Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 44 (2003), No. 2, 309-321.

Harmonic φ-Morphisms

C. L. Bejan **M. Benyounes**

Universitatea "Al. I. Cuza", Seminar Matematic
6600 Iasi, Romania
e-mail: bejan@math.tuiasi.ro

Université de Brest, Mathématiques
6 av. Le Gorgeu, 29200 Brest, France
e-mail: Michele.Benyounes@univ-brest.fr

Abstract. By extending the main result of [6], we characterize the harmonicity of any φ-morphism $\Phi : TM \to TN$, covering a map $\varphi : M \to N$, between Riemannian manifolds, when the tangent bundles carry the complete lift metric. By following the pattern of (classical) harmonic morphisms [1], [3], we introduce in a natural way the notion of harmonic φ-morphism and give a characterization that corresponds to the one obtained in [4], [8]. One of the properties is that φ is a harmonic morphism if and only if $\mathrm{d}\varphi$ is a harmonic φ-morphism. We end with some examples and applications to (1,1)-tensor fields.

MSC 2000: 53C20, 58E20, 53C55
Keywords: φ-morphism, harmonic maps and morphisms

Introduction

A distinguished class of harmonic maps is the class of harmonic morphisms, these are defined as maps between (semi)-Riemannian manifolds which pull back local harmonic functions to local harmonic functions. We refer to [1] as the first monograph on this topic.

Let $\varphi : (M, g) \to (N, h)$ be a map between Riemannian manifolds. From vector bundles category theory $\Phi : TM \to TN$ is a φ-morphism, provided the fibre restriction $\Phi_p : T_pM \to T_{\varphi(p)}N$ is linear at any $p \in M$. Thus Φ determines a 1-form $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ with values in the pull-back bundle $\varphi^{-1}TN$. We prove that $\Phi : (TM, g^c) \to (TN, h^c)$ is a harmonic map between semi-Riemannian manifolds (where c denotes the complete lift defined in [11]), if and only if $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ is coclosed w.r.t. the pull back connection $\nabla^{\varphi^{-1}TN}$. When we particularize Φ to a (1,1)-tensor field K on M, viewed as a map $K : (TM, g^c) \to (TN, h^c)$, then its harmonicity was characterized in [6]. Here we characterize the (1,1)-tensor field K of maximal rank which are harmonic morphisms and we find that K has to be the identity up to a non-zero constant factor. This restrictive condition is a reason to introduce *harmonic φ-morphisms,* by following the pattern of (classical) harmonic morphisms. If ∇ is a linear connection of $\varphi^{-1}TN$, compatible with Φ, then we call Φ a *harmonic φ-morphism* w.r.t. ∇, provided any harmonic local function f on N has the pull-back $\mathrm{d}f \circ \Phi$ coclosed on M. We prove that $\varphi : (M, g) \to (N, h)$ is a harmonic morphism if and only if $\mathrm{d}\varphi : TM \to TN$ is a harmonic φ-morphism w.r.t. $\nabla^{\varphi^{-1}TN}$. Different fom the behaviour of the harmonic maps, the composition of two harmonic morphisms is a harmonic morphism. We provide a class of connections w.r.t. which the same property is valid for φ-morphisms. Corresponding to the characterization of [4], [8] of harmonic morphisms as horizontally weakly conformal harmonic maps, we characterize here the harmonic φ-morphisms.

At the end, we apply the notion of harmonic φ-morphism to certain classes of (1,1)-tensor fields (almost complex and almost product structures and the Ricci (1,1)-tensor field).

Throughout the paper, all data are smooth and we assume the Einstein convention on the summing of repeated indices.

Acknowledgement. First author warmly thanks John C. Wood for useful discussions and suggestions during her visit at Leeds University. Both authors are very indebted to the referee for all his suggestions and useful ideas.

1. Preliminaries

To fix notations, let $\varphi : (M, g) \to (N, h)$ be a map between Riemannian manifolds, with ∇^M and ∇^N the corresponding Levi-Civita connections and let $\Phi : TM \to TN$ be a φ-morphism.

A linear connection D of a vector bundle $E \to M$, defines the exterior derivative d and the coderivative δ of any bundle valued 1-form $\omega \in \mathcal{A}^1(E)$, respectively by

$$\mathrm{d}\omega(X, Y) = D\omega(X, Y) - D\omega(Y, X)\,, \tag{1.1}$$

where $D\omega(X, Y) = (D_X\omega)Y,\ \forall X,\ Y \in \Gamma(TM)$ and

$$\delta\omega = -\mathrm{div}\,\omega = -\mathrm{trace}\,D\omega\,. \tag{1.2}$$

ω is called *harmonic* if it is both closed ($\mathrm{d}\omega = 0$) and coclosed ($\delta\omega = 0$). Note that d and δ depend on D and therefore the closure, coclosure and harmonicity properties of ω also depend on D. In particular, when $E = \varphi^{-1}TN$ (resp. $E = M \times \mathbb{R}$) carries a linear connection ∇ (resp. standard connection on the trivial bundle), then $\mathrm{d}\Phi$ (resp. $\mathrm{d}(\theta \circ \Phi)$) denotes the exterior derivative of $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ (resp. $\theta \circ \Phi \in \mathcal{A}^1(M)$, for any $\theta \in \mathcal{A}^1(N)$). We distinguish between $\mathrm{d}\Phi \in \mathcal{A}^2(\varphi^{-1}TN)$ and the tangent map $\mathrm{d}\Phi : TTM \to TTN$. Which we use will be clear from the context.

