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Generalized Hodge Classes on the Moduli Space of Curves

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Abstract. On the moduli space of curves we consider the cohomology classes $\mu_j(s), s \in \mathbb{N}, s \geq 2$, which can be viewed as a generalization of the Hodge classes λ_i defined by Mumford in [6]. Following the methods used in this paper, we prove that the $\mu_j(s)$ belong to the tautological ring of the moduli space.

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1. Introduction

Let g and n be non-negative integers such that $n > 2 - 2g$. We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus g curves and by $\mathcal{M}_{g,n}$ its subspace parametrizing smooth curves. More generally, if P is a set with n elements, we shall consider the space $\overline{\mathcal{M}}_{g,P}$ whose elements are stable genus g curves whose marked points are indexed by P. For any g and P, $|P| = n$, in the range above, the collection of all moduli spaces is naturally equipped with some relevant maps among them: these maps lead to the construction of the tautological ring $T^*_{g,P}$ ([1], [3], [5]). We briefly recall some basic definitions to set the essential notation we shall use in the sequel.

First of all, consider the universal curve

$$\pi : \overline{\mathcal{M}}_{g,P\bigcup\{q\}} \to \overline{\mathcal{M}}_{g,P}. \tag{1}$$

We denote by σ_p, $p \in P$, the canonical section of π and by D_p, $p \in P$, the corresponding divisor in $\overline{\mathcal{M}}_{g,P\cup\{q\}}$. The relative dualizing sheaf ω_π of the map in (1) yields to define the

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classes

$$\psi_p = c_1(\sigma_p^*(\omega_\pi)), \ p \in P. \tag{2}$$

Note that the push-forward in (2) is well defined since the Poincaré duality with rational coefficients holds for the smooth orbifold $\overline{\mathcal{M}}_{g,P}$.

Take, now, the cohomology class

$$K = c_1\Big(\omega_\pi\Big(\sum_{p\in P} D_p\Big)\Big).$$

Following [2], the Mumford classes in $H^{2m}(\overline{\mathcal{M}}_{g,P};\mathbb{Q})$ are defined as

$$\kappa_m = \pi_*(K^{m+1}).$$

For $P=\emptyset$ their analogue was first introduced by Mumford in [6]. Another generalization of Mumford's κ_m's to the case of n-pointed curves is given by the classes

$$\widetilde{\kappa}_m = \pi_*(c_1(\omega_\pi)^{m+1}).$$

The set of ψ_p's, κ_m's, and $\widetilde{\kappa}_m$'s is called the set of *Mumford-Morita-Miller classes.* As it is shown in [2], the following relation holds:

$$\kappa_m = \widetilde{\kappa}_m + \sum_{p\in P} \psi_p^m. \tag{3}$$

Another family of morphisms among the moduli spaces can be described through the collection of graphs whose properties are given in [1]. With the same notation adopted in the above paper, for any such graph G, choose an ordering of the $l(v)$ half-edges of G emanating from each vertex v. Then consider the morphisms

$$\xi_G : \prod_{v\in V} \overline{\mathcal{M}}_{g(v),l(v)} \to \overline{\mathcal{M}}_{g,P}, \tag{4}$$

where $g(v)$ are non-negative integers which label each vertex of G. A point in the domain of ξ_G is the datum of an $l(v)$-pointed curve C_v for each v; the image point is the P-labelled genus g curve that one obtains by identifying the marked points of C_v which correspond to those half-edges of G linked by an edge. By its definition, the map ξ_G does not depend on the ordering chosen for the half-edges issuing from each vertex.

These morphisms allow to define the tautological ring $T^*_{g,P}$ which is generated by the classes

$$\xi_{G,*}\big(\otimes_{v\in V_G} p_v\big),$$

where p_v is a monomial in the κ or ψ_p classes of $\overline{\mathcal{M}}_{g(v),l(v)}$.

