). □

We apply again the exactness of the sequence (5.1) and obtain the following criterion for realizability.

Corollary 5.2. *Let* A, B *be objects in* $\mathcal{T}$. *Then every exact sequence* $0 \to \mathsf{H}(B) \to E \to \mathsf{H}(A) \to 0$ *in* $\mathrm{Mod}(\Gamma)$ *is realizable if and only if* $\mathcal{O}_{\Omega_{\mathsf{H}}A,B} = 0$.

6. Applications

6.1. Tate cohomology

We fix a finite group G and a field k. Let $\mathcal{T} = \underline{\mathrm{Mod}}(kG)$ be the stable module category over the group algebra kG and let $\mathsf{H} = \widehat{H}^*(G, -)$ be the functor which assigns to each kG-module its Tate cohomology. Note that $\mathsf{H} = \underline{\mathrm{Hom}}_{kG}(k, -)_*$ and consequently the H-projective objects of $\underline{\mathrm{Mod}}(kG)$ are the direct summands of arbitrary coproducts of objects in $\{\Sigma^n k \mid n \in \mathbb{Z}\}$, where Σ is the suspension functor of the triangulated category $\underline{\mathrm{Mod}}(kG)$.

Theorem 1 tells us when all maps starting in a fixed module $\widehat{H}^*(G, A)$ over the Tate cohomology ring are realizable. It is clear that Theorem 1 is the reformulation of our general Theorem 3.3 in this setting. Moreover, Theorem 1 has an analogue for maps ending in a fixed module $\widehat{H}^*(G, B)$ which is a consequence of Theorem 4.2.

Next we apply Corollary 5.2 and consider extensions over the Tate cohomology ring. We assume for simplicity that G is a p-group where p denotes the characteristic of the field k.

Corollary 6.1. *For a* kG*-module* A *the following are equivalent:*

(i) *Each exact sequence* $0 \to \widehat{H}^*(G, B) \to E \to \widehat{H}^*(G, A) \to 0$ *of graded* $\widehat{H}^*(G, k)$*-modules is realizable (for every* kG*-module* B*).*

(ii) *Each exact sequence* $0 \to \widehat{H}^*(G, B) \to E \to \widehat{H}^*(G, A) \to 0$ *of graded* $\widehat{H}^*(G, k)$*-modules is of the form* $\widehat{H}^*(G, \varepsilon)$ *for some exact sequence* $\varepsilon\colon 0 \to B \to C \to A \to 0$ *of* kG*-modules (for every* kG*-module* B*).*

(iii) $\mathrm{pd}_{\widehat{H}^*(G,k)} \widehat{H}^*(G, A) \leq 2$.

Proof. We combine Theorem 3.3 and Corollary 5.2. The condition $\mathcal{O}_{\Omega_{\mathsf{H}}A,-} = 0$ is equivalent to $\widehat{H}^*(G, \Omega_{\mathsf{H}}A) \cong \Omega\widehat{H}^*(G, A)$ being of projective dimension at most 1, since k generates the stable module category $\underline{\mathrm{Mod}}(kG)$ by our assumption on G. In addition one observes that triangles $B \to C \to A \to \Sigma B$ in $\underline{\mathrm{Mod}}(kG)$ correspond to exact sequences $0 \to B \to C' \to A \to 0$ in $\mathrm{Mod}(kG)$ via the isomorphism $\underline{\mathrm{Hom}}_{kG}(A, \Sigma B) \cong \mathrm{Ext}^1_{kG}(A, B)$. □

Remark 6.2. In Theorem 1 and all its applications, the question arises when a kG-module belongs to the subcategory which is generated by the trivial representation k. It is known that k generates the stable category of the principal block $B_0(kG)$ if and only if the centralizer of each element of order p in G is p-nilpotent [2]. In particular, k generates all kG-modules if G is a p-group. If k generates the principal block, we have for any module A in the stable category of the principal block that all maps out of $\widehat{H}^*(G,A)$ are realizible if and only if the projective dimension of $\widehat{H}^*(G,A)$ over the Tate cohomology ring is at most 1.

6.2. Derived categories

We consider the derived category $\mathcal{D}(\mathrm{Mod}(\Lambda))$ of unbounded complexes of modules over some ring Λ. Le $\mathsf{H}\colon \mathcal{D}(\mathrm{Mod}(\Lambda)) \to \mathrm{Mod}(\Lambda)$ be the functor which takes the cohomology H^0A in degree 0 of a complex A. The following result is another application of Theorem 3.3.

Corollary 6.3. *For any complex A in $\mathcal{D}(\mathrm{Mod}(\Lambda))$ the following are equivalent:*
(i) *For any complex B in $\mathcal{D}(\mathrm{Mod}(\Lambda))$, all maps $H^0A \to H^0B$ are realizable.*
(ii) $\mathcal{O}_{A,-} = 0$.
(iii) $\mathrm{pd}_\Lambda H^0A \le 1$.
In particular, Λ is right hereditary if and only if $\mathcal{O}_{-,-}$ vanishes.

