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Asymptotically Equal Generalized Distances: Induced Topologies and *p*-Energy of a Curve

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Introduction

We consider the framework of generalized metric spaces (S, σ) where S is a non-empty set and

$$\sigma : S \times S \to [0, +\infty]$$

is a map such that $\sigma(x, x) = 0$. Briefly, σ is called a *generalized distance.* Therefore in general, σ satisfies neither symmetry nor the triangle inequality, yet it expresses the intuitive idea of a "distance", i.e. the estimate of the "gauge" between two points.

General metric spaces were studied by Menger, Bouligand, Busemann, Pauc, Carathéodory, Blumenthal and recently by Alexandrov and Gromov ([18], [4], [5], [19], [1], [15]).

By using the weak metric structure it is possible to give a notion of convergence. If σ satisfies the separation property $(\sigma(x, y) = 0 \Leftrightarrow x = y)$, i.e (S, σ) is a *semimetric space*, where the generalized distance is not necessarily symmetric, then it is possible to define four topologies. Moreover if σ is "continuous", then (S, σ) is a Hausdorff topological space.

Here a particular generalized distance $\sigma = \sigma_r$ $(r \geq 1)$ is considered, which is defined on the set S of the Lebesgue measurable subsets of $\mathbf{R}^n$

$$\sigma_r(A, B) = \Big(r \int_{A\Delta B} [\mathrm{dist}(x, \partial B)]^{r-1} dx\Big)^{1/r}, \quad r \geq 1$$

(where $A\Delta B$ is the symmetric difference of A and B). Observe that σ_1 is the Nikodým distance.

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The map σ_r, introduced by E. De Giorgi ([13]) as a generalization of σ_2 and considered by Almgren-Taylor-Wang ([2]), is used in order to study the generalized minimizing motions (e.g. following the mean curvature) ([14],[20])

Other examples come from the study of the Lipschitz manifolds, which has suggested us the generalizations presented in [9].

We shall give a suitable notion of asymptotically equal generalized distances and study some of its properties. If σ is asymptotically equal to ρ and they induce a topology, then the two topologies coincide.

As in [9], if $\gamma : [a, b] \to S$ is a (parameterized) curve of S, it is possible to define three functionals $\mathcal{E}_h(\sigma, p)$ for $h = 1, 2, 3$ and $p \geq 1$, called *p-energy of the curve* γ, which generalize the usual concept. In this paper we prove that, if the generalized distances σ and ϱ are asymptotically equal, then, for $h = 2, 3$,

$$\mathcal{E}_h(\sigma, p)(\gamma) = \mathcal{E}_h(\varrho, p)(\gamma) \qquad \forall\, p \geq 1$$

when γ has a finite energy for some $p_0 > 1$. The statement is true for every continuous curve γ if (S, σ) is a topological space.

The results (some of which are in [10]) answer a question proposed in a talk by E. De Giorgi, who in valuable discussions has drawn our attention to the problems of interactions among topology, differential geometry and calculus of variations.

1. Topology induced by a generalized distance

1.1. Let S be a set and

$$\sigma : S \times S \to [0, +\infty]$$

a map such that $\sigma(x, y) = 0$ if, and only if, $x = y$. In Blumenthal's language [3], (S, σ) is a *semimetric space*, where the generalized distance is not necessarily symmetric. For simplicity of writing we put

$$\sigma(x, y) = xy.$$

All that was said in [3](Ch.1, §6) for a semimetric space (with a symmetric distance) can be easily adapted to the space (S, σ) with a not symmetric distance. We give the basic concepts.

1.2. An element $x \in S$ is called an *L-limit* (left-limit) of a sequence (x_k) of elements of S (briefly $x_k \underset{L}{\longrightarrow} x$ or $x_k \underset{\sigma,L}{\longrightarrow} x$) if, and only if,

$$\lim_k x_k x = 0,$$

$(x_k x)$ being a sequence of non-negative real numbers. Observe that the limit may not be unique.

1.3. Let E be a subset of S. An element $x \in S$ is called an *L-accumulation point* of E provided that, for each positive number ε, there is a point $y \in E$ such that $0 < yx < \varepsilon$.

The subset E is *L-closed* if it contains each one of its accumulation points. E is *L-open* provided its complement $C(E)$ is L-closed. The family of all L-open sets, defined above, is closed under arbitrary unions and finite intersections; therefore, it forms a topology, named $\mathcal{T}_L(\sigma)$ or briefly $\mathcal{T}_L$.

