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Actions of Hopf Algebras on Fully Bounded Noetherian Rings

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Abstract. Let k be a commutative ring, H a finitely generated projective Hopf algebra over k and R a k-algebra which is a left H-module algebra. Assume that for every H-invariant left ideal I of R and every $x + I \in (R/I)^H$ there exists $s \in R^H$, such that $s - x \in I$. The main result of the paper is that R is left FBN if and only if R is left Noetherian and R^H is left FBN. This result generalizes [4, Theorem 8] and [6, Theorem 2.3].

0. Introduction

A ring A is left bounded if every essential left ideal of A contains a nonzero two-sided ideal. The ring A is left fully bounded if for every prime ideal P of A, A/P is left bounded. We say that A is left FBN if it is left Noetherian and left fully bounded. The best known class of left FBN rings are left Noetherian P.I. rings. Right FBN rings are defined in a symmetric fashion.

Let k be a field, G a finite group and R an associative unitary k-algebra which is also a right G-module. Assume that the following condition holds:

($\star$) For every G-invariant right ideal I of R and every $x + I \in (R/I)^G$, there exists $r \in R^G$, such that $r - x \in I$.

Then J. J. Garcia and A. Del Rio [6, Theorem 2.3] have shown that R is right FBN if and only if R is right Noetherian and R^G is right FBN.

If there exists an $r \in R$ such that $tr(r) = 1$ (this is the case if $|G|$, the order of G is invertible in R), then condition ($\star$) holds. So [6, Theorem 2.3] gives a positive answer to Fisher and Osterburg's question for right Noetherian rings [4, Question 7 page 367].

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Let k be a commutative ring, H a finitely generated projective Hopf algebra over k and R a right Noetherian left H-module algebra with an element of trace 1. Then Dăscălescu, Kelarev and Torrecillas proved that R is right FBN if and only if the subalgebra of invariants R^H is right FBN [4, Theorem 8]. This result generalizes partially [6, Theorem 2.3], since an example in [6] shows that condition ($\star$) doesn't imply that R has an element of trace 1. Our aim is to generalize [4, Theorem 8] and [6, Theorem 2.3] in the case where the action comes from a finitely generated projective Hopf algebra over k.

Throughout the paper, k is a commutative ring, H is a Hopf k-algebra with comultiplication Δ, counit ϵ, and antipode s and R is an H-module algebra, i.e. an associative unitary k-algebra which is also a left H-module such that $h.(ab) = \sum_h (h_1.a)(h_2.b)$ for all $h \in H$ and $a, b \in R$. We denote by $R\#H$ the associated smash product. The expression rh means $r\#h$. The multiplication in $R\#H$ is defined by the rule $(ah)(bg) = \sum_h a(h_1.b)(h_2g)$.

The group algebra kG of a finite group G is a finite-dimensional cocommutative Hopf algebra and $R\#kG$ is the usual skew group algebra $R\#G$.

For further informations about Hopf algebras and the ring $R\#H$, the reader is referred to [1, 8, 13].

In the remainder of the paper, all modules are left modules. An R-module M which is an H-module such that $h.(am) = \sum_h (h_1.a)(h_2.m)$ is an $R\#H$-module. Conversely, if M is an $R\#H$-module, M may be thought of as an R-module with an action of H such that the above formula holds. It is clear that R is an $R\#H$-module defined by $(ah).b = a(h.b); a, b \in R$. If M is an H-module, denote by $M^H = \{m \in M \mid h.m = \epsilon(h)m \quad \forall h \in H\}$ the subspace of invariant elements of M. Clearly, R^H is a subring of R called the fixed subring of R (or the subring of invariants of R). The elements of R^H commute with H. If P is an $R\#H$-module, P^H is an R^H-module with trivial H-action.

From now on, H is a finitely-generated projective k-module. Let us denote by $x_1, x_2, \ldots, x_n$ a generator set for H. We know from [7, Proposition 1.1] that H has a nonzero left integral and that the antipode s is a bijective antimorphism of algebras and an antimorphism of coalgebras. Also, $R\#H$ is finitely generated R-free module with generators $x_1, x_2, \ldots, x_n$. If R is left Noetherian, then clearly so is $R\#H$.

