Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 42 (2001), No. 2, 547-555.

On the Permutation Products of Manifolds

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Abstract. In this paper it is proven the following conjecture: If G is a subgroup of the permutation group S_n and M is a 2-dimensional real manifold, then M^n/G is a manifold if and only if $G = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_r}$ where $S_{m_1}, \ldots, S_{m_r}$ are permutation groups of partition of $\{1, 2, \ldots, n\}$ into r subsets with cardinalities $m_1, \ldots, m_r$, and M^n is the topological product of n copies of M.

MSC 2000: 53C99, 20B35
Keywords: permutation products on manifolds, cyclic products on manifolds, permutation group

1. Introduction

Let M be a nonvoid set and let n be a positive integer. In the Cartesian product M^n we define a relation $\approx$ such that $(x_1, \ldots, x_n) \approx (y_1, \ldots, y_n)$ if there exists a permutation $\vartheta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $y_i = x_{\vartheta(i)} \quad (1 \leq i \leq n)$. This is an equivalence relation. The class represented by $(x_1, \ldots, x_n)$ will be denoted by $(x_1, \ldots, x_n)/\approx$ and the set $M^n/\approx$ will be denoted by $M^{(n)}$. The set $M^{(n)}$ is called a *permutation product* of M and it was mainly studied in [3].

If M is a topological space, then $M^{(n)}$ is also a topological space. The space $(R^m)^{(n)}$ $(n \geq 2)$ is a manifold only for $m = 2$. If $m = 2$, then $(R^2)^{(n)} = C^{(n)}$ is homeomorphic to C^m. Indeed, using the fact that the field C is algebraically closed, the mapping $\varphi : C^{(n)} \to C^n$ defined by

$$\varphi((z_1, \ldots, z_n)/\approx) = (\sigma_1, \sigma_2, \ldots, \sigma_n)$$

is a bijection, where $\sigma_i (1 \leq i \leq n)$ is the i-th symmetric function of $z_1, \ldots, z_n$. The mapping φ is also a homeomorphism. Moreover, if M is a 2-dimensional manifold, then $M^{(n)}$ is a manifold. In [1] it is proven that if M is orientable, i.e. if M is a 1-dimensional complex manifold, then $M^{(n)}$ is a complex manifold. If $dim M \neq 2$, then $M^{(n)}$ is not a manifold.

0138-4821/93 \$ 2.50

Now let $dimM = 2$ and let us consider a subgroup G of the permutation group S_n. Then define a relation $\approx$ in M^n by

$$(x_1, x_2, \ldots, x_n) \approx (x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(n)})$$

if and only if $\tau \in G$. The factor space $M^n/\approx$ will be denoted by M^n/G. In [2] it was given the following conjecture which is true for any $n \leq 4$.

Conjecture. *Let $G \leq S_n$ and M be a 2-dimensional real manifold. Then M^n/G is a manifold if and only if $G = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_r}$ where $S_{m_1}, \ldots, S_{m_r}$ are permutation groups of a partition of $\{1, 2, \ldots, n\}$ on r subsets with cardinalities $m_1, \ldots, m_r$.*

In this paper the conjecture will be proved. We remark that in the special case when G is the cyclic group, then M^n/G is called a *cyclic product* of M and denoted by $M^{[n]}$. The cyclic product is not a manifold if $n > 2$ [3] while if $n = 2$, then it is a manifold which is identical to $M^{(2)}$. Note also that if $dimM \geq 3$, then M^n/G is not a manifold whenever G is a cyclic group [3].

2. Proof of the conjecture

Let G be a subgroup of the permutation group S_n of the set of n elements $\{a_1, a_2, \ldots, a_n\}$ which is not of type $S_{m_1} \times S_{m_2} \times \cdots \times S_{m_k}$ and let M^n/G be the corresponding factor-space. We shall prove that M^n/G is not a manifold.

The group G defines a partition of the set $\{a_1, a_2, \ldots, a_n\}$ as follows: if a is an element of $\{a_1, a_2, \ldots, a_n\}$ then we define

$$T_a = \{f(a)|\ f \text{ is a permutation of } G\}.$$

Of course, we have that T_a and T_b are either nonintersecting or equal. Hence they give a partition of the set $\{a_1, a_2, \ldots, a_n\}$ as claimed.