Notation. Throughout this note, a pair $(\Phi,\ \nabla)$ will denote any φ-morphism $\Phi : TM \to TN$ and any linear connection ∇ on $\varphi^{-1}TN$.

Examples of Φ. (i) Obviously, $\mathrm{d}\varphi : TM \to TN$ is a φ-morphism; (ii) any (1,1)-tensor field on M determines a 1_M-morphism $K : TM \to TM$, where 1_M denotes the identity map of M.

Examples of ∇. (i) Let $\nabla^{\varphi^{-1}TN}$ be the unique (see [3]) linear connection on $\varphi^{-1}TN$, which satisfies

$$\nabla_X^{\varphi^{-1}TN}\varphi^* U = \varphi^* \nabla^N_{\mathrm{d}\varphi X} U \qquad \forall X \in \Gamma(TM), \qquad U \in \Gamma(TN), \tag{1.3}$$

where

$$\varphi^* U \in \Gamma(\varphi^{-1}TN), \qquad (\varphi^* U)_p = U_{\varphi(p)}, \qquad \forall p \in M. \tag{1.4}$$

(ii) Define ∇^Φ to be the linear connection on $\varphi^{-1}TN$ which satisfies

$$\nabla_X^\Phi \varphi^* U = \varphi^* \nabla^N_{\Phi X} U, \qquad \forall X \in \Gamma(TM), \qquad U \in \Gamma(TN). \tag{1.5}$$

The existence and uniqueness of ∇^Φ are proved as in [2, pp. 4] by replacing $\mathrm{d}\varphi$ with Φ.

Obviously (ii) generalizes (i) since $\nabla^{\varphi^{-1}TN} = \nabla^{\mathrm{d}\varphi}$.

From a straightforward calculation, we obtain:

Lemma 1.1.

(a) *The following conditions are equivalent for a pair* $(\Phi,\ \nabla)$:

$$\begin{aligned} &\nabla\Phi \quad \textit{is symmetric} \;\Leftrightarrow\; \Phi \in \mathcal{A}^1(\varphi^{-1}TN) \quad \textit{is closed} \;\Leftrightarrow \\ &\Leftrightarrow\; \nabla_X(\Phi Y) - \nabla_Y(\Phi X) = \Phi[X,\ Y], \qquad X,\ Y \in \Gamma(TM); \end{aligned} \tag{1.6}$$

(b) *The pair* $(\mathrm{d}\varphi,\ \nabla^{\varphi^{-1}TN})$ *satisfies* (1.6);

(c) *The pair* $(\Phi,\ \nabla^\Phi)$ *satisfies* (1.6) *if and only if* $[\Phi X,\ \Phi Y] = \Phi[X,\ Y],\ \forall X,\ Y \in \Gamma(TM)$.

Note that not every pair $(\Phi,\ \nabla)$ satisfies these conditions, for example if (M, g) is a Riemannian manifold with ∇ the Levi-Civita connection, then the pairs $(fI,\ \nabla)$ do not satisfy (1.6), where f is a non-constant function on M and $I : TM \to TM$ is the identity.

Formula. *For any φ-morphisms $\Phi,\ \Psi : TM \to TN$, the pair $(\Phi,\ \nabla^\Psi)$ satisfies:*

$$\nabla^\Psi(\theta \circ \Phi) = \theta \circ \nabla^\Psi \Phi + \nabla^N \theta(\Psi\cdot,\ \Phi\cdot), \qquad \forall \theta \in \mathcal{A}^1(N). \tag{1.7}$$

Proof.

$$\nabla^\Psi(\theta \circ \Phi)(X,\ Y) = (\nabla_X^\Psi(\theta \circ \Phi))Y = (\nabla_X^\Psi(\theta \circ \Phi(Y)) - \theta \circ \Phi(\nabla_X^M Y) =$$

$$= (\nabla^N_{\Psi X}\theta)\Phi Y + \theta(\nabla_X^\Psi(\Phi Y)) - \theta \circ \Phi(\nabla_X^M Y) = (\nabla^N_{\Psi X}\theta)\Phi Y + \theta(\nabla_X^\Psi \Phi)Y =$$

$$= \nabla^N\theta(\Psi X,\ \Phi Y) + \theta \circ \nabla^\Psi\Phi(X,\ Y), \qquad \forall X,\ Y \in \Gamma(TM).$$

2. Compatible pairs

This section is devoted to a certain class of pairs (Φ, ∇), for which Φ and ∇ are related by a certain compatibility relation.

More precisely, the above Formula leads us to the following:

Definition 2.1. *A pair* (Φ, ∇) *is called compatible if it satisfies the following compatibility relation*

$$\nabla(\theta \circ \Phi) = \theta \circ \nabla\Phi + \nabla^N \theta(\Phi\cdot, \Phi\cdot), \qquad \forall \theta \in \mathcal{A}^1(N). \tag{2.1}$$

Example. Let $\Phi, \Psi : TM \to TN$ be φ-morphisms. Then the pair (Φ, ∇^Ψ) is compatible if and only if $\Phi = \Psi$. In particular, $(\mathrm{d}\varphi, \nabla^{\varphi^{-1}TN})$ is compatible.

