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On the Affine Convexity of Convex Curves and Hypersurfaces

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1. Introduction

It is well known that there are interesting analogies between the euclidean geometry and the equiaffine geometry of plane curves. As an example we would like to mention that there exists an affine rectification for arbitrary convex curves analogous to the euclidean rectification by inscribing parabola polygons into the (generalized) tangent bundle of the curve (see [7]). Hereby each parabola arc (with vanishing affine curvature k) connecting its final support elements replaces a segment (with vanishing euclidean curvature κ) connecting its endpoints.

Now a closed straight polygon Π in the euclidean plane is a convex curve if and only if the lines carrying the segments are supporting Π. In analogy to this we shall call a closed convex parabola polygon $\mathcal{P}$ *affinely convex* if and only if the parabolas carrying the segments of $\mathcal{P}$ are supporting $\mathcal{P}$. After a characterization of such affinely convex parabola polygons in Section 2 we consider in Section 3 as limiting case smooth ovals C in the plane and define them to be affinely convex if and only if every hyperosculating parabola of C is supporting C. One main result is now the characterization of affinely convex ovals by the condition $k \geqq 0$ in analogy to the euclidean characterization of smooth convex curves as simply closed curves with $\kappa \geqq 0$. (We owe partial results in this direction to T. Carleman [3]). Moreover, each parabola arc connecting two support elements of an affinely convex oval either is a part of this oval or it does not intersect the latter in its interior, in analogy to the euclidean case. This fact is an improvement of an old result of K. Reidemeister [11].

Unfortunately ovaloids F in the d-space ($d \geqq 3$) don't possess hyperosculating paraboloids in general. Therefore the notion of affine convexity may be defined for them in Section 4 only in an analytic way, namely by the claim that their affine curvatures $k_1, \ldots, k_{d-1}$ are non-negative everywhere. But it turns out that this definition (made in analogy to the euclidean geometry) has interesting consequences: The affine surface area of affinely convex ovaloids is a strictly monotone increasing (with respect to inclusion) and a continuous (with respect

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to the Hausdorff topology) functional, properties which are not valid for general ovaloids. But these properties are also valid for affinely convex ovals and parabola polygons. All these things follow from the fact that the convex domains resp. bodies bounded by these curves resp. hypersurfaces belong to a special class for which C. Petty [10] had proved the indicated properties.

2. Affinely convex parabola polygons

We begin with

Definition 2.1. *The union of a finite set of parabola arcs* $\mathcal{P}_{l-1l}$ *in the affine plane* $\mathbb{R}_2$*, situated on the pairwise different parabolas* P_{l-1l} *and determined by their final support elements (points and tangent lines)* (x_{l-1}, t_{l-1}) *and* (x_l, t_l) $(l = 1, \dots, k)$ *with*

$$x_k = x_0, \quad t_k = t_0, \tag{1}$$

is called a parabola polygon $\mathcal{P}$*:*

$$\mathcal{P} := \bigcup_{l=1}^{k} \mathcal{P}_{l-1l} \tag{2}$$

if $\mathcal{P}$ *is a simply closed convex curve.*

We say that $\mathcal{P}$ is *inscribed into a simply closed plane convex curve* C if all support elements (x_l, t_l) of $\mathcal{P}$ belong to the generalized tangent bundle $\mathcal{T}(C)$ of C consisting of all support elements of C (points and support lines) $(l = 1, \dots, k)$.

Definition 2.2. *A parabola polygon* $\mathcal{P}$ *is affinely convex if any parabola* P_{l-1l} *of* $\mathcal{P}$ *supports the curve* $\mathcal{P}$*, i.e. if* $\mathcal{P}$ *is contained in the closed convex halfspace bounded by* P_{l-1l} $(l = 1, \dots, k)$.

For example a regular parabola polygon, inscribed into a circle, is affinely convex as well as any of its equiaffine images (see Fig. 1 for a regular parabola triangle). Otherwise, if two adjacent arcs of a parabola polygon $\mathcal{P}$ are inscribed into one branch of a hyperbola H then $\mathcal{P}$ cannot be affinely convex (see Fig. 2).

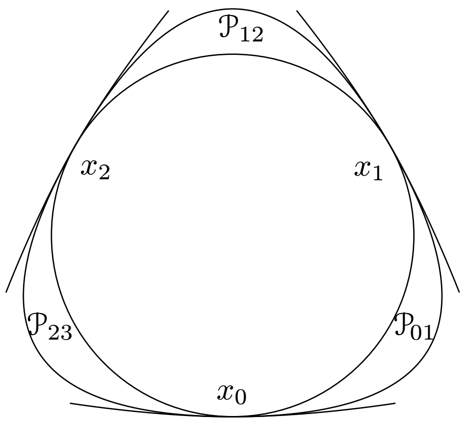

Figure 1

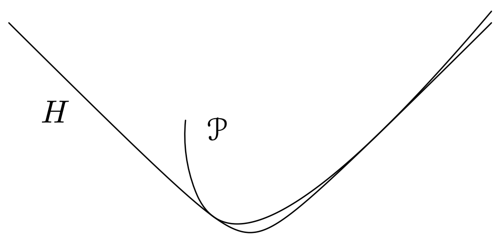

Figure 2

Now it is possible to characterize affinely convex parabola polygons in the following manner:

Proposition 2.3. *A parabola polygon $\mathcal{P}$ is affinely convex if and only if the following conditions are fulfilled:*
i) *The parabolas P_{l-1l} and P_{ll+1} of the arcs $\mathcal{P}_{l-1l}$ and $\mathcal{P}_{ll+1}$ of $\mathcal{P}$ do not only touch at x_l but also osculate at this point of second order, i.e. they have three infinitesimally neighbouring points in common there ($l = 1, \dots, k$).*
ii) *The direction angles φ_{l-1l} of the axes of the parabolas P_{l-1l} of $\mathcal{P}$ are ordered by (w.l.o.g.)*

$$0 \leqq \varphi_{01} < \varphi_{12} < \cdots < \varphi_{k-1k} < 2\pi \tag{3}$$

if the arcs $\mathcal{P}_{l-1l}$ of $\mathcal{P}$ are ordered in the positive sense ($l = 1, \dots, k$).

Proof. At first we shall show that the conditions i) and ii) are necessary. Indeed, as by the affine convexity of $\mathcal{P}$ the parabola P_{l-1l} is supporting in particular the arc $\mathcal{P}_{ll+1}$ of $\mathcal{P}$ the euclidean curvature $\kappa^{(l-1l)}(x_l)$ of P_{l-1l} at x_l cannot exceed the euclidean curvature $\kappa^{(ll+1)}(x_l)$ of $\mathcal{P}_{ll+1}$ at x_l:

$$\kappa^{(l-1l)}(x_l) \leqq \kappa^{(ll+1)}(x_l). \tag{4}$$

In the same way, as P_{ll+1} is supporting in particular $\mathcal{P}_{l-1l}$, we find

$$\kappa^{(ll+1)}(x_l) \leqq \kappa^{(l-1l)}(x_l) \tag{5}$$

such that by (4) and (5)

$$\kappa^{(l-1l)}(x_l) = \kappa^{(ll+1)}(x_l)$$

or – equivalently –

$$\frac{d^2x^{(l-1l)}}{d\sigma^2}(\sigma_l) = \frac{d^2x^{(ll+1)}}{d\sigma^2}(\sigma_l) \tag{6}$$

besides

$$x^{(l-1l)}(\sigma_l) = x^{(ll+1)}(\sigma_l), \quad \frac{dx^{(l-1l)}}{d\sigma}(\sigma_l) = \frac{dx^{(ll+1)}}{d\sigma}(\sigma_l)$$

which yields property i) ($x = x^{(l-1l)}(\sigma)$ resp. $x = x^{(ll+1)}(\sigma)$ are suitable representations of P_{l-1l} resp. P_{ll+1} with the help of the euclidean arclength σ).