For these sets T_a we define subgroups

$$G_a = \{f|f \in G \text{ and } f(x) = x \text{ for any } x \notin T_a\}$$

and

$$G'_a = \{f|f \in G \text{ and } f(T_a) = T_a\}.$$

It is obvious that $G'_a = G$.

Now we will define some notions and will prove some of their properties.

Definition 1. *The cycle of a_j with respect to a given permutation f is the finite set $\{f^s(a_j)|s = 1, 2, \ldots\}$ which is denoted by $C(a_j, f)$. The degree of the cycle is the number of its elements.*

Each element of the set $\{a_1, a_2, \ldots, a_n\}$ appears in exactly one cycle with respect to f and any two cycles are nonintersecting or they coincide. If the number of elements of the cycle is 1, then it is called trivial.

Property 1. *If p is a prime divisor of the degree of the cycle $C(a,f)$, i.e. if $|C(a,f)| = p^\alpha m$ such that $(p,m)=1$, then the cycle contains subcycles of degrees p^i for $i=1,2,\dots,\alpha-1$.*

Proof. If $s = p^{\alpha-i}m$, then the cycle $C(a,f^s)$ is a subcycle of degree p^i of $C(a,f)$.

Property 2. *If $C(a,f)$ and $C(b,f)$ are two nonintersecting cycles with respect to f with degrees p and q, $(p,q)=1$, then there exists $g \in G$ such that $C(a,g) = C(a,f)$, $g = f^i$ for some $i \in N$ and $C(b,g) = \{b\}$ is trivial.*

Proof. If we put $g = f^q$, then it is easy to verify that the previous requirements are satisfied.

Property 3. *If the degree of $C(a_i,f)$ is $q > 2$ and there exists a permutation h of G_{a_i} such that $h(c) = d$, $h(d) = c$ and $h(x) = x$ for x different from c and d, where $c, d \in C(a_i,f)$ – (in other words, h is the transposition (cd) (thereby there exists a positive integer p such that $f^p(c) = d$)) – and if $(p,q) = 1$, then $\{g|_{C(a_i,f)} : g \in G_{a_i}$ and $g(C(a_i,f)) \subseteq C(a_i,f)\} = \{g|_{C(a_i,f)} : g \in G_{a_i}\}$ and this is the group of all permutations of the cycle $C(a_i,f)$.*

Proof. It is sufficient to prove that for each pair of two elements u and v of $C(a_i,f)$ there exists h_1 of G_{a_i} such that $h_1(u) = v$ and $h_1(v) = u$ and for any other element x of $C(a_i,f)$, $h_1(x) = x$. It is sufficient to prove that this subgroup contains all transpositions in order to be the group of all permutations. If $g = f^p$, then $C(a_i,f) = C(a_i,g)$. There exist $n_1, n_2 \in Z_q$, $n_1 \neq n_2$, such that $g^{n_1}(u) = d$ and $g^{n_2}(v) = d$. We can assume that $n_1 < n_2$ since the opposite case can be discussed analogously. Let us define $g_1 = (g \circ h)^{n_2 - n_1 - 1} \circ g^{n_1}$. Then $g_1(v) = d$, $g_1(u) = c$ and for any x different from u and v it holds $g_1(x) = x$. Let us define $h_1 = g_1^{q-1} \circ h \circ g_1 = g_1^{-1} \circ h \circ g_1$, where the second equality holds because $g_1^q = id$. Thus we obtain

$$h_1(u) = g_1^{-1}(h(g_1(u))) = g_1^{-1}(h(c)) = v, \quad h_1(v) = g_1^{-1}(h(g_1(v))) = g_1^{-1}(h(d)) = u$$

and

$$h_1(x) = g_1^{-1}(h(g_1(x))) = g_1^{-1}(g_1(x)) = x$$

for any x different from u and v.

Property 4. *If the degree q $(q > 2)$ of $C(a_i,f)$ is prime number and if there exists a permutation h of G_{a_i} such that $h(c) = d$, $h(d) = c$ and $h(x) = x$ for x different from c and d, where $c, d \in C(a_i,f)$, then $\{g|_{C(a_i,f)} : g \in G_{a_i}\}$ is the group of all permutations.*

Proof. There exists $p \in N$, $p < q$, such that $f^p(u) = v$. Because q is a prime and $p < q$, it follows that $(p,q) = 1$. Now the proof follows from Property 3.