Lemma 2.2. *Any compatible pair* (Φ, ∇) *satisfies*

$$\mathrm{d}(\theta \circ \Phi) = \theta \circ \mathrm{d}\Phi + \mathrm{d}\theta(\Phi\cdot, \Phi\cdot), \qquad \forall \theta \in \mathcal{A}^1(N); \tag{2.2}$$

$$\nabla(\mathrm{d}f \circ \Phi)(X, Y) = [(\nabla_X \Phi)Y]f + \nabla^N \mathrm{d}f(\Phi X, \Phi Y), \forall X, Y \in \Gamma(TM), f \in \mathcal{F}(N); \tag{2.3}$$

$$\delta(\mathrm{d}f \circ \Phi) = (\delta\Phi)f + \mathrm{trace}\nabla^N \mathrm{d}f(\Phi\cdot, \Phi\cdot), \qquad \forall f \in \mathcal{F}(N). \tag{2.4}$$

Proof. (2.1) and (1.1) yield (2.2). From (2.1) applied to any exact form θ, it follows (2.3). We end the proof by obtaining (2.4) from (2.3) and (1.2).

Proposition 2.3. *Let* (Φ, ∇) *be a compatible pair.*

(i) *Let* $\Phi \in \mathcal{A}(\varphi^{-1}TM)$ *be closed. Then* $\theta \circ \Phi \in \mathcal{A}^1(M)$ *is closed if and only if* $\nabla^N\theta$ *is symmetric in its two variables restricted to the image of* Φ.

In particular, if θ is closed, so is $\theta \circ \Phi$ and the converse holds if rank $\Phi = \dim N$;

(ii) $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ *is closed if and only if* $\theta \circ \Phi \in \mathcal{A}^1(M)$ *is closed whenever* $\theta \in \mathcal{A}^1(N)$ *is closed*;

(iii) $\nabla\Phi$ *is symmetric if and only if* $\nabla(\mathrm{d}f \circ \Phi)$ *is so.*

Proof. (i) follows from (2.2) and (1.1). We derive (ii) and (iii) from (2.3) and the symmetry of the Hessian $\nabla^N \mathrm{d}f$, $f \in \mathcal{F}(N)$, which complete the proof.

As a consequence of Proposition 2.3, (ii), we obtain:

Corollary 2.4. *The following assertions are equivalent for any compatible pair* (Φ, ∇) *with closed* $\Phi \in \mathcal{A}^{-1}(\varphi^{-1}TN)$:

(i) *For any harmonic local function* $f : \mathcal{U} \subset (N, h) \to \mathbb{R}$, *the pull-back* $\mathrm{d}f \circ \Phi$ *is coclosed on M*;

(ii) Φ *pulls back any harmonic* 1*-form* θ *on* N *to a harmonic form* $\theta \circ \Phi$ *on M.*

3. Harmonic φ-morphisms

The main notion of this note is naturally introduced in this section, as being suggested by Corollary 2.4; note however that we do not require Φ to be closed.

Definition 3.1. *Let (Φ, ∇) be a compatible pair. Then we define Φ to be a harmonic φ-morphism (w.r.t. ∇), or briefly a ∇-harmonic φ-morphism, if φ pulls back any harmonic 1-form θ on N to a harmonic form $\theta \circ \varphi$ on M.*

An example of a large class of harmonic φ-morphisms is given by the following:

Theorem 3.2. *Any map $\varphi : (M, g) \to (N, h)$ is a harmonic morphism if and only if* $\mathrm{d}\varphi : TM \to TN$ *is a harmonic* φ-morphism *w.r.t.* $\nabla^{\varphi^{-1}TN}$.

The proof follows from Lemma 1.1 (b) and Corollary 2.4.

In order to characterize the harmonic φ-morphisms by analogy with the harmonic morphisms [4], [8], we state first the following:

Lemma 3.3 *Let $\varphi : (M^m, g) \to (N^n, h)$ and $\Phi : TM \to TN$ be as above. If $H_p = [\mathrm{Ker}\,\Phi_p]^\perp$ denotes the horizontal space at $p \in M$, then the following assertions are equivalent*:

(i) *For any $p \in M^m$, either $\Phi_p = 0$ or Φ_p is surjective and there exists a positive function $\lambda \in \mathcal{F}(M^m)$, called the dilation, such that*:

$$h(\Phi X, \Phi Y) = \lambda g(X, Y), \qquad \forall X, Y \in H_p\,; \tag{3.1}$$

(ii) *There exists a positive function $\lambda \in \mathcal{F}(M^m)$ such that*

$$g^{ij}\Phi_i^\alpha \Phi_j^\beta = \lambda h^{\alpha\beta} \tag{3.2}$$

for any local frames $\left\{\dfrac{\partial}{\partial x^i},\ i = \overline{1,m}\right\}$ *and* $\left\{\dfrac{\partial}{\partial x^\alpha},\ \alpha = \overline{1,n}\right\}$ *on M^m and N^n, respectively.*

The proof follows from the following algebraic result:

Fact. [3, pp. 41] Let $F : U \to W$ be a non-constant linear map between Euclidean spaces. By the identification $V^* = V$ and $W^* = W$, the adjoint $F^* : W \to V$ is given by $< F^*(w), v > = < F(v), w >,\ \forall v \in V,\ w \in W$. Then F satisfies Lemma 3.3 (i) if and only if F^* embeds W conformally in $(\mathrm{Ker} F)^\perp \subset V$.

Example. On any almost Hermitian (resp. Riemannian almost product) manifold, the almost complex (resp. almost product) structure satisfies the equivalent conditions of Lemma 3.3.