Now it is clear that the (different) conics in the projective plane $\mathbb{P}_2 := \mathbb{R}_2 \cup \{l_\infty\}$, resulting after completion of the parabolas P_{l-1l} and P_{ll+1} in $\mathbb{R}_2$ by their improper points i_{l-1l} and i_{ll+1} (where they touch the improper line l_∞), may have only one further point in common besides the triple point x_l ($l = 1, \dots, k$), not lying on l_∞. This additional point of intersection must exist by topological reasons such that we have the situation as indicated in Fig. 3 where the axes a_{l-1l} resp. a_{ll+1} of P_{l-1l} resp. P_{ll+1} passing through the point x_l with direction angles φ_{l-1l} resp. φ_{ll+1} satisfy

$$\varphi_{l-1l} < \varphi_{ll+1} \quad (l = 1, \dots, k).$$

A suitable numeration of the parabola arcs then shows the validity of ii).

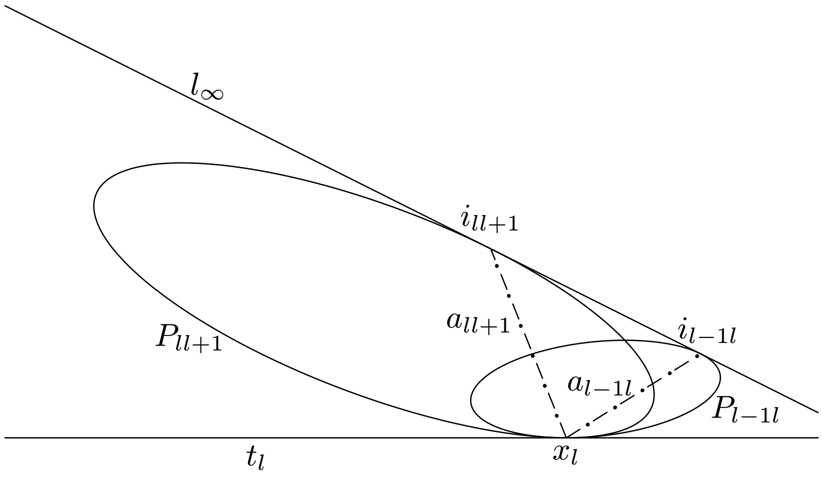

Figure 3

Conversely, we will assume now that the parabola polygon $\mathcal{P}$ fulfils the conditions i) and ii), and we want to prove that $\mathcal{P}$ must be affinely convex. For this reason we fix the parabola P_{l-1l} carrying the arc $\mathcal{P}_{l-1l}$ of $\mathcal{P}$ and consider at first the part P^+_{l-1l} of P_{l-1l} connecting the points x_l and i_{l-1l} of P_{l-1l} in the positive sense. Then P^+_{l-1l} meets the segment $\mathcal{P}_{ll+1} \cup P^+_{ll+1}$ of P_{ll+1} between x_l and i_{ll+1} only at x_l since the conics P_{l-1l} and P_{ll+1} have except the triple point x_l (see i)) only one further point in common which lies out of P^+_{l-1l} because of ii) (compare Fig. 3!).

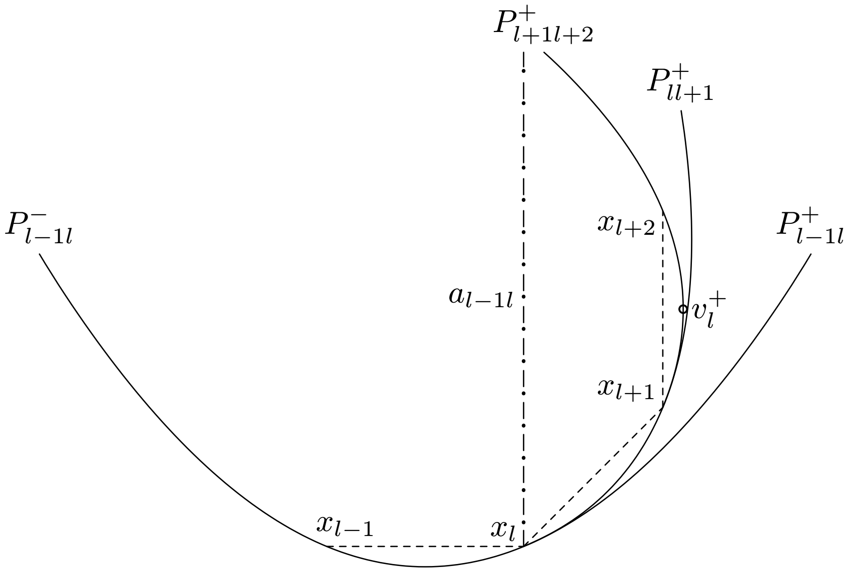

Figure 4

Now we consider the point v^+_l of $\mathcal{P}$ with (oriented) tangent line parallel to the axis a_{l-1l} of P_{l-1l} through x_l. If v^+_l lies on $\mathcal{P}_{ll+1}$ we are sure that the arc $\mathcal{P}^+_l$ of $\mathcal{P}$ from x_l to v^+_l (in a positive sense) does not meet the part P^-_{l-1l} of P_{l-1l} from x_l to i_{l-1l} (in a negative sense) with exception of x_l because $\mathcal{P}^+_l$ and P^-_{l-1l} are separated by a_{l-1l}. Therefore P_{l-1l} supports the part $\mathcal{P}_l^+$ of $\mathcal{P}$. In the other case where v_l^+ lies on $\mathcal{P}_{mm+1}$ with $l < m$ we iterate the preceding considerations from l to m and see that P_{l-1l} supports $\mathcal{P}_l^+$ too. In the same way we find that that P_{l-1l} supports the arc $\mathcal{P}_l^-$ of $\mathcal{P}$ from x_l to the point v_l^- (in a negative sense) with an oriented line in the opposite direction to the direction of a_{l-1l}. Since it is trivial that the remainder $\mathcal{P} \setminus \left(\mathcal{P}_l^+ \cup \mathcal{P}_l^-\right)$ of $\mathcal{P}$ is also supported by P_{l-1l} we finally know that P_{l-1l} supports $\mathcal{P}$ (see Fig. 4). As this is true for $l = 1, \dots, k$ the parabola polygon $\mathcal{P}$ must be affinely convex by Definition 2.2 which completes the proof of Proposition 2.3. □

3. Affinely convex ovals

In this section we consider ovals instead of parabola polygons, so to speak as limiting case of the latter ones. An oval C is a simply closed plane curve $x = x(\sigma)$ of class $\mathcal{C}_2$ in $\mathbb{R}_2$ with positive curvature

$$\kappa := \left(\frac{dx}{d\sigma}, \frac{d^2x}{d\sigma^2}\right) \tag{7}$$

everywhere (σ euclidean arclength parameter of C). It is well-known that it is possible to introduce for C an *equiaffine arclength parameter* s by

$$s := \int_{\sigma_0}^{\sigma} \kappa^{\frac{1}{3}}\, d\sigma \tag{8}$$

whence by (7) for a curve of class $\mathcal{C}_3$

$$\left(\frac{dx}{ds}, \frac{d^2x}{ds^2}\right) = 1. \tag{9}$$

Supposing now that C be of class $\mathcal{C}_4$ with respect to σ the curve C is of class $\mathcal{C}_3$ with respect to s and differentiation of (9) yields

$$\left(\frac{dx}{ds}, \frac{d^3x}{ds^3}\right) = 0. \tag{10}$$

Thus, after introduction of the so-called *affine normal vector*

$$y := \frac{d^2x}{ds^2} \tag{11}$$

we may define the coefficient k in the relation

$$\frac{d^3x}{ds^3} = \frac{dy}{ds} = -k\,\frac{dx}{ds} \tag{12}$$

(see (10)) as *affine curvature* k of the oval C (in analogy to the equation $\frac{dn}{d\sigma} = -\kappa\,\frac{dx}{d\sigma}$ for the euclidean curvature κ of C where n is the inner unit normal vector).