Property 4'. *If the degree of $C(a_i,f)$ is q and if there exists a permutation h of G_{a_i} such that $h(c) = d$, $h(d) = c$ and $h(x) = x$ for x different from c and d where $c, d \in C(a_i,f)$ and* *if $f(c) = d$ where c and d are neighbor with respect to the cycle $C(a_i, f)$, then $\{g|_{C(a_i,f)} : g \in G_{a_i}\}$ is the group of all permutations on $C(a_i, f)$.*

The proof is analogous to that of Property 4. In this case it is $p = 1$. As a consequence of Properties 3, 4 and 4', we obtain:

Property 5. *If $C(a, f)$ is a cycle with respect to f of degree $p > 2$ and the transposition $(uv) \in G$ for $u, v \in C(a, f)$ and $(p, s) = p' > 1$, where s is positive integer such that $f^s(u) = v$, then $\{g|_{C(a_i,f)} : g \in G_{a_i}\}$ contains all bijections of the subcycles $C(a_i, f^{p/p'})$, $a_i \in C(a, f)$, and those obtained by the previous ones by cyclic permutation with f. Furthermore, we have*

$$\{g|C(a_i, f^{p/p'}) : g \in G_{a_i}\} = \{g|C(a_i, f^{p/p'}) : g \in G_{a_i},\ \ g(C(a_i, f^{p/p'})) \subseteq C(a_i, f^{p/p'})\}.$$

Proof. The row of the cycle $C(a_i, f^{p/p'})$ is equal to $p/p' = q$. Since $(s, q) = 1$ and using Property 3, it follows that

$$\{g|_{C(a_i,f^{p/p'})} : g \in G_{a_i}\} = \{g|_{C(a_i,f^{p/p'})} : g \in G_{a_i},\ \ g(C(a_i, f^{p/p'})) \subseteq C(a_i, f^{p/p'})\}$$

which is the group of all permutations of the cycle (subcycle) $C(a_i, f^{p/p'})$. Each bijection from $\{g|_{C(a_i,f^{p/p'})} : g \in G_{a_i}\}$ belongs also to $\{g|_{C(a_i,f)} : g \in G_{a_i}\}$ and hence it follows that $\{g|_{C(a_i,f)} : g \in G_{a_i}\}$ contains all the bijections of the set of elements of the cycle $C(a_i, f^{p/p'})$, which should be proven.

Note that it may happen to exist f such that $C(x, g) = C(x, f)$ but $f^s \neq g$ for any s.

Definition 2. *If $C(x, g) = C(x, f)$ for any x and there exists s such that $f^s|_{C(x,g)} = g|_{C(x,g)}$, then we say that f and g act similarly on their joint cycle $C(x, g)$.*

The following question appears naturally. If the degree of the cycle of x with respect to g is minimal as in Property 6, is it possible to find f of G which does not act similarly with g to the cycle $C(x, g)$? Indeed, the following property holds.

Property 6. *If $C(x, f)$ is a cycle which contains x, with the smallest possible degree q, i.e. it does not exist a cycle $C(x, g)$ for $g \in G$ of degree smaller than q and bigger than 2, and if $C(x, f) = C(x, g)$, then there exists a positive integer s such that $f^s|_{C(x,f)} = g|_{C(x,f)}$, i.e. f and g act similarly on the cycle $C(x, f)$.*

Remark. If $q = 2$, then it should be $g = f$. In this case q is a prime number.