The main result of this article states that if for every H-invariant left ideal I of R and every $x + I \in (R/I)^H$ there exists $s \in R^H$ such that $s - x \in I$, then R is left FBN if and only if R is left Noetherian and R^H is left FBN. The main tool to prove this result is the basic fact that R has a canonical structure of $R\#H$-module such that $Hom_{R\#H}(R, R)$ is isomorphic to R^H. We use the same techniques as in [6].

1. Preliminary results

We recall briefly some basic definitions. Let A be a ring, P and M two A-modules. We say that M is

- *finitely P-generated* if there exists an epimorphism $P^{(I)} \to M$ for some finite set I;
- *P-faithful* if $Hom_A(P, M') \neq 0$, for every nonzero submodule M' of M.

If M is finitely generated, clearly M is finitely A-generated.

For every subset X of M (resp. of $Hom_A(P, M)$), we set

$$l_A(X) = \{a \in A \mid am = 0 \text{ for all } m \in M\} \quad (\text{resp. } \ l_P(X) = \cap_{f \in X} Ker f).$$

Let A be a ring. An A-module M is said to be *quasi-projective* if for every submodule N of M and every homomorphism $f : M \to M/N$ there is an endomorphism $g : M \to M$ such that $p \circ g = f$ where $p : M \to M/N$ is the canonical epimorphism.

If R is finitely generated as R^H-module and if M is a finitely generated R-module, then M is a finitely generated R-faithful R^H-module.

A subset I of R is *H-invariant* if $H.I \subseteq I$. Clearly, the H-invariant left ideals of R are just the $R\#H$-submodules of R. If I is an H-invariant two-sided ideal of R, then R/I is an H-module algebra.

The left integral space of H is defined by

$$\int_H = \{t \in H \mid ht = \epsilon(h)t \ \text{ for all } h \in H\}.$$

We always fix an element $0 \neq t \in \int_H$. Let M be an $R\#H$-module. If $m \in M$, the H-submodule Hm of M is a finitely generated k-submodule of M containing m. More precisely, Hm is generated over k by the $x_i m$.

Lemma 1.1. *An $R\#H$-module is finitely generated as $R\#H$-module if and only if it is finitely generated as R-module.*

Proof. Let M be an $R\#H$-module finitely generated as $R\#H$-module. For every $m \in M$, $(R\#H)m = R(Hm) = \sum R(x_i m)$. So M is generated as R-module by the $x_i m_j$; $1 \leq i \leq n$, $1 \leq j \leq l$; where $m_1, m_2, \ldots, m_l \in M$ is a generator set for M as $R\#H$-module. □

The following lemma is the analogue of Năstăsescu and Dăscălescu's result [9] used in the proof of [6, Theorem 2.3].

Lemma 1.2. *If R is left FBN, then so is $R\#H$.*

Proof. By Lemma 1.1, $R\#H$ is finitely R-generated. By [6, Corollary 1.9], R is an FBN left R-module. Let M be a finitely generated $R\#H$-module. Then M is a finitely generated $R\#H$-faithful R-module. Consider the subset $\mathcal{M} = Hom_{R\#H}(R\#H, M)$ of $Hom_R(R\#H, M)$. By [6, Corollary 1.8], there exists a finite subset F of $\mathcal{M}$ such that $l_{R\#H}(\mathcal{M}) = l_{R\#H}(F)$. Since $R\#H$ is left Noetherian, the result follows from [6, Theorem 1.2]. □

2. The main results

We continue with the preceding notations. The map $\tilde{t} : R \to R$ given by $\tilde{t}(r) = t.r$ is an R^H-bimodule morphism with values in R^H. Consider the following two conditions:

(C_1) For every H-invariant left ideal I of R and every $x + I \in (R/I)^H$, there exists $s \in R^H$, such that $s - x \in I$.

(C_2) There exists an $r \in R$, such that $\tilde{t}(r) = 1$.

Lemma 2.1. $(C_2) \Rightarrow (C_1)$.

Proof. Let $r \in R$ such that $\tilde{t}(r) = 1$, I be an H-invariant left ideal of R and $x + I \in (R/I)^H$. Then $\tilde{t}(rx) - x = \tilde{t}(rx) - \tilde{t}(r)x = t.(rx) - (t.r)x = \sum_t (t_1.r)(t_2.x - \epsilon(t_2)x) \in I$. Since $\tilde{t}(rx) \in R^H$, the result follows. □

An example in [6] shows that (C_1) doesn't imply (C_2).