As limit case we have now to replace the parabola P_{l-1l} containing the parabola arc from the support element $(x(s_{l-1}), t(s_{l-1}))$ of C to the support element $(x(s_l), t(s_l))$ with the join a_l of $t(s_{l-1}) \cap t(s_l)$ and $\frac{1}{2}(x(s_{l-1}) + x(s_l))$ as an axis by the parabola P_{s_0} hyperosculating C

at $x(s_0)$ of third order with the axis a_{s_0} through $x(s_0)$. This is obvious because P_{s_0} has four infinitesimally neighbouring points in common with C there. P_{s_0} may be represented by

$$x = z(s) := x(s_0) + (s - s_0)\frac{dx}{ds}(s_0) + \frac{1}{2}(s - s_0)^2\frac{d^2x}{ds^2}(s_0) \tag{13}$$

where s is also an affine arclength parameter because of

$$\left(\frac{dz}{ds}, \frac{d^2z}{ds^2}\right) = \left(\frac{dx}{ds}(s_0) + (s - s_0)\frac{d^2x}{ds^2}(s_0), \frac{d^2x}{ds^2}(s_0)\right) = \left(\frac{dx}{ds}(s_0), \frac{d^2x}{ds^2}(s_0)\right) = 1 \tag{14}$$

(see (9)). It has indeed the property of hyperosculation since

$$z(s_0) = x(s_0), \quad \frac{d^\nu z}{ds^\nu}(s_0) = \frac{d^\nu x}{ds^\nu}(s_0) \quad (\nu = 1, 2) \tag{15}$$

whence by (8),

$$\frac{dx}{ds} = \kappa^{-\frac{1}{3}} \cdot \frac{dx}{d\sigma}, \quad \frac{d^2x}{ds^2} = -\frac{1}{3}\kappa^{-\frac{5}{3}}\frac{d\kappa}{d\sigma} \cdot \frac{dx}{d\sigma} + \kappa^{\frac{1}{3}} \cdot n$$

and

$$\frac{d^2x}{d\sigma^2} = \kappa n, \quad \frac{d^3x}{d\sigma^3} = -\kappa^2 \cdot \frac{dx}{d\sigma} + \frac{d\kappa}{d\sigma} \cdot n$$

(n inner unit normal vector) we get

$$z(s_0) = x(s_0), \quad \frac{d^\mu z}{d\sigma^\mu}(s_0) = \frac{d^\mu x}{d\sigma^\mu}(s_0) \quad (\mu = 1, 2, 3) \tag{16}$$

(compare condition (6) for the osculation of P_{l-1l} and P_{ll+1} !). These facts motivate in analogy to Definition 2.2 the

Definition 3.1. *An oval C of class $\mathcal{C}_4$ is called affinely convex if any hyperosculating parabola P_{s_0} of C supports C (in the same sense as in Definition 2.2).*

We have to mention that 1940 T. Carleman [3] made the same definition (in a local sense) for ovals of class $\mathcal{C}_4$ which were called *courbes paraboliquement convexes.* Using the representation $\xi_2 = \xi_2(\xi_1)$ for the points $x = (\xi_1, \xi_2)$ of C this author found as a necessary condition for such a curve

$$\frac{d^2}{d\xi_1^2}\left(\left(\frac{d^2\xi_2}{d\xi_1^2}\right)^{-\frac{2}{3}}\right) \leqq 0. \tag{17}$$

Since we have

$$-\frac{1}{2}\frac{d^2}{d\xi_1^2}\left(\left(\frac{d^2\xi_2}{d\xi_1^2}\right)^{-\frac{2}{3}}\right) = k \tag{18}$$

(see [1], p. 14 (83)) this condition (17) is equivalent to $k \geqq 0$ for all points of C. Carleman even proved in addition that conversely the stronger condition $k > 0$ for the oval C implies that all its hyperosculationg parabolas P_{s_0} have the property

$$C \cap P_{s_0} = \{x(s_0)\}. \tag{19}$$

We shall now prove the stronger

Theorem 3.2. *An oval C of class $\mathcal{C}_4$ is affinely convex if and only if the affine curvature k of C is nonnegative everywhere*:

$$k \geqq 0. \tag{20}$$

Proof. In the first part we show that the condition (20) is necessary. For this purpose we use the representation

$$(z - x(s_0), \frac{d^2x}{ds^2}(s_0))^2 + 2(z - x(s_0), \frac{dx}{ds}(s_0)) \leqq 0 \tag{21}$$

of the closed convex region bounded by P_{s_0} following from (13) and (9). As C is supported by P_{s_0} as an affinely convex oval we have

$$A(s) := (x(s) - x(s_0), \frac{d^2x}{ds^2}(s_0))^2 + 2(x(s) - x(s_0), \frac{dx}{ds}(s_0)) \leqq 0 \tag{22}$$

for all s whence in particular using (12)

$$A(s_0) = \frac{dA}{ds}(s_0) = \frac{d^2A}{ds^2}(s_0) = \frac{d^3A}{ds^3}(s_0) = 0, \quad \frac{d^4A}{ds^4}(s_0) = -6k(s_0)\ ^1. \tag{23}$$

We make now the assumption

$$\frac{d^4A}{ds^4}(s_0) > 0 \tag{24}$$

such that because of $\frac{d^3A}{ds^3}(s_0) = 0$ the relation

$$\frac{d^3A}{ds^3}(s) > 0 \quad (s_0 < s < s_0 + \delta) \tag{25}$$

holds for a suitable $\delta > 0$. On the other hand iterated application of the mean value theorem together with (22) and (23) yields

$$\frac{dA}{ds}(s_0 + \delta_1) \leqq 0, \quad \frac{d^2A}{ds^2}(s_0 + \delta_2) \leqq 0 \text{ and } \frac{d^3A}{ds^3}(s_0 + \delta_3) \leqq 0$$

[1] In order to get $\frac{d^4A}{ds^4}(s_0)$ we have used the rule $\frac{d}{ds}(f \cdot g)(s_0) = f(s_0) \cdot \frac{dg}{ds}(s_0)$ if f is continuous at s_0 and g is differentiable at s_0 with $g(s_0) = 0$.

for suitable $0 < \delta_3 < \delta_2 < \delta_1 < \delta$ contradicting (25). Thus the assumption (24) is wrong with the consequence that by (23) indeed $k(s_0) \geqq 0$ is valid for every s_0 [2].