Proof. Suppose that g acts on $\{1, 2, \dots, q\}$ cyclically. Without loss of generality we can suppose that $x = 1$ and $y = 2$, i.e. $g(i) = i+1$ for $i < q$ and $g(q) = 1$. Let $C(1, f) = C(1, g)$ and assume that f and g do not act similarly on this common cycle, i.e. there is no number s such that $f^s = g$. Let j be the smallest number from $\{1, 2, \dots, q\}$ such that $f(j) > g(j)$, i.e. $f(1) = g(1)$, $f(2) = g(2)$, ..., $f(j-1) = g(j-1)$, and $f(j) > g(j)$. Therefore it holds $g^{q-f(j)}(j) = 1$ and $q > f(j) - g(j)$. Let $h = g^{q-f(j)} \circ f^{j-1}$. Then it is very easy to verify that $h(1) = 1$. Indeed, $h(1) = g^{q-f(j)} \circ f^{j-1}(1) = g^{q-f(j)} \circ g^{j-1}(1) = g^{q-f(j)}(j) = 1$. Here we have used that $f^{j-1}(1) = j = g^{j-1}(1)$. From the fact that there is no s such that $f^s = g$, it follows that h is not identity mapping on $\{1, 2, \ldots, q\}$. This means that there exists k such that $h(k) \neq k$. Hence the cycle $C(k, h)$ has degree smaller than q. This is a contradiction to the choice of q.

Property 6'. *If $C(a_i, f) = C(a_i, g)$, $i = 1, 2, \ldots, s$, are cycles of prime order q with the property $C(a_i, f) \cap C(a_i, g) = \emptyset$ for $i \neq j$, such that f and g act similarly over each cycle and if $f(x) = g(x) = x$ for any $x \notin C(a_i, f)$, $i = 1, \ldots, s$, then*

$$M^n/{<}f{>} = M^n/{<}g{>} = M^n/{<}f, g{>} \cong M^{n-qs} \times (M^s)^{(q)}$$

where $(M^s)^{(q)}$ denotes the q-th cyclic product on M^s.

Proof. Since f and g act similarly, it follows that f can be generated from g and conversely, i.e. it holds

$${<}f{>} = {<}g{>} = {<}f, g{>} \cong Z_q.$$

If we have in mind how f and g act on M^n, we just obtain that

$$M^n/{<}f{>} = M^n/{<}g{>} \cong M^{n-qs} \times (M^s)^{(q)}.$$

Property 7. *If $C(a, f)$ is a cycle with respect to f with degree m, the transpositions (u_1v_1) and (u_2v_2) belong to G such that $u_1, v_1, u_2, v_2 \in C(a, f)$, and s_1, s_2 are divisors of m such that $f^{s_1}(u_1) = v_1$, $f^{s_2}(u_2) = v_2$, then the following properties are satisfied:*

1. *If $s_1 = s_2$, then ${<}f|_{C(a,f)}, (u_1v_1){>} = {<}f|_{C(a,f)}, (u_2v_2){>}$.*
2. *If $(s_1, s_2) = 1$, then ${<}f|_{C(a,f)}, (u_1v_1), (u_2v_2){>} \cong S_m$.*
3. *If $(s_1, s_2) = s_3 > 1$, then ${<}f|_{C(a,f)}, (u_1v_1), (u_2v_2){>} \cong {<}f|_{C(a,f)}, (u_3v_3){>}$ where u_3, v_3 are elements of the cycle $C(a, f)$ such that $f^{s_3}(u_3) = v_3$ and s_3 is a divisor of m.*

Proof. 1. Using the fact that there exist $t, s \in N$ such that $f^t(u_1) = u_2$, $f^t(v_1) = v_2$ and $f^s(u_1) = v_1$, we obtain that $(u_2v_2) = f^t \circ (u_1v_1)$. From the last equality it follows that ${<}f|_{C(a,f)}, (u_1v_1){>} = {<}f|_{C(a,f)}, (u_2v_2){>}$.
2. Under the previous assumptions there exist n_1 and $n_2 \in Z_m$ such that $n_2s_2 = n_1s_1 + 1$ and hence $f^{n_2s_2}(a) = f(f^{n_1s_1}(a))$, i.e. the points $f^{n_2s_2}(a)$ and $f^{n_1s_1}(a)$ are neighbors in the cycle $C(a, f)$. Moreover we will prove that the transpositions $(af^{n_2s_2}(a))$ and $(af^{n_1s_1}(a))$ belong to G. Indeed, there exists s such that $f^s(a) = u_1$, $f^{s_1}(f^s(a)) = f^{s_1}(u_1) = v_1 = f^s(f^{s_1}(a))$ and hence we obtain that $(u_1v_1) = f^s \circ (af^{s_1}(a)) \in G$. Since G is a group, it follows that $(af^{s_1}(a)) \in G$. Note that the point a is an arbitrary point from the cycle $C(a, f)$ and hence $(f^t(a)f^t(f^{s_1}(a))) \in G$. Finally, we obtain