In [6], Mumford also introduces the classes

$$\lambda_i = c_i(\omega_\pi) \in H^*(\overline{\mathcal{M}}_g;\mathbb{Q}), \tag{5}$$

which are called *Hodge classes.* He proves that they belong to the tautological ring by applying the Grothendieck-Riemann-Roch Theorem to the universal curve

$$\pi : \overline{\mathcal{M}}_{g,q} \to \overline{\mathcal{M}}_g.$$

In the next section we consider a slight generalization of the Hodge classes and show that they can be expressed in terms of tautological ones.

2. Generalized Hodge classes are tautological

Let us consider the relative dualizing sheaf ω_π, with π the morphism introduced in (1), and the vector bundles

$$\mathbb{E}_s = \pi_*(\omega_\pi^s),\ s \geq 1.$$

Definition 2.1. *The Chern classes of the vector bundle $\mathbb{E}_s$ are called generalized Hodge classes and denoted by $\mu_j(s)$.*

Obviously, the $\mu_j(1)$'s are exactly the classes introduced in (5). By Definition 2.1 we observe that the $\mu_j(s)$'s are zero up to genus 1 unless $\mu_1(1) = \lambda_1$. In fact,

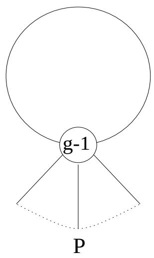
$$\mathrm{rk}(\mathbb{E}_s) = \begin{cases} g & s = 1, \\ (2s-1)(g-1) & s \geq 2. \end{cases}$$

Let us introduce some additional notation to state the main result of this section. Denote by $ch(\mathbb{E}_s)$ the Chern character of $\mathbb{E}_s$. We shall also use Bernoulli numbers B_n and Bernoulli polynomials $B_n(u)$. Their definition is given via the following identities:

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} B_n \frac{x^n}{n!}, \tag{6}$$

$$e^{ux}\frac{x}{e^x - 1} = \sum_{n \geq 0} B_n(u) \frac{x^n}{n!},$$

where x and u are formal variables.
Consider, now, the graphs G_1 and $G_{h,A}$ ($h \geq 0$, $A \subset P$, $2h-1+|A| > 0$, $2(g-h)-1+|A^c| > 0$)



G_1

$G_{h,A}$

Next, as defined in (4), the morphisms associated with the graphs above will be denoted by

$$\xi_{G_1} : \overline{\mathcal{M}}_{g-1, P \cup \{q_1, q_2\}} \to \overline{\mathcal{M}}_{g,P}$$

and

$$\xi_{G_{h,A}} : \overline{\mathcal{M}}_{h, A \cup \{r_1\}} \times \overline{\mathcal{M}}_{g-h, A^c \cup \{r_2\}} \to \overline{\mathcal{M}}_{g,P}.$$

As recalled in the Introduction, the Hodge classes can be expressed in terms of tautological classes. This is proved via Mumford's result (see [6]), which is stated here for the moduli space of pointed curves.

Theorem 2.1. *In* $H^*(\overline{\mathcal{M}}_{g,P};\mathbb{Q})$

$$ch(\mathbb{E}_1) = g + \frac{1}{2}\sum_{m\geq 1}\frac{B_{2m}}{(2m)!}\Big\{\widetilde{\kappa}_{2m-1} +$$

$$\xi_{G_1,*}(\psi_{q_1}^{2m-2} - \psi_{q_1}^{2m-3}\psi_{q_2} + \cdots + \psi_{q_2}^{2m-2}) +$$

$$\sum_{h=0}^{g}\sum_{A\subset P}\xi_{G_{h,A},*}(\psi_{r_1}^{2m-2}\otimes 1 - \psi_{r_1}^{2m-3}\otimes\psi_{r_2} + \ldots + 1\otimes\psi_{r_2}^{2m-2})\Big\}.$$

Notice that the relations in Theorem 2.1 involve only Mumford classes $\widetilde{\kappa}_m$ for m odd.

The same methods adopted by Mumford to prove Theorem 2.1 give analogous relations among the $\widetilde{\kappa}$ and the $\mu_j(s)$. In particular, the following holds.