Now we prove that (20) is also sufficient for the oval C to be affinely convex. Therefore we introduce at first the *affine evolvents* E_τ of C with the representation

$$w(s,\tau) := x(s) + (\tau - s)\frac{dx}{ds}(s) + \frac{1}{2}(\tau - s)^2\frac{d^2x}{ds^2}(s) \quad (s_0 \leqq s \leqq \tau) \tag{26}$$

depending on a parameter τ such that $\tau - s_0$ is the affine arclength of the curve which is the union of the arc of C from $x(s_0)$ to $x(s)$ and the arc of P_s from $x(s)$ to $w(s,\tau)$. E_τ is a curve of class $\mathcal{C}_1$ with respect to s which joins the points $w(s_0,\tau)$ and $w(\tau,\tau) = x(\tau)$. Moreover this curve penetrates each parabola P_s either in a stationary or in a transversal manner, a consequence of the relation

$$\left(\frac{\partial w}{\partial \tau}(s,\tau), \frac{\partial w}{\partial s}(s,\tau)\right) = \left(\frac{dx}{ds}(s) + (\tau - s)\frac{d^2x}{ds^2}(s), \frac{1}{2}(\tau - s)^2\frac{d^3x}{ds^3}(s)\right) = \frac{1}{2}(\tau - s)^3 k(s) \geqq 0 \tag{27}$$

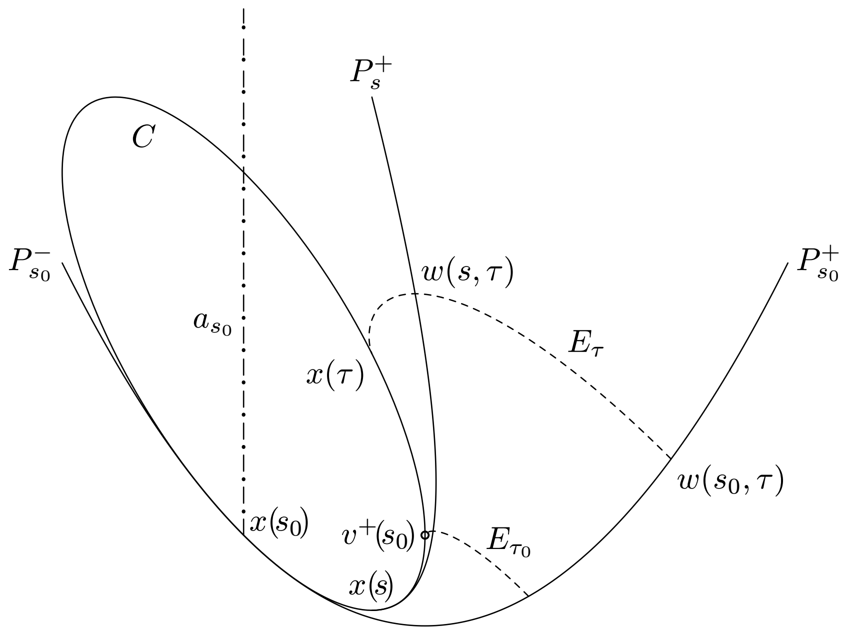

Figure 5

(see (26) and (20)). Therefore we are sure that the arc $C_{s_0}^+$ of C from $x(s_0)$ to the point $v^+(s_0)$ with (oriented) tangent line parallel to the axis a_{s_0} of P_{s_0} is supported by the part $P_{s_0}^+$ of P_{s_0} from $x(s_0)$ to its improper point i_{s_0} on the "convex side". Moreover $C_{s_0}^+$ does not meet the part $P_{s_0}^-$ of P_{s_0} from i_{s_0} to $x(s_0)$ with exception of $x(s_0)$ because $C_{s_0}^+$ and $P_{s_0}^-$ are separated by a_{s_0}. Thus P_{s_0} supports the part $C_{s_0}^+$ of C. In the same way we find that P_{s_0} also supports the arc $C_{s_0}^-$ of C from $x(s_0)$ to the point $v^-(s_0)$ with (oriented) tangent line in the opposite direction to that of a_{s_0}. Finally P_{s_0} supports trivially the arc $C \setminus (C_{s_0}^+ \cup C_{s_0}^-)$ of C

[2] This can also be seen after application of a formula of J. Merza ([8], théorème 2): $k(s_0) = \lim_{s \to s_0} \frac{8\bar{d}(s)}{(s-s_0)^4}$ where the (nonnegative) affine distance $\bar{d}(s)$ of $x(s)$ from P_{s_0} is given by $x(s) = z(s_1) + \bar{d}(s)y(s_0)$ if the curve $C : x = x(s)$ is sufficiently smooth.

and these facts altogether complete the proof of Theorem 3.2 if $x(s_0)$ is arbitrarily chosen on C (see Fig. 5 and compare it with Fig. 4). □

Corollary 3.3. *An oval C of class $\mathcal{C}_4$ is affinely convex if and only if the direction angles $\varphi(s)$ of the axes a_s of the hyperosculating parabolas P_s of C have the property*

$$\frac{d\varphi}{ds}(s) \geqq 0 \tag{28}$$

(*compare Proposition* 2.3 ii)).

This is an immediate consequence of the known fact that the direction of the axes of a parabola P_s equals the direction of the (constant) affine normal vectors of P_s which by (13) and (15) equals the direction of the affine normal vector $y(s)$ of C at $x(s)$. This remark namely yields

$$\frac{d\varphi}{ds} = \frac{d}{ds}\left(\arctan\frac{\eta_2}{\eta_1}\right) = \frac{(y, \frac{dy}{ds})}{\eta_1^2+\eta_2^2} = \frac{k}{\eta_1^2+\eta_2^2} \geqq 0$$

for all s (see (12)) for $y = (\eta_1, \eta_2) \neq 0$.

For example every ellipse (with affine normals intersecting in its midpoint lying on its convex side) is affinely convex, and every arc of a branch of a hyperbola (with affine normals intersecting in its midpoint lying on its concave side) cannot be part of an affinely convex oval. Now affinely convex ovals have the following "convexity property" which was an important tool in the early considerations of P. Böhmer [2] and H. Mohrmann [9] for (closed and open) ovals with $k > 0$ or $k < 0$ everywhere (see also W. Blaschke [1], p. 47–49):

Theorem 3.4. *If $\mathcal{P}_{s_0s_1}$ is the parabola arc with the support elements $(x(s_0), t(s_0))$ and $(x(s_1), t(s_1))$ of an affinely convex oval C of class $\mathcal{C}_4$ as final support elements*
$(s_0 < s_1,\ \measuredangle(t(s_0), t(s_1)) < \pi)$ then either
i) $\mathcal{P}_{s_0s_1} \cap C = \{x(s_0), x(s_1)\}$ *or*
ii) $\mathcal{P}_{s_0s_1}$ *is a part of C.*