$$\begin{aligned}(af^{n_1s_1}(a)) &= (af^{s_1}(a)) \circ (f^{s_1}(a)f^{2s_1}(a)) \circ (f^{2s_1}(a)f^{3s_1}(a)) \circ \cdots \\ &\cdots \circ (f^{(n_1-1)s_1}(a)f^{n_1s_1}(a)) \circ (f^{(n_1-1)s_1}(a)f^{(n_1-2)s_1}(a)) \circ \cdots \\ &\cdots \circ (f^{2s_1}(a)f^{s_1}(a)) \circ (f^{s_1}(a)a) \in G.\end{aligned}$$

Analogously, it verifies that $(af^{n_2s_2}(a)) \in G$. Thus we have

$$(f^{n_2s_2}(a)f^{n_1s_1}(a)) = (af^{n_2s_2}(a))(af^{n_1s_1}(a))(af^{n_2s_2}(a)) \in G.$$

Now using the Property 4', we obtain that each bijection of the cycle $C(a,f)$ is generated. This completes the proof of 2.

3. Analogously there exist n_1 and $n_2 \in Z_m$ such that $n_1s_1 + s_3 = n_2s_2$ and hence it follows that

$$f^{n_2s_2}(a) = f^{s_3}(f^{n_1s_1}(a)).$$

Suppose that $f^{n_2s_2}(a) = a_2$ and $f^{n_1s_1}(a) = a_1$. The points a_1 and a_2 belong to $C(a,f)$, hence we have

$$(a_1a_2) = (aa_1) \circ (a_1a_2) \circ (aa_2) \in G$$

and $f^{s_3}(a_2) = a_1$. The transpositions $(af^{s_2}(a))$ and $(af^{s_1}(a))$ can be obtained by composing transpositions of the form $(bf^{s_3}(b))$. Moreover, for the transposition $(af^{s_2}(a))$ it holds $s_2 = ps_3$ for a positive integer p. Hence it holds that

$$\begin{aligned}(af^{s_2}(a)) = (af^{s_3}(a)) \circ (f^{s_3}(a)f^{2s_3}(a)) \circ \cdots \circ (f^{(p-1)s_3}(a)f^{ps_3}(a)) \circ \\ \circ (f^{(p-1)s_3}(a)f^{(p-2)s_3}(a)) \circ (f^{(p-2)s_3}(a)f^{(p-3)s_3}(a)) \circ \cdots \circ (f^{s_3}(a)a)\end{aligned}$$

and analogously for $(af^{s_1}(a))$. This means that each bijection which can be generated by $f|_{C(a,f)}$, $(af^{s_2}(a))$ and $(af^{s_1}(a))$ can also be generated only by $f|_{C(a,f)}$ and $(af^{s_3}(a))$. The converse holds, too. From $(af^{s_2}(a)) \in G$ and from the fact that a and $f^{s_i}(a)$ are neighbors with respect to the cycle $C(a,f^{s_i})$ according to Property 4', it follows that each bijection restricted on the cycle $C(a,f^{s_i})$ belongs to G, and hence also the transposition $(af^{n_is_i}(a))$ belongs to G for $i = 1,2$. By substituting $f^{n_1s_1}(a)$ instead of a we get

$$\begin{gathered}(f^{n_1s_1}(a)f^{n_2s_2}(f^{n_1s_1}(a))) \in G, \\ (f^{n_1s_1}(a)f^{s_3}(a)) \in G, \\ (a, f^{s_3}(a)) = (af^{n_1s_1}(a)) \circ (f^{n_1s_1}(a)f^{s_3}(a)) \circ (af^{n_1s_1}(a)) \in G\end{gathered}$$

and hence 3. is proven.

Remark. From Property 7 we obtain the following conclusion. If G restricted on the cycle $C(a,f)$ does not contain all its bijections, then all bijections restricted on the cycle $C(a,f)$ obtained by G, can be obtained by f and the transposition (uv) (if such exists) for $u,v \in C(a,f)$ such that $s|p$ and $(s,p) > 1$ is the smallest one with that property, where s is the smallest positive integer which satisfies $f^s(u) = v$.