Properties. *Let $\Phi : TM \to TN$ be a φ-morphism which satisfies the equivalent conditions of Lemma* 3.3. *Then:*

(a) $\operatorname{rank}\Phi = \dim N \leq \dim M$ *on an open subset on* M;

(b) $\Phi_p = 0$ *at any* $p \in M$, *where rank* $\Phi_p < \dim N$.

Proof. From Lemma 3.3 (i), at any $p \in M$, either $\lambda(p) = 0$ and then rank $\Phi = 0$, or $\lambda(p) \neq 0$ and rank $\Phi = \dim N$.

The proof given in [3, pp. 42] to characterize the harmonic morphisms can be easily adapted here (by replacing $d\varphi$ with Φ) such that from (2.4) and (3.2) we obtain

Theorem 3.4. *Let* (Φ, ∇) *be a compatible pair. Then* Φ *is a* ∇*-harmonic* φ*-morphism if and only if* Φ *satisfies the equivalent conditions of Lemma* 3.3 *and* $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ *is coclosed (w.r.t.* ∇*).*

Corresponding to the composition property of the harmonic morphisms, the harmonic φ-morphisms have the following behaviour:

Proposition 3.5. *For* $i = \overline{1,2}$, *let* $\varphi_i : M_i \to M_{i+1}$ *be a map between Riemannian manifolds and let* $\Phi_i : TM_i \to TM_{i+1}$ *be a harmonic* φ_i*-morphism with* $\Phi_i \in \mathcal{A}^2(\varphi^{-1}TM_{i+1})$ *closed w.r.t.* ∇^{Φ_i}. *Then* $\Phi_2 \circ \Phi_1$ *is a* $\nabla^{\Phi_2 \circ \Phi_1}$*-harmonic* $\varphi_2 \circ \varphi_1$*-morphism.*

Proof. From Lemma 1.1 (c), $\Phi_2 \circ \Phi_1$ is closed (since $\Phi_i \in \mathcal{A}^1(\varphi_i^{-1}TM_{i+1})$, $i = \overline{1,2}$ are so) We remark that both pairs $(\Phi_i, \nabla^{\Phi_i})$, $i = \overline{1,2}$, are compatible from the Example following Definition 2.1. Let θ be a harmonic 1-form, then by applying Corollary 2.4 (ii) to Φ_2, we see that $\theta \circ \Phi_2$ is harmonic; applying it to Φ_1 then shows that $\theta \circ \Phi_2 \circ \Phi_1$ is harmonic.

Due to Theorem 3.4, the main notion of this note, introduced by Definition 3.1 for compatible pairs, may be extended to arbitrary pairs, as follows:

Definition 3.6. *Let the pair* (Φ, ∇) *be arbitrary, then we define* Φ *to be a generalized harmonic* φ*-morphism (w.r.t.* ∇*, or briefly a* ∇*-harmonic* φ*-morphism), if* Φ *satisfies the equivalent conditions of Lemma* 3.3 *and* $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ *is coclosed w.r.t.* ∇.

The class of harmonic φ-morphisms is larger than the one provided by Theorem 3.2 and moreover, the pair (Φ, ∇) need not be compatible, as one can see from the following:

Example. Let $M = N = \mathbb{R}^2$ be the Euclidean space with the canonical coordinates (x^1, x^2) and let φ be the identity map of $\mathbb{R}^2$. If $\Phi : T\mathbb{R}^2 \to T\mathbb{R}^2$ is the φ-morphism defined such that $\Phi\left(\frac{\partial}{\partial x^1}\right) = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$ and $\Phi\left(\frac{\partial}{\partial x^2}\right) = -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}$, then $\delta\Phi = 0$ w.r.t. the flat connection of $\mathbb{R}^2$ and Φ satisfies (3.1) with dilation $\lambda(x^1, x^2) = (x^1)^2 + (x^2)^2$. It follows that Φ is a generalized harmonic φ-morphism, which is not the tangent map of φ.

From Theorem 3.4 we note that the coclosure condition in Definition 3.6 is necessary for any harmonic φ-morphism. We characterize this condition in a special case, by giving a geometrical interpretation of it.

First we recall that some geometrical objects on a manifold M that can be lifted on TM by the vertical and complete lifts.

Definition 3.7. [11] *Let $\pi : TM \to M$ be the canonical projection. If $f \in \mathcal{F}(M)$, $X \in \Gamma(TM)$, $g \in \mathcal{T}_2^0(M)$ and ∇ denote respectively a real function, a vector field, a $(0,2)$-tensor field and a linear connection on M, then their vertical and complete lifts on TM are defined by:*

$$f^v,\ f^c \in \mathcal{F}(TM), \quad f^v = f \circ \pi, \quad f^c = \mathrm{d}f; \tag{3.3}$$

$$\begin{cases} X^v, X^c \in \Gamma(TTM), & X^v f^v = 0 \\ X^v f^c = (Xf)^v = X^c f^v, & X^c f^c = (Xf)^c; \end{cases} \tag{3.4}$$

$$\begin{cases} g^v, g^c \in \mathcal{T}_2^0(TM), \quad g^v(X^v, Y^v) = g^v(X^v, Y^c) = g^v(X^c, Y^v) = 0 \\ g^v(X^c, Y^c) = (g(X,Y))^v \quad \text{and} \quad g^c(X^v, Y^v) = 0, \\ g^c(X^v, Y^c) = (g(X,Y))^v = g^c(X^c, Y^v), \quad g^c(X^c, Y^c) = (g(X,Y))^c; \end{cases} \tag{3.5}$$

$$\begin{cases} \nabla^c_{X^v} Y^v = 0, \qquad \nabla^c_{X^v} Y^c = (\nabla_X Y)^v = \nabla^c_{X^c} Y^v, \\ \nabla^c_{X^c} Y^c = (\nabla_X Y)^c, \qquad \forall Y \in \Gamma(TM). \end{cases} \tag{3.6}$$

Remarks. (i) Any tensor field on TM may be expressed locally in terms of the vertical and complete lifts of some tensors on M;

(ii) $\Gamma(TTM) = \mathrm{span}\{X^v,\ X^c : X \in \Gamma(TM\}$;

(iii) If $(M,\ g)$ is a Riemannian manifold of m-dimension and ∇ is its Levi-Civita connection, then g^c is a semi-Riemannian metric on TM of signature $(m,\ m)$ and ∇^c is its Levi-Civita connection.