Proof. For the proof of Theorem 3.4 we carry out a slight modification together with an improvement of a proof of K. Reidemeister [11]. For this purpose we introduce in the plane $\mathbb{R}_2$ an affine coordinate system $\{\xi_1, \xi_2\}$ in such a manner that we get $x(s_0) = (0,1),\ x(s_1) = (1,0)$ and $t(s_0) : \xi_1 = 0,\ t(s_1) : \xi_2 = 0$. Then $\mathcal{P}_{s_0s_1}$ may be represented by

$$\xi_1 = \zeta_1(\tau) := \tau^2,\ \ \xi_2 = \zeta_2(\tau) := (1-\tau)^2 \quad (0 \leqq \tau \leqq 1), \tag{29}$$

and its affine arclength parameter equals $4^{\frac{1}{3}}\tau$. Furthermore the arc $\mathcal{C}$ of C between $x(s_0)$ and $x(s_1)$ may be represented by $x = x(s)$ with $s_0 \leqq s_1$ or

$$\xi_1 = \xi_1(\tau),\ \ \xi_2 = \xi_2(\tau) \quad (0 \leqq \tau \leqq 1) \tag{30}$$

with

$$\tau := \frac{s - s_0}{L(\mathcal{C})} \quad (s_0 \leqq s \leqq s_1,\ L(\mathcal{C}) := s_1 - s_0). \tag{31}$$

Here ξ_1 and ξ_2 are functions of τ of class $\mathcal{C}_3$ on $[0,1]$ having the properties

$$\xi_1(0)=0,\ \xi_2(0)=1,\ \xi_1(1)=1,\ \xi_2(1)=0 \tag{32}$$

and

$$\frac{d\xi_1}{d\tau}(0)=0,\ \frac{d\xi_2}{d\tau}(1)=0. \tag{33}$$

After introduction of $f_1:=\xi_1-\zeta_1,\ f_2:=\xi_2-\zeta_2$, two functions of class $\mathcal{C}_3$ on $[0,1]$, we get using (29), (32) and (33)

$$f_1(0)=f_1(1)=\frac{df_1}{d\tau}(0)=0, \tag{34}$$

$$f_2(0)=f_2(1)=\frac{df_2}{d\tau}(1)=0 \tag{35}$$

as well as by (12), (20) and (31)

$$\frac{d^3f_1}{d\tau^3}(\tau)=\frac{d^3\xi_1}{d\tau^3}(\tau)=-k(\tau)L(\mathcal{C})^2\frac{d\xi_1}{d\tau}(\tau)\leqq 0, \tag{36}$$

$$\frac{d^3f_2}{d\tau^3}(\tau)=\frac{d^3\xi_2}{d\tau^3}(\tau)=-k(\tau)L(\mathcal{C})^2\frac{d\xi_2}{d\tau}(\tau)\geqq 0 \quad (0\leqq\tau\leqq 1). \tag{37}$$

But (36) means that $\frac{d^2f_1}{d\tau^2}$ is a monotone decreasing function, either positive on $[0,1]$ or positive on $[0,\alpha)$, zero on $[\alpha,\beta]$ and negative on $(\beta,1]$ $(0\leqq\alpha\leqq\beta\leqq 1)$ or negative on $[0,1]$. Therefore the function f_1 itself is either strictly convex or strictly convex on $[0,\alpha)$, linear on $[\alpha,\beta]$ and strictly concave on $(\beta,1]$ or strictly concave. But such a function only fits into the boundary conditions (34) if it is either positive in $(0,1)$ such that

$$\xi_1(\tau)>\tau^2 \quad (0<\tau<1) \tag{38}$$

or if it vanishes identically such that

$$\xi_1(\tau)\equiv\tau^2 \tag{39}$$

(see (29)). In the same way we find that the function $\bar{f}_2(\tau):=f_2(1-\tau)$ $(0\leqq\tau\leqq 1)$ which fulfils the same inequality and boundary conditions as f_1 because of (37) and (35) either has the property $\bar{f}_2(\tau)>0$ $(0<\tau<1)$ or $\bar{f}_2(\tau)\equiv 0$ such that by (29) either

$$\xi_2(\tau)>(1-\tau)^2 \quad (0<\tau<1) \quad \text{or} \tag{40}$$

$$\xi_2(\tau)\equiv(1-\tau)^2 \tag{41}$$

holds.

Geometrically the results (38),(39) and (40),(41) may be interpreted as follows: In the case $\xi_1(\tau) > \tau^2$ and $\xi_2(\tau) > (1-\tau)^2$ $(0 < \tau < 1)$ every point $(\xi_1(\tau'), \xi_2(\tau'))$ $(\tau' = \frac{s'-s_0}{s_1-s_0}$ with $s_0 < s' < s_1$, see (31)) of $\mathcal{C}$ lies in the open quadrant

$$Q_{\tau'} := \{(\xi_1, \xi_2) \in \mathbb{R}_2 | \ \xi_1 > {\tau'}^2, \ \xi_2 > (1-\tau')^2\} \tag{42}$$

of $\mathbb{R}_2$ for which we have $\mathcal{P}_{s_0 s_1} \cap Q_{\tau'} = \emptyset$ whence

$$\mathcal{P}_{s_0 s_1} \cap \mathcal{C} = \{x(s_0), x(s_1)\} \tag{43}$$

follows. As trivially $\mathcal{P}_{s_0 s_1} \cap (C \setminus \mathcal{C}) = \emptyset$ indeed i) holds. The same fact is true in the cases $\xi_1(\tau) > \tau^2$ and $\xi_2(\tau) \equiv (1-\tau)^2$ as well as $\xi_1(\tau) \equiv \tau^2$ and $\xi_2(\tau) > (1-\tau)^2$ for $0 < \tau < 1$. Since in the last case $\xi_1(\tau) \equiv \tau^2$ and $\xi_2(\tau) \equiv (1-\tau)^2$ the parabola arc $\mathcal{P}_{s_0 s_1}$ is a part of C such that ii) holds the proof of Theorem 3.4 is complete. □

Remark 3.5. We have proved more than claimed in i) or ii) of Theorem 3.4, namely:
The arc $\mathcal{C}$ of C between $x(s_0)$ and $x(s_1)$ lies in the closed (convex) region bounded by the segment $x(s_0)x(s_1)$ and the parabola arc $\mathcal{P}_{s_0 s_1}$.
Moreover it should be mentioned that in this theorem $\mathcal{C}$ may be replaced by an affinely convex oval arc from $x(s_0)$ to $x(s_1)$ with tangent lines at the endpoints intersecting within the closed triangle with the vertices $x(s_0)$, $x(s_1)$ and $t(s_0) \cap t(s_1)$ (i.e. (33) may be replaced by the inequalities $\frac{d\xi_1}{d\tau}(0) \geqq 0$, $\frac{d\xi_2}{d\tau}(1) \leqq 0$) (compare [1], p. 47–48).

4. Affinely convex ovaloids

Unfortunately it is not possible to extend Definition 3.1 of affinely convex ovals to ovaloids F (compact orientable hypersurfaces in $\mathbb{R}_d$ $(d > 2)$ of class $\mathcal{C}_2$ with positive Gauss curvature H_{d-1} everywhere and bijective Gauss map $F \to S^{d-1}$ because a hyperosculating paraboloid for F at a point x of F only exists in the case where the so-called cubic form $A(x)$ of F at x vanishes (see [1], p. 107 (29) and p. 114 (78) for $d = 3$). Moreover the (elliptic) paraboloids are not the only equiaffine analogues for the hyperplanes in the euclidean differential geometry, the same role are playing more generally all the improper affine hyperspheres (see [1], p. 209–210 for $d = 3$). For these reasons we shall define affinely convex ovaloids in a purely analytical manner keeping the consistency with the case $d = 2$ (see Theorem 3.2) by generalization of (20) to the case $d > 2$. Before doing so we note some basic facts of the equiaffine differential geometry of ovaloids $F : x = x(u^1, \dots, u^{d-1})$ in $\mathbb{R}_d$ of class $\mathcal{C}_5$ with positive Gauss curvature

$$H_{d-1} := \frac{(-\frac{\partial n}{\partial u^1}, \dots, -\frac{\partial n}{\partial u^{d-1}}, n)}{(\frac{\partial x}{\partial u^1}, \dots, \frac{\partial x}{\partial u^{d-1}}, n)} = \frac{\det(<n, \frac{\partial^2 x}{\partial u^i \partial u^j}>)}{\det(<\frac{\partial x}{\partial u^i}, \frac{\partial x}{\partial u^j}>)} \tag{44}$$