6.3. An extended Milnor sequence

We consider again the five-term exact sequence (2.1)

$$0 \longrightarrow \mathrm{Ph}_{\mathsf{H}}(A,B) \longrightarrow \mathcal{T}(A,B) \xrightarrow{\mathsf{H}_{A,B}} \mathrm{Hom}_\Gamma\big(\mathsf{H}(A),\mathsf{H}(B)\big) \xrightarrow{\vartheta_{A,B}}$$
$$\mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A,B) \xrightarrow{\zeta_{A,B}} \mathrm{Ph}^2_{\mathsf{H}}(\Sigma^{-1}A,B) \longrightarrow 0\ .$$

This sequence generalizes and extends some sequences which appear in the literature. Whenever there is an appropriate model for the triangulated category $\mathcal{T}$, the sequence (2.1) is induced from the Roos spectral sequence

$$E_2^{p,q} = \varprojlim{}^p \mathrm{Ext}^q(A_\alpha, B) \implies \mathrm{Ext}^n(\varinjlim A_\alpha, B)$$

where $\{A_\alpha\}$ is a filtered system of compact objects having A as its colimit. This spectral sequence has been studied by Christensen (see [6]) in relation to purity and phantom maps in the derived category of a ring. Here, we follow Benson and Gnacadja (see [4]) and work in the stable category $\mathcal{T} = \underline{\mathrm{Mod}}(kG)$ over a group algebra kG. We choose for $\mathsf{H}\colon \underline{\mathrm{Mod}}(kG) \to \mathrm{Mod}(\Gamma)$ the functor sending a module A to $\underline{\mathrm{Hom}}_{kG}(-,A)|_{\underline{\mathrm{mod}}(kG)}$. It turns out that our map

$$\vartheta_{A,B}\colon \mathrm{Hom}_\Gamma\big(\mathsf{H}(A),\mathsf{H}(B)\big) \longrightarrow \mathrm{Ph}_{\mathsf{H}}(\Omega_{\mathsf{H}}A,B)$$

can be identified with the transgression

$$d_2^{0,1}\colon \varprojlim \underline{\mathrm{Hom}}_{kG}(A_\alpha,B) \longrightarrow \mathrm{PExt}^2_{kG}(A,\Sigma^{-1}B)$$

appearing in [4].

Let us be more precise. We denote for kG-modules A and B by $\mathrm{PExt}^1_{kG}(A,B)$ the space of pure exact sequences $0 \to B \to E \to A \to 0$ in the category of kG-modules. The isomorphism $\mathrm{Ext}^1_{kG}(A,B) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{kG}(A,\Sigma B)$ induces an isomorphism

$$\mathrm{PExt}^1_{kG}(A,B) \xrightarrow{\cong} \underline{\mathrm{Ph}}_{kG}(A,\Sigma B) = \mathrm{Ph}_{\mathsf{H}}(A,\Sigma B)$$

onto the space of phantom maps $A \to \Sigma B$ in $\underline{\mathrm{Mod}}(kG)$. Let $\mathrm{P}\Omega A$ be the kernel of a pure epimorphism $P \to A$ where P is pure-projective, hence $\mathrm{P}\Omega A \cong \Omega_{\mathsf{H}} A$ in $\underline{\mathrm{Mod}}(kG)$. Then we obtain an isomorphism

$$\mathrm{PExt}^2_{kG}(A,B) \xrightarrow{\cong} \mathrm{PExt}^1_{kG}(\mathrm{P}\Omega A,B) \xrightarrow{\cong} \underline{\mathrm{Ph}}_{kG}(\Omega_{\mathsf{H}} A,\Sigma B).$$

Now write $A = \varinjlim A_\alpha$ as a filtered colimit of finitely presented kG-modules. Then we have an isomorphism $\varinjlim \mathsf{H}(A_\alpha) \xrightarrow{\cong} \mathsf{H}(A)$ which induces an isomorphism

$$\mathrm{Hom}_\Gamma\big(\mathsf{H}(A),\mathsf{H}(B)\big) \xrightarrow{\cong} \varprojlim \underline{\mathrm{Hom}}_{kG}(A_\alpha,B).$$

Having made these identifications, we obtain from the five-term exact sequence (2.1) the extended Milnor sequence which appears in Theorem 1.2 of [4].

Corollary 6.4. *For kG-modules $A = \varinjlim A_\alpha$ and B, there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \underline{\mathrm{Ph}}_{kG}(A,B) \longrightarrow \underline{\mathrm{Hom}}_{kG}(A,B) \longrightarrow \varprojlim \underline{\mathrm{Hom}}_{kG}(A_\alpha,B) \\ \xrightarrow{d_2^{0,1}} \mathrm{PExt}^2_{kG}(A,\Sigma^{-1}B) \longrightarrow \underline{\mathrm{Ph}}^2_{kG}(\Sigma^{-1}A,B) \longrightarrow 0\ . \end{aligned}$$

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