Local coordinates 3.8. If $(M,\ g)$ is a Riemannian manifold of m-dimension, let (x^i) and $(x^i,\ y^i)$ be local coordinates which induce the local frames

$$\left\{\frac{\partial}{\partial x^i} : i = \overline{1,m}\right\} \qquad \text{and} \qquad \left\{\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x^i}\right)^c, \frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i}\right)^v : i = \overline{1,m}\right\}$$

on M and TM, respectively. With respect to the last local frame, we have the following local expression:

$$g^c = \begin{pmatrix} y^k \dfrac{\partial g_{ij}}{\partial x^k} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}, \qquad (g^c)^{-1} = \begin{pmatrix} 0 & g^{ij} \\ g^{ij} & y^k \dfrac{\partial g^{ij}}{\partial x^k} \end{pmatrix}. \tag{3.7}$$

Now we characterize the coclosure condition in a special case:

Theorem 3.9. *Let $\varphi : (M,g) \to (N,h)$ be a map between Riemannian manifolds. Then any φ-morphism $\Phi : (TM, g^c) \to (TN, h^c)$ is a harmonic map if and only if $\Phi \in \mathcal{A}^1(\varphi^{-1}TN)$ is coclosed w.r.t. $\nabla^{\varphi^{-1}TN}$.*

This theorem is a consequence of the following

Formula. $\tau(\Phi) = 2(\delta\Phi)^v$.

Proof. The second fundamental form of Φ is given by, for all $U, V \in \Gamma(TTM)$:

$$\nabla d\Phi(U,V) = \nabla_U^{\Phi^{-1}TTN} d\Phi(V) - d\Phi\left(\overset{M}{\nabla^c_U} V\right) = \overset{N}{\nabla^c}_{d\Phi(U)} d\Phi(V) - d\Phi\left(\overset{M}{\nabla^c_U} V\right). \tag{3.8}$$

Let m and n be the dimensions of M and N, respectively. Similarly to the local coordinates which induce the local frames

$$\left\{\frac{\partial}{\partial u^\alpha} : \alpha = \overline{1,n}\right\} \quad \text{and} \quad \left\{\frac{\partial}{\partial u^\alpha} = \left(\frac{\partial}{\partial u^\alpha}\right)^c, \frac{\partial}{\partial v^\alpha} = \left(\frac{\partial}{\partial u^\alpha}\right)^v : i = \overline{1,m}\right\}$$

on N and TN, respectively. Then the map $\Phi : TM \to TN$ is given in local coordinates by

$$\Phi(x,y) = \left(\varphi^\alpha(x), \Phi_i^\alpha(x) y^i\right), \qquad \alpha = \overline{1,n}. \tag{3.9}$$

where $x = (x^1, \ldots, x^m),\ y = (y^1, \ldots, y^n)$. Then:

$$\begin{aligned} d\Phi\left(\frac{\partial}{\partial x^i}\right) &= \frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial}{\partial u^\alpha} + \frac{\partial \Phi_k^\alpha}{\partial x^i} y^k \frac{\partial}{\partial v^\alpha}; \\ d\Phi\left(\frac{\partial}{\partial y^i}\right) &= \Phi_i^\alpha \frac{\partial}{\partial v^\alpha}, \qquad i = \overline{1,m}. \end{aligned} \tag{3.10}$$

From (3.8), (3.10) and (3.6), it follows:

$$\nabla d\Phi\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0, \qquad i, j = \overline{1,m}. \tag{3.11}$$

By using (3.7), (3.8), (3.11) and the symmetry of $\nabla d\Phi$, we obtain the local expression of the tensor field $\tau = \operatorname{trace} \nabla d\Phi$:

$$\tau = 2g^{ij} \nabla d\Phi\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^i}\right). \tag{3.12}$$

Let $\overset{M}{\Gamma^k_{ij}}$ and $\overset{N}{\Gamma^\gamma_{\alpha\beta}}$ denote the Christoffel symbols of $\overset{M}{\nabla}$ and $\overset{N}{\nabla}$, respectively. Then (3.8), (3.10) and (3.6) yield:

$$\tau = 2g^{ij}\left(\Phi_j^\alpha \frac{\partial \varphi^\beta}{\partial x^i} \overset{N}{\Gamma^\gamma_{\alpha\beta}} - \overset{M}{\Gamma^k_{ij}} \Phi_k^\alpha\right)\frac{\partial}{\partial v^\gamma}. \tag{3.13}$$

On the other hand, from (1.2) we infer the local expression:

$$\delta\Phi = g^{ij}\left(\nabla^{\varphi^{-1}TN}_{\frac{\partial}{\partial x^i}} \Phi\right)\frac{\partial}{\partial x^j}. \tag{3.14}$$

From (1.3), by a straightforward calculation, we obtain:

$$\delta\Phi = g^{ij}\left(\Phi_j^\alpha \frac{\partial \varphi^\beta}{\partial x^i} \overset{N}{\Gamma^\gamma_{\alpha\beta}} - \overset{M}{\Gamma^k_{ij}} \Phi_k^\alpha\right)\frac{\partial}{\partial u^\gamma}. \tag{3.15}$$

Then (3.13) and (3.15) yield the formula.