(n inner unit normal vector of F, compare (7)). On F there exists the equiaffinely invariant (positive definite) Blaschke metric tensor field with (symmetric) coefficients

$$G_{ij} := \frac{(\frac{\partial x}{\partial u^1}, \dots, \frac{\partial x}{\partial u^{d-1}}, \frac{\partial^2 x}{\partial u^i \partial u^j})}{(\det((\frac{\partial x}{\partial u^1}, \dots, \frac{\partial x}{\partial u^{d-1}}, \frac{\partial^2 x}{\partial u^i \partial u^j})))^{\frac{1}{d+1}}} = \frac{<n, \frac{\partial^2 x}{\partial u^i \partial u^j}>}{H_{d-1}^{\frac{1}{d+1}}} \quad (i, j = 1, \dots, d-1), \tag{45}$$

and it is possible to define there a typical (twice differentiable) *affine normal vector* field y by

$$y := \frac{1}{d-1}\Delta x \tag{46}$$

(Δ Beltrami operator with respect to G_{ij}, compare (11)). For this vector field we have the affine Weingarten equations

$$\frac{\partial y}{\partial u^i} = -B_i^j \frac{\partial x}{\partial u^j} \quad (i = 1, \dots, d-1) \tag{47}$$

with the integrability conditions of Ricci

$$B_{ik} := B_i^j G_{jk} = B_k^j G_{ji} = B_{ki} \quad (i, k = 1, \dots, d-1) \tag{48}$$

such that the affine shape operator with the coefficients B_i^j $(i, j = 1, \dots, d-1)$ has $d-1$ real eigenvalues called the *affine principal curvatures* $k_1, \dots, k_{d-1}$ of F (compare (12)).

After these preparations we may make

Definition 4.1. *An ovaloid F of class $\mathcal{C}_5$ is called affinely convex if its affine principal curvatures satisfy the conditions*

$$k_1 \geqq 0, \dots, k_{d-1} \geqq 0. \tag{49}$$

Now we have at first to study the properties of the so-called (negative) *curvature image* $\hat{F} : x = -y(u^1, \dots, u^{d-1})$ of F with the convex hull $\mathrm{conv}\hat{F}$ in the case that F is affinely convex. We find

Proposition 4.2. *Let F be an affinely convex ovaloid of class $\mathcal{C}_5$ with the curvature image $\hat{F}$. Then:*

i) *$\hat{F}$ is a part of the boundary of $\hat{K} := \mathrm{conv}\hat{F}$.*

ii) *The support function $h_{\hat{K}}$ of $\hat{K}$ equals the $\frac{1}{d+1}$-th power of the Gauss curvature of the ovaloid F:*

$$h_{\hat{K}}(-n) = H_{d-1}^{\frac{1}{d+1}}(x(n)) \quad (n \in S^{d-1}) \tag{50}$$

(*see* (44)) [3].

Proof. i) It suffices to show that for any point $-y_0 = -y(u^1_{(0)}, \dots, u^{d-1}_{(0)})$ of $\hat{F}$ the whole set $\hat{F}$ and a fortiori $\hat{K}$ lies in the (convex) halfspace $<n_0, z + y_0> \ \geqq 0$ of $\mathbb{R}_d$ where $n_0 := n(u^1_{(0)}, \dots, u^{d-1}_{(0)})$ such that $-y_0$ cannot be a point in the interior of $\hat{K}$. For this purpose we choose an arbitrary point $-y_1 = -y(u^1_{(1)}, \dots, u^{d-1}_{(1)}) \in \hat{F}$ and join $n_1 := n(u^1_{(1)}, \dots, u^{d-1}_{(1)})$ with n_0 by the arc $n(\sigma) = n(u^1(\sigma), \dots, u^{d-1}(\sigma))$ of a great circle of S^{d-1} where σ is an (euclidean) arclength parameter $(0 \leqq \sigma \leqq \sigma_1 := \measuredangle(n_0, n_1) \leqq \pi)$ and $u^i(\sigma)$ are functions of σ of class $\mathcal{C}_4$.

[3] In the special case $\overset{a}{H}_{d-1} := k_1 \cdots k_{d-1} > 0$, i.e. $k_1 > 0, \dots, k_{d-1} > 0$ this was previously proved in [4], p. 265–266.

To this arc there corresponds a curve $x(\sigma) = x(u^1(\sigma), \dots, u^{d-1}(\sigma))$ on F (with this spherical image) of class $\mathcal{C}_4$ as well as a curve $-y(\sigma) = -y(u^1(\sigma), \dots, u^{d-1}(\sigma))$ on $\hat{F}$ of class $\mathcal{C}_2$. By construction we have

$$n_0 = <n(\sigma), n_0> n(\sigma) + <\frac{dn}{d\sigma}(\sigma), n_0> \frac{dn}{d\sigma}(\sigma) \tag{51}$$

with

$$<\frac{dn}{d\sigma}(\sigma), n_0> \leqq 0 \quad (0 \leqq \sigma \leqq \sigma_1). \tag{52}$$

Hence by (51), (47), (45), (48), (52) and (49)

$$\frac{d}{d\sigma} <n_0, -y(\sigma) + y_0> = <n_0, -\frac{\partial y}{\partial u^i}\frac{du^i}{d\sigma}> = <\frac{dn}{d\sigma}, n_0><\frac{\partial n}{\partial u^k}\frac{du^k}{d\sigma}, B_i^j \frac{\partial x}{\partial u^j}\frac{du^i}{d\sigma}>$$

$$= -<\frac{dn}{d\sigma}, n_0> B_i^j G_{jk} H_{d-1}^{\frac{1}{d+1}} \frac{du^i}{d\sigma}\frac{du^k}{d\sigma} = -<\frac{dn}{d\sigma}, n_0> H_{d-1}^{\frac{1}{d+1}} B_{ik}\frac{du^i}{d\sigma}\frac{du^k}{d\sigma} \geqq 0$$

$(0 \leqq \sigma \leqq \sigma_1)$ whence because of $<n_0, -y(0) + y_0> = 0$ after integration indeed $<n_0, -y(\sigma_1) + y_0> = <n_0, -y_1 + y_0> \geqq 0$ follows.