Now let us return to the proof of the conjecture. Note that G acts transitively on T_a for any $a \in \{a_1,\ldots,a_n\}$. Moreover, since G is not a group of the form $S_{m_1} \times S_{m_2} \times \cdots \times S_{m_k}$, at least one of the subgroups G_a or G''_a is not of the form S_{m_i}, where $G''_a = G|_{T_a} = \{f|_{T_a} : f \in G\}$. Thus we have two cases:

1. There exists g such that there are at least two nonintersecting minimal cycles $C(a,g)$ and $C(b,g)$ with the same degree q (here "minimal" is meant in the sense of Property 6 even if the degree can be 2 in this case), on which g acts simultaneously, and thereby there is no $h \in G$ such that $h(x) = x$ for any $x \notin C(a,g)$, and $h(x) \neq x$ for any $x \in C(a,g)$. Indeed, h moves only the points of the cycle $C(a,g)$ and thereby the cycle $C(a,h)$ is a subcycle of $C(a,g)$.

2. There is no $g \in G$ with the previous property.

In both cases we will prove that M^n/G is not a manifold, where M is a 2-dimensional manifold.

Case 1.

Assume that the degree of the cycles $C(a,g)$ and $C(b,g)$ is 2. There exist u, v, x, y such that the composition $(uv)(xy)$ enters in the decomposition of g but the transpositions (uv) and (xy) are not elements of G. Let g be chosen such that the number of its cycles of degree 2 over which g acts simultaneously is $r > 1$. The number r will be called *pairwise degree* of g. Let g be chosen with the smallest possible value of r.

If we choose a point $x = (x_1, x_2, \dots, x_n) \in M^n$ such that the coordinates corresponding to the same cycle of degree 2 of g are equal, i.e. $x_{g(i)} = x_i$ for all indices i with the property $g^2(i) = i$, then the points of different cycles are different and the remaining points are completely different. Here g is a bijection on the index set $\{1, 2, \dots, n\}$.

The set $G^g = \{f \in G|\ f$ acts invariantly on the point $x\} = \{f \in G|f$ acts invariantly on any cycle of $g\}$ is a group. Therefore, the minimality of r implies that if g and f have the same pairwise degree r, then $f = g$. Hence we obtain that $G^g = \{id, g\}$.

Now we note that the tangent space at the point corresponding to x in the factor space is homeomorphic to

$$(R^2)^n/G^g \cong ((R^2)^r)^{[2]} \times R^2 \times \dots \times R^2$$

which is not homeomorphic to R^{2n} and hence the space M^n/G is not a manifold.

If the cycles $C(a,g)$ and $C(b,g)$ have degree $q > 2$ with the previous property and if we have in mind Properties 1 and 2, then $C(a,g)$ and $C(b,g)$ can be chosen such that q is a prime number. Further, from Property 4 we can conclude that there is no subcycle of degree 2, i.e. transposition for none of the previous cycles. Otherwise, Property 4 would imply that any bijection on the corresponding cycle could be obtained and hence g would not satisfy the conditions from 1, i.e. there does exist h as in 1.

Further, let us assume that g is chosen such that there exists the smallest possible number $r > 1$ for nonintersecting cycles on which g acts simultaneously as above.

We choose a point $x = (x_1, x_2, \dots, x_n) \in M^n$ such that all coordinates corresponding to the same cycle of the previous r cycles of g are equal. There the points of different cycles are different and the remaining points are completely different. In this case the tangent space at the point $x^{\approx} \in M^n/G$ which corresponds to x is homeomorphic to $(R^2)^n/G^g$, where $G^g = \{f|f(C(x',g)) = C(x',g)$ for any cycle $C(x',g)$ of $g\}$. By the minimality of q and r and from Property 6, it follows that $G^g = \langle g \rangle = Z_r$, since any $f \in G^g$ acts similarly to g on any cycle of g, i.e. over any such cycle $f^s = g$ restricted on it, for some positive integer s. Thus we get that

$$(R^2)^n/G^g \cong ((R^2)^r)^{[q]} \times R^2 \times \dots \times R^2$$

where R^2 appears $n - qr$ times. This space is not homeomorphic to R^{2n} and thus we obtain that M^n/G is not a manifold.