Remark. If in particular Φ is a (1,1)-tensor field on M (as in Section 1, Example of Φ), then the main theorem of [6] is obtained.

4. Applications to (1,1)-tensor fields

In the remaining part of the paper, the results of Section 3 are applied in the case when $(M, g) = (N, h)$ and φ is the identity map 1_M of M. Hence any φ-morphism Φ becomes a (1,1)-tensor field $K : TM \to TM$ which is given in the local coordinates (3.8) by

$$K(x, y) = (K^1, \ldots, K^m; K^{m+1}, \ldots, K^{2m}) = (x^1, \ldots, x^m; K^1_j y^j, \ldots, K^m_j y^j) \tag{4.1}$$

where $x = (x^1, \ldots, x^m)$, $y = (y^1, \ldots, y^m)$, $K\dfrac{\partial}{\partial x^j} = K^i_j \dfrac{\partial}{\partial x^j}$ and $K^i_j = K^i_j(x)$, $i, j = \overline{1, m}$.

As we mentioned in the last remark of Section 3, the class of all harmonic (1,1)-tensor fields $K : (TM, g^c) \to (TM, g^c)$ was studied in [6]. Here we determine the subclass of all (1,1)-tensor fields $K : (TM, g^c) \to (TM, g^c)$ which are harmonic morphisms. First we recall one of the equivalent definitions provided in [1].

Definition 4.1 *Let $F : (A, g) \to (B, h)$ be a map between semi-Riemannian manifolds and let $a \in A$. Then F is called:*

(i) *weakly conformal at a, if there is $\Lambda(a) \in R$ such that:*

$$h(\mathrm{d}F_a(U), \mathrm{d}F_a(V)) = \Lambda(a) g(U, V), \qquad \forall U, V \in T_a A\,, \tag{4.2}$$

(Λ(a) *is called the conformality factor*).

(ii) *horizontally weakly conformal at a if there is $\Lambda(a) \in R$ such that for any local frame $\{Z_\alpha : \alpha = \overline{1, n}\}$ on B:*

$$g(\operatorname{grad} F^\alpha, \operatorname{grad} F^\beta) = \Lambda(a) h^{\alpha\beta}, \qquad \alpha, \beta = \overline{1, n}\,, \tag{4.3}$$

where $n = \dim B$.

(iii) *(horizontally) weakly conformal on A if F is (horizontally) weakly conformal at all $a \in A$.*

Proposition 4.2. [1] *Let $F : (A, g) \to (B, h)$ be a map between semi-Riemannian manifolds and let $a \in A$. Then F is weakly conformal at a if and only if one of the following holds*:

(i) $\mathrm{d}F_a = 0$;

(ii) $\mathrm{d}F_a$ *maps $T_a A$ conformally onto its image, i.e. there exists $\Lambda(a) \neq 0$ such that* (4.2) *holds good*;

(iii) *the image of $\mathrm{d}F_a$ is non-zero and lightlike w.r.t. the semi-Riemannian metric h, i.e. h restricted to the image of $\mathrm{d}F_a$ is zero.*

From (4.1), Definition 4.1 and Proposition 4.2, we obtain:

Lemma 4.3. *Let $K : (TM, g^c) \to (TM, g^c)$ be a* (1-1)*-tensor field of maximal rank (i.e.* $\operatorname{rank} K = \dim M$*). Then the following assertions are equivalent*:

(a) *K is weakly conformal*;

(b) *K is horizontally conformal*;

(c) *at any $a \in A$, K satisfies condition* (ii) *of Proposition* 4.2.

Lemma 4.4. *Let K be a* (1,1)*-tensor field on M and let $m = \dim M$. Then $K : (TM, g^c) \to (TM, g^c)$ is horizontally weakly conformal of maximal rank and of conformality factor $\Lambda \in \mathcal{F}(TM)$ if and only if* (4.4) *and* (4.5) *are satisfied*:

$$K = \Lambda I, \tag{4.4}$$

where I is the identity tensor field and

$$\textit{either} \quad \Lambda = 1 \quad \textit{or} \quad \Lambda \in \mathcal{F}(M), \quad \Lambda(x) \neq 0 \tag{4.5}$$

and at any $x \in M$, $2\dfrac{\partial \ln|\Lambda - 1|}{\partial x^i} = \operatorname{trace}\left(\mathcal{L}_{\frac{\partial}{\partial x^i}} g\right)$, $i = \overline{1, m}$, where L denotes Lie derivative.

Remark. Actually, (4.5) says that Λ depends only on $x \in M$ and not on $(x, y) \in TM$.