ii) In order to see (50) we choose an arbitrary direction given by the unit vector $n_0 \in S^{d-1}$ which may be considered as image of the Gauss map of a point $x_0 \in F$ with the corresponding point $-y_0 \in \hat{F}$. As we have seen in the proof of i) the hyperplane $<n_0, z + y_0> = 0$ through $-y_0$ with normal direction n_0 supports $\hat{K}$ such that the (positive) distance $< -n_0, -y_0>$ of this hyperplane from the origin yields the value of the support function $h_{\hat{K}}$ of $\hat{K}$ for $-n_0$:

$$h_{\hat{K}}(-n_0) = <n_0, y_0>. \tag{53}$$

But from the well-known affine Gauss equations

$$\frac{\partial^2 x}{\partial u^i \partial u^j} = \Gamma_{ij}^k \frac{\partial x}{\partial u^k} + A_{ij}^k \frac{\partial x}{\partial u^k} + G_{ij}\, y \quad (i, j = 1, \dots, d-1) \tag{54}$$

where Γ_{ij}^k resp. A_{ij}^k are the Christoffel symbols for the metric (45) resp. the coefficients of the cubic form of F $(i, j, k = 1, \dots, d-1)$ we may conclude after scalar multiplication by n

$$<n, \frac{\partial^2 x}{\partial u^i \partial u^j}> = G_{ij} <n, y>. \tag{55}$$

Now the comparison of (55) and (45) yields

$$<n, y> = H_{d-1}^{\frac{1}{d+1}} \tag{56}$$

which together with (53) provides the relation (50), claimed in ii). □

One important consequence of Proposition 4.2 is the fact that the convex body K bounded by an affinely convex ovaloid F is *of elliptic type* in the sense of

Definition 4.3. *A convex body K of dimension d in $\mathbb{R}_d$ is called to be of elliptic type if there exists a positive and continuous "curvature function" ρ for K on S^{d-1}, characterized by the equation*

$$d\,V(K,\dots,K,L) = \int_{S^{d-1}} h_L \cdot \rho \, d\omega \tag{57}$$

for every convex "test body" L in $\mathbb{R}_d$, with the property

$$\rho^{-\frac{1}{d+1}} = h_M \tag{58}$$

for a suitable convex body M in $\mathbb{R}_d$ ($V(K,\dots,K,L)$ mixed volume, $d\omega$ surface area element of S^{d-1}).

Namely if K is bounded by an ovaloid F then

$$\rho = H_{d-1}^{-1} \tag{59}$$

because of $d\,V(K,\dots,K,L) = \int_F h_L \; d\mathcal{H}^{d-1} = \int_{S^{d-1}} h_L H_{d-1}^{-1} \; d\omega$ (see (44), $\mathcal{H}^{d-1}$ Hausdorff measure on F) for all convex bodies L (see (50) and (58)).

Now it is our aim to prove interesting results for the so-called *affine surface area* $A(F)$ of an ovaloid F in $\mathbb{R}_d$ of class $\mathcal{C}_5$:

$$A(F) =: \int_F (\frac{\partial x}{\partial u^1}, \dots, \frac{\partial x}{\partial u^{d-1}}, y) \; du^1 \cdots du^{d-1} = \int_F H_{d-1}^{\frac{1}{d+1}} \; d\mathcal{H}^{d-1} = \int_{S^{d-1}} H_{d-1}^{-\frac{d}{d+1}} \; d\omega \tag{60}$$

(compare (8)) in the special case where the ovaloids are affinely convex. In this case the enclosed convex bodies are of elliptic type and the application of results of C. Petty (see [10], Theorem 3.21 and Theorems 3.12, 2.6) for bodies of elliptic type provides:

Theorem 4.4. *Let A be the functional, defined by $F \mapsto A(F)$ for each affinely convex ovaloid F of class $\mathcal{C}_5$ in $\mathbb{R}_d$. Then*
i) *A is strictly monotone increasing, i.e.*

$$F' \subseteqq \operatorname{conv} F = K, \; F' \neq F \Rightarrow A(F') < A(F), \quad \textit{and} \tag{61}$$

ii) *A is continuous, i.e.*

$$F = \lim_{\nu\to\infty} F_\nu \Rightarrow A(F) = \lim_{\nu\to\infty} A(F_\nu). \tag{62}$$

We have to mention that (61) is not true in general just as (62) where we have only upper semicontinuity:

$$F = \lim_{\nu\to\infty} F_\nu \Rightarrow A(F) \geqq \limsup_{\nu\to\infty} A(F_\nu) \tag{63}$$

(see [6], Proposition 9.2). For the sake of completeness we shall outline the

Proof of Theorem 4.4. i) We use the so-called "reflected Hölder inequality"

$$\int_{S^{d-1}} f \cdot g \, d\omega \geqq \left(\int_{S^{d-1}} f^{-d} \, d\omega \right)^{-\frac{1}{d}} \cdot \left(\int_{S^{d-1}} g^{\frac{d}{d+1}} \, d\omega \right)^{\frac{d+1}{d}} \tag{64}$$

for positive and continuous functions f, g on S^{d-1} with equality if and only if

$$g^{\frac{d}{d+1}} = c \cdot f^{-d} \quad (c \text{ constant with } c > 0). \tag{65}$$

Then the application of (65), (50), (64), (59) and (57) together with the monotoneity of the mixed volume yields

$$\left(\int_{s^{d-1}} h_{\hat{K}}^{-d} \, d\omega \right)^{-\frac{1}{d}} \left(\int_{S^{d-1}} H_{d-1}^{-\frac{d}{d+1}} \, d\omega \right)^{\frac{d+1}{d}} = \int_{S^{d-1}} h_{\hat{K}} H_{d-1}^{-1} \, d\omega = d\, V(K, \ldots, K, \hat{K}) \geqq$$

$$d\, V(K', \ldots, K', \hat{K}) = \int_{S^{d-1}} h_{\hat{K}} H'^{-1}_{d-1} \, d\omega \geqq \left(\int_{S^{d-1}} h_{\hat{K}}^{-d} \, d\omega \right)^{-\frac{1}{d}} \left(\int_{S^{d-1}} H'^{-\frac{d}{d+1}}_{d-1} \, d\omega \right)^{\frac{d+1}{d}} \tag{66}$$

($K' := \operatorname{conv} F'$) whence by (60) indeed $A(F) \geqq A(F')$. Equality here implies equality in the second inequality of (66) whence by (65) and (50) $H'^{-1}_{d-1} = c^{\frac{d+1}{d}} h_{\hat{K}}^{-(d+1)} = c^{\frac{d+1}{d}} H_{d-1}^{-1}$ and then by the equality in the first inequality of (66) $c = 1$ and

$$H'_{d-1} = H_{d-1} \tag{67}$$

follows. But an old theorem of Minkowski says that (67) implies the relation $K' = K + a$ (a constant with $a \in \mathbb{R}_d$) and thus indeed $K' = K$ because of $K' \subseteqq K$.

For this proof of i) the assumption of the affine convexity of the smaller ovaloid F' may be omitted. Modifications of part i) of Theorem 4.4 may be found in [5].

ii) The second part of Theorem 4.4 may also be proved by reduction to the corresponding property of the mixed volume. Because of the validity of (63) we have only to show the lower semicontinuity of the functional A:

$$F = \lim_{\nu \to \infty} F_\nu \Rightarrow A(F) \leqq \liminf_{\nu \to \infty} A(F_\nu). \tag{68}$$