Theorem 2.2. *The generalized Hodge classes* $\mu_j(s), s\in\mathbb{N}, s\geq 2$, *belong to the tautological ring* $T^*_{g,P}$.

Proof. Let us apply the Grothendieck-Riemann-Roch Theorem to the morphism

$$\pi : \overline{\mathcal{M}}_{g,P\cup\{q\}} \to \overline{\mathcal{M}}_{g,P}$$

and to the vector bundle $\mathbb{E}_s, s\geq 2$. By the same arguments expounded in [6] and [4], we get

$$ch(\mathbb{E}_s) = \sum_{m\geq 1}\frac{B_m(s)}{m!}\widetilde{\kappa}_{m-1} + \frac{1}{2}\sum_{m\geq 1}\frac{B_{2m}}{(2m)!}\,\cdot$$

$$\cdot\Big\{\xi_{G_1,*}(\psi_{q_1}^{2m-2} - \psi_{q_1}^{2m-3}\psi_{q_2} + \ldots + \psi_{q_2}^{2m-2}) + \tag{7}$$

$$\sum_{h=0}^{g}\sum_{A\subset P}\xi_{G_{h,A},*}(\psi_{r_1}^{2m-2}\otimes 1 - \psi_{r_1}^{2m-2}\otimes\psi_{r_2} + \ldots + 1\otimes\psi_{r_2}^{2m-2})\Big\},$$

where $B_m(s)$ is the m-th Bernoulli polynomial evaluated at s. Since the generalized Hodge classes can be expressed in terms of Chern characters, the result follows. □

From Theorem 2.2 we deduce relations for the $\widetilde{\kappa}_m$ classes for m even, whereas in Theorem 2.1 no information was given for these classes. More explicitly, we have

Corollary 2.3. *For each* $s\geq 2$, *the subring of* $H^*(\mathcal{M}_{g,P};\mathbb{Q})$ *generated by the classes* $\widetilde{\kappa}_i$ *equals the subring generated by the generalized Hodge classes* $\mu_j(s)$.

Proof. By Theorem 2.2, we get

$$ch_0(\mathbb{E}_s) = B_1(s) = (s - \frac{1}{2})(2g-2) = (2s-1)(g-1),$$

and, for $t \geq 1$,

$$ch_t(\mathbb{E}_s) = \begin{cases} \frac{B_{t+1}(s)}{(t+1)!}\widetilde{\kappa}_t + \delta_t & t \equiv 1 \bmod 2, \\ \frac{B_{t+1}(s)}{(t+1)!}\widetilde{\kappa}_t & \text{otherwise}, \end{cases} \tag{8}$$

where

$$\begin{aligned} \delta_t : &= (-1)^{t-1}\frac{B_{t+1}}{(t+1)!}\{\frac{1}{2}\xi_{G_0,*}(\psi_{q_1}^{t-1} - \psi_{q_1}^{t-2}\psi_{q_2} + \ldots + \psi_{q_2}^{t-1}) \\ &+ \frac{1}{2}\sum_{h=0}^{g}\sum_{A\subset P}\xi_{G_{h,A},*}(\psi_{r_1}^{t-1}\otimes 1 - \psi_{r_1}^{t-2}\otimes\psi_{r_2} + \ldots + 1\otimes\psi_{r_2}^{t-1})\}. \end{aligned}$$

Thus the claim follows from properties of Bernoulli polynomials. Indeed, since

$$\int_y^{y+1} e^{tx}\frac{x}{e^x - 1}dt = e^{yx},$$

we have

$$\int_y^{y+1} B_n(t)dt = y^n;$$

hence

$$B_n(s+1) - B_n(s) = ns^{n-1},$$

for each $n \geq 0$ and $s \geq 2$. Accordingly,

$$B_n(s) = n[(s-1)^{n-1} + \ldots + 1] + B_n. \tag{9}$$

Since Bernoulli numbers are not integers for $n \geq 1$, (9) shows that the Bernoulli polynomials are not zero for each integer s, $s \geq 2$. Therefore the relations in (8) can be inverted. □