Lemma 2.2. *The following statements are equivalent.*
(a) *R is $R\#H$-quasi-projective.*
(b) *Condition (C_1) is satisfied.*
(c) *For every H-invariant left ideal I of R, $(R/I)^H = (R^H + I)/I$.*

Proof. The equivalence (b) $\Leftrightarrow$ (c) is obvious.

(a)$\Rightarrow$(b) Let I be an H-invariant left ideal of R and $x+I \in (R/I)^H$. Then right multiplication by $x+I$ is an $R\#H$- morphism $f : R \to R/I$. Let $\pi : R \to R/I$ be the canonical epimorphism. Since R is $R\#H$-quasi-projective, there exists $g \in Hom_{R\#H}(R, R)$ such that $\pi \circ g = f$. Take $s = g(1)$, then $s \in R^H$ and $s - x \in I$.

(b)$\Rightarrow$(a) Let $f : R \to R/I$ be an $R\#H$-morphism, where I is an $R\#H$-submodule of R. Then I is an H-invariant left ideal of R and $f(1) + I \in (R/I)^H$. Let $s \in R^H$, such that $f(1)+I = s+I$ and $g : R \to R$ be the right multiplication by s map. Then $g \in Hom_{R\#H}(R, R)$ and if we denote by π the canonical epimorphism $R \to R/I$, then $\pi \circ g = f$. □

Lemma 2.3. *Let M be an $R\#H$-module.*
(a) *The map $f \mapsto f(1)$ defines an isomorphism of R^H-modules between $Hom_{R\#H}(R, M)$ and M^H.*
(b) *$End_{R\#H}(R)$ is isomorphic to R^H.*
(c) *R is R^H-isomorphic to $(R\#H)^H$, where $R\#H$ is considered as left $R\#H$-module via left multiplication.*

Proof. (a) and (b) follow from [11, Corollary 3.5] and [12, Definition 3.1].

(c) The map $R \to tR$; $r \mapsto tr$ is an R^H-isomorphism. By [8, Proof of Theorem 8.3.3 page 139], $tR = (R\#H)^H$. Note that in [8, 11, 12], k is a field but there is no problem with k being now only a commutative ring. □

We can now state the main theorem of the paper.

Theorem 2.4. *Assume condition (C_1) holds. Then the following statements are equivalent:*
(a) *R is left FBN.*
(b) *R is left Noetherian and R^H is left FBN.*

Proof. By assumption and Lemma 2.2, R is $R\#H$-quasi-projective.

(a)$\Rightarrow$(b) Assume that R is left FBN. By Lemma 1.2 , $R\#H$ is left FBN too and, by [6, Corollary 1.9], R is FBN as $R\#H$-module. Now [6, Theorem 1.7] and Lemma 2.3 (b) imply that $R^H \simeq End_{R\#H}(R)$ is left FBN.

(b)⇒(a) Since R is $R\#H$-quasi-projective and $R\#H$ is left Noetherian, Lemma 2.3 and [2, Corollary 4.11] imply that R is a Noetherian R^H-module. So R is finitely generated as R^H-module. Let M be a finitely generated R-module. Then M is a finitely generated R-faithful R^H-module. Consider the subset $\mathcal{M} = Hom_R(R, M)$ of $Hom_{R^H}(R, M)$. Since R^H is left FBN, there exists a finite subset F of $\mathcal{M}$ such that $l_R(F) = l_R(\mathcal{M})$ (see [6, Corollary 1.8]). Since R is left Noetherian, the result follows from [6, Theorem 1.2]. □

Corollary 2.5. *Assume that* $1 \in \tilde{t}(R)$. *Then the following statements are equivalent.*

(a) *R is left FBN.*

(b) *R is left Noetherian and R^H is left FBN.*

Proof. By Lemma 2.1, condition (C_1) is satisfied. □

We close the paper by the following remark:

Remark 2.6. *If the map $\tilde{t}$ is surjective, $1 \in \tilde{t}(R)$. If k is a field, then H is a finite-dimensional Hopf algebra over k. If H is semisimple, then the map $\tilde{t}$ is surjective* [8, page 55]. *If H has a finite global dimension, H is semisimple* [3, Corollary 1.7]. *If k has characteristic* 0*, then H is semisimple if and only if s is involutive* [10, Theorem 5.4]. *If H is cocommutative or commutative, then s is involutive.*

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