Instead of (66) we use for this purpose the inequality

$$d\, V(K, \ldots, K, L^*) = \int_{S^{d-1}} h_{L^*} H_{d-1}^{-1} \, d\omega \geqq$$

$$\left(\int_{S^{d-1}} h_{L^*}^{-d} \, d\omega \right)^{-\frac{1}{d}} \left(\int_{S^{d-1}} H_{d-1}^{-\frac{d}{d+1}} \, d\omega \right)^{\frac{d+1}{d}} = (d\, V(L))^{-\frac{1}{d}} A(F)^{\frac{d+1}{d}} = (d\, \kappa_d)^{-\frac{1}{d}} A(F)^{\frac{d+1}{d}} \tag{69}$$

for any convex body L in $\mathbb{R}_d$ whose centroid lies in the origin:

$$c(L) = 0 \tag{70}$$

and whose volume equals the volume of the d-dimensional unit ball:

$$V(L) = \kappa_d \tag{71}$$

(L^* polar body of L with respect to the origin). Equality holds in (69) for the body

$$L_K := \left(\frac{d\kappa_d}{A(F)}\right)^{\frac{1}{d}} \hat{K}^* \tag{72}$$

since by (50)

$$h_{L_K^*}^{-d} = \frac{d\kappa_d}{A(F)} h_{\hat{K}}^{-d} = \frac{d\kappa_d}{A(F)} H_{d-1}^{-\frac{d}{d+1}} \tag{73}$$

(see (65)), and by (73)

$$V(L_K) = \frac{1}{d}\int_{S^{d-1}} h_{L_K^*}^{-d}\, d\omega = \frac{\kappa_d}{A(F)}\int_{S^{d-1}} H_{d-1}^{-\frac{d}{d+1}}\, d\omega = \kappa_d. \tag{74}$$

Hereby we also have by (50) and a theorem of Minkowski

$$c(L_K) = \left(\frac{d\kappa_d}{A(F)}\right)^{\frac{1}{d}} \frac{1}{V(\hat{K}^*)}\int_{S^{d-1}} \frac{h_{\hat{K}}^{-(d+1)}}{d+1}(-n)\, d\omega = (\cdots)\int_{S^{d-1}} (-n) H_{d-1}^{-1}\, d\omega = 0. \tag{75}$$

Now let be $F = \lim_{\nu\to\infty} F_\nu$ (in the Hausdorff sense) with the convex bodies $K_\nu :=$ conv F_ν of elliptic type. Then we have likewise the equalities

$$d\,V(K_\nu, \dots, K_\nu, L_{K_\nu}^*) = (d\kappa_d)^{-\frac{1}{d}} A(F_\nu)^{\frac{d+1}{d}} \tag{76}$$

with

$$V(L_{K_\nu}) = \kappa_d,\;\; c(L_{K_\nu}) = 0 \quad (\nu = 1, 2, \dots) \tag{77}$$

as for K (see (69), (73), (74) and (75)). We consider the sequence $\{A(F_\nu)\}_{\nu\in\mathbb{N}}$ which has a convergent subsequence $\{A(F_{\nu'}\}_{\nu'\in\mathbb{N}}$ with

$$\liminf_{\nu\to\infty} A(F_\nu) = \lim_{\nu'\to\infty} A(F_{\nu'}). \tag{78}$$

The application of Blaschke's selection theorem to the bodies $L_{K_{\nu'}}$ which is possible because of the normalization (77) (for details see [6], proof of Lemma 6.3) yields the convergence of a subsequence of $\{L_{K_{\nu'}}\}_{\nu'\in\mathbb{N}}$ such that we may assume, without changing the notation,

$$\lim_{\nu'\to\infty} L_{K_{\nu'}}^* =: L_0^* \quad \text{with} \quad V(L_0) = \kappa_d,\; c(L_0) = 0. \tag{79}$$

Now we are passing to a limit for $\nu' \to \infty$ in the equation (76) from which we may conclude using (78), (79), (69) and the continuity of the mixed volume:

$$(d\,\kappa_d)^{-\frac{1}{d}}(\liminf_{\nu\to\infty}(A(F_\nu))^{\frac{d+1}{d}} = d\,V(\lim_{\nu'\to\infty} K_{\nu'}, \dots, \lim_{\nu'\to\infty} K_{\nu'}, \lim_{\nu'\to\infty} L_{K_{\nu'}}^*) =$$

$$= d\,V(K, \dots , K, L_0^*) \geqq (d\,\kappa_d)^{-\frac{1}{d}} A(F)^{\frac{d+1}{d}}$$

(compare (66)) which indeed equals the claim (68) of ii). □

We end with the final

Remark 4.5. i) Proposition 4.2 and its consequence, Theorem 4.4, may be proved exactly in the same way for affinely convex ovals C in $\mathbb{R}_2$ with the affine perimeter

$$L(C) = \int_C \kappa^{\frac{1}{3}}\, d\sigma = \int_{S^1} \kappa^{-\frac{2}{3}}\, d\omega \tag{80}$$

(see (8)).
ii) Theorem 4.4 is also valid for affinely convex parabola polygons $\mathcal{P}$ (see Definition 2.2) in $\mathbb{R}_2$ with the affine perimeter (80) since their enclosed convex domains $\mathcal{K}$ are of elliptic type (see Definition 4.3).

The reason of this behaviour is the fact that $\mathcal{P}$ possesses the positive and continuous curvature function ρ with

$$\rho\Big|_{\mathcal{P}_{l-1l}} = \frac{1}{\kappa^{(l-1l)}} \tag{81}$$

(see (6)) and with

$$<n(x_l), y^{(l-1l)}> \;=\; <n(x_l), y^{(ll+1)}> \;= \rho(x_l)^{-\frac{1}{3}} \quad (l = 1, \dots , k) \tag{82}$$

(see (56)) such that $\rho^{-\frac{1}{3}}$ must be the support function of the solid convex polygon

$$\hat{\mathcal{K}} := \text{ conv } \{-y^{(01)}, \dots , -y^{(k-1k)}\} \tag{83}$$

with the vertices $-y^{(01)}, \dots , -y^{(k-1k)}$, the endpoints of the negative affine normal vectors of the parabola arcs $\mathcal{P}_{01}, \dots , \mathcal{P}_{k-1k}$.

References

[1] Blaschke, W.: *Vorlesungen über Differentialgeometrie II.* Springer-Verlag, Berlin 1923.

[2] Böhmer, P.: *Über elliptisch-konvexe Ovale.* Math. Ann. **60** (1905), 256–262.

[3] Carleman, T.: *Sur les courbes paraboliquement convexes.* Vierteljschr. Naturf. Ges. Zürich **85** Beiblatt (1940), 61–63.

[4] Leichtweiß, K.: *Über einige Eigenschaften der Affinoberfläche beliebiger konvexer Körper.* Results Math. **13** (1988), 255–282. Zbl 0645.53004

[5] Leichtweiß, K.: *Bemerkungen zur Monotonie der Affinoberfläche von Eihyperflächen.* Math. Nachr. **147** (1990), 47–60. Zbl 0708.53009

[6] Leichtweiß, K.: *Affine Geometry of Convex Bodies.* Verlag Joh. Ambros. Barth, Heidelberg 1998. Zbl 0899.52005

[7] Leichtweiß, K.: *On the affine rectification of convex curves*. Beitr. Alg. Geom. **40** (1999), 185–193. Zbl 0154.9696

[8] Merza, J.: *Sur la courbure affine des courbes*. Publ. Math. Debrecen **7** (1960), 65–71. Zbl 0096.36801

[9] Mohrmann. H.: *Über beständig elliptisch, parabolisch oder hyperbolisch gekrümmte Kurven*. Math. Ann. **72** (1912), 285–291.

[10] Petty, C.: *Geominimal surface area*. Geom. Dedicata **3** (1974), 77–97. Zbl 0285.52006

[11] Reidemeister, K.: *Über affine Geometrie XXXI*: *Beständig elliptisch oder hyperbolisch gekrümmte Eilinien*. Math. Z. **10** (1921), 318–320.

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