Proof. Let K^α, $\alpha = \overline{1, 2m}$, be defined as in (4.1). Then:

$$g^c(\operatorname{grad} K^\alpha, U) = UK^\alpha, \qquad \forall U \in \Gamma(TTM), \qquad \alpha = \overline{1, 2m}. \tag{4.6}$$

Replacing all instances of U by $\left(\dfrac{\partial}{\partial x^i}\right)^v, \left(\dfrac{\partial}{\partial x^i}\right)^c$, $i = \overline{1, m}$, in turn we obtain:

$$\operatorname{grad} K^s = g^{sj} \frac{\partial}{\partial y^j}\,; \tag{4.7}$$

$$\operatorname{grad} K^{m+s} = K^s_i g^{ik} \frac{\partial}{\partial x^k} + y^k \left(g^{ij} \frac{\partial K^s_h}{\partial x^i} + \frac{\partial g^{jl}}{\partial x^h} K^s_l \right) \frac{\partial}{\partial y^j}, \qquad s = \overline{1, m}.$$

From (4.7) and (3.7) one obtains:

$$g^c(\operatorname{grad} K^t, \ \operatorname{grad} K^s) = 0; \tag{4.8}$$

$$g^c(\operatorname{grad} K^t, \ K^{m+s}) = K^s_i g^{it}, \qquad s, t = \overline{1, m}\,. \tag{4.9}$$

From (4.3), Lemma 4.3 and (3.7), K is horizontally weakly conformal if and only if there exists $\Lambda \in \mathcal{F}(TM)$ such that (4.8), (4.10) and (4.11) hold good, where

$$g^c(\operatorname{grad} K^t, \ \operatorname{grad} K^{m+s}) = \Lambda g^{st}; \tag{4.10}$$

$$g^c(\operatorname{grad} K^{m+t}, \ \operatorname{grad} K^{m+s}) = \Lambda y^k \frac{\partial g^{st}}{\partial x^k}, \qquad s, t = \overline{1, m}\,. \tag{4.11}$$

Note that (4.8) is always satisfied and that (4.10) is equivalent to (4.4) by virtue of (4.9). Hence the previous assertion combined with Lemma 4.3 and Proposition 4.2 ensure that K is horizontally weakly conformal of maximal rank and dilation $\Lambda \in \mathcal{F}(TM)$ if and only if (4.4) and (4.11) are satisfied for $\Lambda(x,y) \neq 0,\ \forall (x,y) \in TM$. Now, the lemma will be a consequence of the following:

Fact. *If* (4.4) *is satisfied for* $\Lambda(x,y) \neq 0,\ \forall (x,y) \in TM$, *then* (4.11) *is equivalent to* (4.5).

To show this fact, we assume (4.4) and then from (4.7) and (3.7) the following equivalence holds:

$$(4.11) \Leftrightarrow$$

$$\Leftrightarrow \Lambda^2 g^{tk} g^{sl} \frac{\partial g_{kl}}{\partial x^h} y^h + \Lambda g^{tk} y^h \left(g^{ij} \frac{\partial \Lambda}{\partial x^i} \delta^s_h + \Lambda \frac{\partial g^{js}}{\partial x^h} \right) g_{hj} +$$

$$+ \Lambda g^{sk} y^h \left(g^{ij} \frac{\partial \Lambda}{\partial x^i} \delta^t_h + \Lambda \frac{\partial g^{it}}{\partial x^h} \right) g_{kj} = \Lambda \frac{\partial g^{ts}}{\partial x^h} y^h \Leftrightarrow$$

$$\Leftrightarrow -\Lambda^2 \frac{\partial g^{ts}}{\partial x^h} y^h + \Lambda y^h \left(g^{it} \frac{\partial \Lambda}{\partial x^i} \delta^s_h + g^{is} \frac{\partial \Lambda}{\partial x^i} \delta^t_s + 2\Lambda \frac{\partial g^{ts}}{\partial x^h} \right) = \Lambda \frac{\partial g^{ts}}{\partial x^h} y^h \Leftrightarrow$$

$$\Leftrightarrow \Lambda^2 \frac{\partial g^{ts}}{\partial x^i} y^i + \Lambda \frac{\partial \Lambda}{\partial x^i} (g^{it} y^s + g^{is} y^t) = \Lambda \frac{\partial g^{ts}}{\partial x^i} y^i \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial \Lambda}{\partial x^i} (g^{it} y^s + g^{is} y^t) = (1 - \Lambda) y^i \frac{\partial g^{ts}}{\partial x^i} \quad (\text{since } \Lambda \neq 0 \text{ at any point of } TM) \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial \Lambda}{\partial x^i} 2 y^i = (\Lambda - 1) y^i \frac{\partial g_{ts}}{\partial x^i} g^{ts} \Leftrightarrow (4.5),$$

which complete the above fact and hence the Lemma.

Example. Let $M = S^2 = \{x = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in \mathbb{R}^3 / \varphi \in [0, 2\pi],\ \theta \in [0, \pi]\}$ be the unit sphere endowed with the metric $g = (\mathrm{d}\theta)^2 + \sin^2\theta(\mathrm{d}\varphi)^2$ induced from $\mathbb{R}^3$. Then the identity I is the only one (1,1)-tensor field $K : (TM, g^c) \to (TM, g^c)$ which are horizontally weakly conformal and of maximal rank.

Theorem 4.5. *Any* (1,1)-*tensor field* $K : (TM, g^c) \to (TM, g^c)$ *of maximal rank is a harmonic morphism (of conformal factor* $\Lambda \in \mathcal{F}(TM)$*) if and only if* K *is the identity tensor field up to a non-zero constant factor* $\Lambda \in \mathbb{R}$, *which satisfies* (4.5)

Proof. As we mentioned in Introduction, the harmonic morphisms are characterized in the Riemannian case by [4], [8] and in the semi-Riemannian case by [4], as to be the maps which are horizontally weakly conformal and harmonic. From Theorem 3.9 and Lemma 4.4, K is a harmonic morphism of maximal rank and of conformal factor $\Lambda \in \mathcal{F}(TM)$ if and only if it satisfies three relations: $\delta K = 0$, (4.4) and (4.5). From (4.4) and $\delta K = 0$ one can see that Λ is constant. Therefore these relations are equivalent to $K = \Lambda I$, with Λ a non-zero constant satisfying (4.5) which complete the proof.