Case 2.
In this case for any cycle $C(x,g)$ there exists $f \in G$ such that $C(x,g) = C(x,f)$ and thereby $f(y) = y$ for any $y \notin C(x,f)$, i.e. any cycle can be considered separately and the cycle $C(x,f)$ is called unical. Because of this argument we obtain that $G_a = G$ for any a. Further we will need the following property.

Property 8. *Under the assumptions in case 2, if for any unical cycle obtained from G any bijection on that cycle is contained in G, then* $G_a = G \cong S_{m_i}$*, where* $m_i = |T_a|$*.*

Proof. Since G acts transitively on T_a, it follows that for any $u, v \in T_a$ there exist f and $g \in G$ such that $g(u) = v$ and $C(u,g) = C(u,f)$, and thereby the cycle $C(u,f)$ is unical. But since $v \in C(u,f)$, there exists a positive integer s such that $f^s(u) = v$. According to the assumption that any bijection on the cycle $C(a,f)$ belongs to G, we obtain that also the transposition (uv) belongs to G. Since u and v are arbitrary elements of the set T_a, it follows that any bijection in T_a can be generated because f acts trivially on the elements not in T_a.

Thus there exists at least one cycle $C(a,f)$ as above for which not all transpositions (uv); $u, v \in C(a,f)$, belong to G. We choose $G(a,f)$ with degree $p > 2$ which is minimal with this property. Note that in this case p may not be prime. According to Properties 3 and 5, there are two possibilities:

a) The cycle $C(a,f)$ does not contain u, v such that the transposition (uv) belongs to G.

b) Suppose the transposition (uv) belongs to G for some $u, v \in C(a,f)$. If s is the smallest positive integer such that $f^s(u) = v$, then $(p,s) = p_1 > 1$.

We consider now both of these possibilities.

a) From the minimality of the degree of the cycle $C(a,f)$ and from Property 6, we obtain that for any $g \in G$ such that $C(a,f) = C(a,g)$, there is a positive integer s such that $g = f^s$, i.e. f and g act similarly on $C(a,f)$.

We choose a point $x = (x_1, \ldots, x_n) \in M^n$ such that all coordinates corresponding to the previous cycle of f are equal. There the points of different cycles are different and the remaining points are completely different.

In this case the tangent space at the point $x^{\approx} \in M^n/G$ which corresponds to x is homeomorphic to $(R^2)^n/G^f$, where $G^f = \{g | g(C(a,f)) = C(a,f)\}$. Using Property 6 and the minimality of p, it follows that $G^f = \langle f \rangle = Z_p$, which implies that M^n/G^f is homeomorphic to $(R^2)^{[p]} \times R^2 \times \cdots \times R^2$. Since this space is not homeomorphic to R^{2n} for $p > 2$, the factor space M^n/G is not a manifold.

b) In this case since G is not of the form $S_{m_1} \times S_{m_2} \times \cdots \times S_{m_k}$ it follows that there exists a cycle $C(a,f)$ such that f contains only the cycle $C(a,f)$ as non-trivial and thereby G does not contain all permutations of that cycle. According to Property 7, there exists a transposition $(uv) \in G$, $u, v \in C(a,f)$ such that for any $g \in G$ such that $g(C(a,f)) = C(a,f)$ is generated by f and (uv), and thereby $f^s(u) = v$, for which $(s,m) = p_1 > 1$ and $p_1 < p$. Thus $G|_{C(a,f)}$ is isomorphic to $\langle f, (uv) \rangle$.

Thus we consider a point $x = (x_1, \dots, x_n) \in M^n$ where the coordinates corresponding to the points of the cycle $C(a, f)$ are equal. There the points of different cycles are different and the remaining points are completely different. Then the tangent space over the corresponding point of M^n/G is homeomorphic to $(R^2)^n/G^f$, where $G^f = \{g \in G | g(C(a, f)) = C(a, f)\} \cong G|_{C(a,f)} \cong <f, (uv)>$. But $(R^2)^n/G^f$ is homeomorphic to $((R^2)^{(p_1)})^{[p/p_1]} \times R^2 \times \cdots \times R^2$. Since $p_1 > 1$ and $p/p_1 > 1$, this space is not a manifold. Thus M^n/G is not a manifold.

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Received October 25, 1999; revised version September 12, 2000