2.1. Examples of relations in the tautological ring

For low m we give explicit relations involving the $\widetilde{\kappa}_m$ classes and the generalized Hodge classes. If we set $m = 1$ in (7), for $s \geq 1$ we get

$$\mu_1(s) = ch_1(\mathbb{E}_s) = \frac{B_2(s)}{2}\widetilde{\kappa}_1 + \frac{1}{12}\delta,$$

where

$$\delta := \frac{1}{2}\xi_{G_1,*}(1) + \frac{1}{2}\sum_{0\leq h\leq [g/2]}\sum_{A\subseteq P}\xi_{G_{h,A},*}(1), \tag{10}$$

and

$$\frac{B_2(s)}{2} = \frac{6s^2 - 6s + 1}{12}.$$

In other words,

$$\mu_1(s) = \frac{6s^2 - 6s + 1}{12}\{\kappa_1 + \sum_{p \in P} \psi_p\} + \frac{1}{12}\delta.$$

Note that for $s = 1$ we get

$$\lambda_1 = ch(\mathbb{E}_1) = \frac{1}{12}(\widetilde{\kappa}_1 + \delta),$$

which coincides with the relation given in [6]: here we have the classes $\widetilde{\kappa}_1$, since we are dealing with pointed curves.

As remarked in Section 2, the classes $\mu_j(s)$ serve to express the classes $\widetilde{\kappa}_m$, m even, in terms of other tautological classes. For instance, we have

$$\widetilde{\kappa}_2 = \frac{3}{B_3(s)}\left[\frac{B_2^2(s)}{4}\widetilde{\kappa}_1^2 + \frac{1}{144}\delta^2 + \frac{B_2(s)}{12}\widetilde{\kappa}_1\delta - 2\mu_2(s)\right],$$

and

$$\begin{aligned}
\widetilde{\kappa}_4 &= \frac{1}{B_5(s)}\left[-20\mu_4(s) + 20B_2(s)\frac{B_4(s)}{4!}\widetilde{\kappa}_1\widetilde{\kappa}_3 + 20B_2(s)\widetilde{\kappa}_1\delta_3\right.\\
&+ -\frac{B_2^2(s)B_3(s)\widetilde{\kappa}_2\widetilde{\kappa}_1^2}{12} + \frac{5}{96}B_2^4(s)\widetilde{\kappa}_1^4 + \frac{5}{36}B_4(s)\widetilde{\kappa}_3\delta\\
&+ \frac{10}{3}\delta_3\delta - \frac{5}{36}B_2(s)B_3(s)\delta\widetilde{\kappa}_1\widetilde{\kappa}_2 + \frac{5}{144}B_2^3(s)\widetilde{\kappa}_1^3\delta - \frac{5}{432}B_3(s)\widetilde{\kappa}_2\delta^2\\
&+ \left.\frac{5}{576}B_2^2(s)\widetilde{\kappa}_1\delta^2 + \frac{5}{5184}B_2(s)\widetilde{\kappa}_1\delta^3 + \frac{5}{124416}\delta^4 + \frac{5}{18}B_3^2(s)\widetilde{\kappa}_2^2\right],
\end{aligned}$$

where

$$B_2(s) = s^2 - s + 1/6,$$

$$B_3(s) = s^3 - (3/2)s^2 + (1/6)s,$$

$$B_4(s) = s^4 - 2s + s^2 - 1/30,$$

$$B_5(s) = s^5 - (5/2)s^4 + (5/3) - s/6,$$

and

$$\begin{aligned}
\delta_3 &= -\frac{1}{1440}\{\xi_{G_1,*}(\psi_{q_1}^2 - \psi_{q_1}\psi_{q_2} + \psi_{q_2}^2)\\
&+ \sum_{h=0}^{g}\sum_{A \subset P}\xi_{G_{h,A},*}(\psi_{r_1}^2 \otimes 1 - \psi_{r_1} \otimes \psi_{r_2} + 1 \otimes \psi_{r_2}^2)\}.
\end{aligned}$$

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