Proposition 4.6. [1] *Let $F : A \to B$ be a weakly conformal map between semi-Riemannian manifolds of the same dimension m which is non-degenerate on a dense subset. Then*

(i) *if $m = 2$, F is harmonic;*

(ii) *if $m \geq 3$, F is harmonic if and only if the conformality factor is constant.*

Remark. From Proposition 4.6 and Lemma 4.3, any (1,1)-tensor field $K : (TM, g^c) \to (TM, g^c)$ of maximal rank which is a harmonic morphism, has a constant conformality factor when $\dim TM \geq 3$ (i.e. $\dim M > 1$). Therefore Theorem 4.5 shows that Proposition 4.6 (ii) holds for any (1,1)-tensor field $K : (TM, g^c) \to (TM, g^c)$ of maximal rank which is a harmonic morphism, even when $\dim M = 1$.

Among the (1,1)-tensor fields of maximal rank, the class of (classical) harmonic morphisms determined by Theorem 4.5 is very restricted, so that we are motivated to study (1,1)-tensor fields which are harmonic φ-morphisms with φ the identity map.

5. Examples of harmonic φ-morphisms

Some of the examples of harmonic (1,1)-tensor fields obtained in [6] turn out to be of maximal rank and moreover turn out to be harmonic φ-morphisms (with φ the identity map) w.r.t. the canonical connection. We note that the notion of harmonic (1,1)-tensor field given in [6] is different of that used in [2]. As they are consequences of Definition 3.6 and Theorem 3.9 all the statements of this section are given without proof.

Proposition 5.1. *The identity tensor field of a Riemannian manifold M is a harmonic 1_M-morphism of dilation one.*

Proposition 5.2. *On any Einstein manifold M, the Ricci* (1,1)-*tensor field Ric*: $TM \to TM$ *is a harmonic* 1_M-*morphism.*

We recall that an almost Hermitian manifold $(M,\ g,\ J)$ is called cosymplectic [10] or semi-Kähler [7], provided $\delta J = 0$.

Proposition 5.3. *An almost Hermitian manifold* $(M,\ g,\ J)$ *is semi-Kähler if and only if J is a harmonic* 1_M-*morphism.*

Next, an almost product Riemannian manifold is defined as a Riemannian manifold (M, g) endowed with an almost product structure P (that is a (1,1)-tensor field $P \neq \pm 1_{TM}$, with $P^2 = 1_{TM}$) such that $g(PX,\ PY) = g(X, Y)$, $\forall X,\ Y \in \Gamma(TM)$. A classification of these manifolds is given in [9].

Proposition 5.4. *If* $(M,\ P,\ g)$ *is an almost product Riemannian manifold lying in the class* $W_1 \oplus W_2 \oplus W_4 \oplus W_5$, [9] *then P is a harmonic* 1_M-*morphism.*

The needed property of being in the given class is $\delta P = 0$.

References

[1] Baird, P.; Wood, J. C.: *Harmonic Morphisms between Riemannian Manifolds.* LMS Monograph Series, Oxford University Press.

[2] Chen, B.-Y.: Nagano, T.: *Harmonic Metrics, Harmonic Tensors and Gauss Maps.* J. Math. Soc. Japan **36** (1984), 295–313. Zbl 0543.58015

[3] Eells, J.; Lemaire, L.: *Selected Topics in Harmonic Maps.* CBMS Regional Conf. Ser. in Math. **50**, AMS Providence 1983.

[4] Fuglede, B.: *Harmonic Morphisms between Riemannian Manifolds.* Ann. Inst Fourier Grenoble **28** (1978), 107–144. Zbl 0339.53026

[5] Fuglede, B.: *Harmonic Morphisms between semi-Riemannian Manifolds.* Ann. Acad Sci. Fennicae **21** (1996), 31–50. Zbl 0847.53013

[6] Garcia-Rio, E.; Vanhecke, L.; Vázquez-Abal, M. E.: *Harmonic Endomorphism Fields.* Illinois J. Math **40** (1996), 23–30. Zbl 0880.53032

[7] Gray, A.; Hervella, L.: *The Sixteen Classes of almost Hermitian Manifolds and their Linear Invriants.* Ann. Mat. Pura Appl. **123** (1980), 35–58. Zbl 0444.53032

[8] Ishihara, T.: *A Mapping of Riemannian Manifolds which Preserves Harmonic Functions.* J. Math. Kyoto Univ. **19** (1979), 215–229. Zbl 0421.31006

[9] Naveira, A.: *A Classification of Riemannian almost Product Manifolds.* Rend. Mat. Roma (VII), vol. **3**(3) (1983), 577–592. Zbl 0538.53045

[10] Salamon, S. M.: *Harmonic and Holomorphic Maps.* Geometry Seminar "Luigi Bianchi" II 1984, L. N. Math. **1164**, Springer 1985, 161–224. Zbl 0591.53031

[11] Yano, K.; Ishihara, T.: *Tangent and Cotangent Bundles.* Pure and Appl. Math. **16**, Marcel Dekker, New York 1973.

Received October 1, 2001