1.4. Let x, y be elements of S. If for any sequence (y_k) of elements of S,

$$(y_k) \xrightarrow[L]{} y \Rightarrow (y_k x) \to yx,$$

then the distance function σ is said to be *L-continuous* at y, x; it is continuous in S provided it is continuous at each pair of points of S.

1.5. If $x \in S$ and ε is a positive number, the subset

$$B_L(x; \varepsilon) = \{y \in S; yx < \varepsilon\}$$

is called the *L-spherical neighborhood* of x with radius ε. Observe that a spherical neighborhood need not be open, nevertheless if σ is an L-continuous distance function, then, for all $x \in S$ and $\varepsilon > 0$, the sets $B_L(x; \varepsilon)$ are L-open and they form a base for the topology $\mathcal{T}_L(\sigma)$.

1.6. All that was said can be repeated interchanging the roles of left and right.
An element $x \in S$ is called an *R-limit* of a sequence (x_k) of elements of S if, and only if,

$$\lim_k xx_k = 0.$$

An element $x \in S$ is called an *R-accumulation point* of E provided that for every positive number ε, there is a point $y \in E$ such that $0 < xy < \varepsilon$. The family of all R-open sets forms a topology on S, named $\mathcal{T}_R$. If σ is R-continuous in x, y, then the sets

$$B_R(x; \varepsilon) = \{y \in S; xy < \varepsilon\}$$

are R-open.

1.7. Example. Let $S = \mathbf{R}$ and

$$\sigma(x, y) = \begin{cases} y - x, & x < y, \\ 0, & x \geq y. \end{cases}$$

The map σ is not symmetric, but satisfies the triangle inequality; moreover it is R-continuous. The spherical neighborhoods are the sets $B_R(x; \varepsilon) = (-\infty, x + \varepsilon]$, which generate on $\mathbf{R}$ the topology of upper semicontinuity. Analogously, $B_L(x; \varepsilon) = [x - \varepsilon, +\infty)$ generate on $\mathbf{R}$ the topology of lower semicontinuity.

1.8. An element $x \in S$ is called a *w-limit* (weak-limit) of a sequence (x_k) of elements of S if, and only if,

$$\min\{\lim_k x_k x, \lim_k xx_k\} = 0.$$

An element $x \in S$ is called a *w-accumulation point* of E provided that, for every positive number ε, there is a point $y \in E$ such that $0 < \min\{yx, xy\} < \varepsilon$. The family of all w-open sets forms the topology $\mathcal{T}_w$.

1.9. Analogously, an element $x \in S$ is called an *s-limit* (strong-limit) of a sequence (x_k) of elements of S if, and only if,

$$\max\{\lim_k x_k x, \lim_k x x_k\} = 0.$$

An element $x \in S$ is called an *s-accumulation point* of E provided that, for every positive number ε, there is a point $y \in E$ such that $0 < \max\{yx, xy\} < \varepsilon$. The family of all s-open sets forms the topology $\mathcal{T}_s$.

The topology $\mathcal{T}_w$ is the weakest of the four topologies, while $\mathcal{T}_s$ is the strongest. In general the four topologies may be distinct even if σ is continuous (with respect to $\mathcal{T}_w$ and hence with respect to the others).

We summarize the results in the following theorem, which was previously known if the distance function is symmetric:

1.10. Theorem. *On a semimetric space (S, σ), where σ is a generalized (not necessarily) symmetric distance, the topologies $\mathcal{T}_h$ $(h = L, R, w, s)$ can be defined. If σ is continuous with respect to $\mathcal{T}_h$, then $(S, \mathcal{T}_h)$ is a Hausdorff space and the balls $B_h(x; \varepsilon)$ form a base for the neighborhoods. Moreover, if σ is continuous with respect to the weak topology $\mathcal{T}_w$, then S is Hausdorff also with respect to the other topologies $\mathcal{T}_h$ $(h = L, R, s)$.*

Examples

1.11. Let $S = \mathbf{R}$ and

$$\sigma(x, y) = \begin{cases} y - x, & y \geq x, \\ 1, & y < x. \end{cases}$$

The w-topology is the Euclidean one, the s-topology is the discrete one, the spherical neighborhoods of $\mathcal{T}_L$ and $\mathcal{T}_R$ are respectively

$$B_L(x; \varepsilon) = [x, x + \varepsilon), \quad B_R(x; \varepsilon) = (x - \varepsilon, x].$$

Because $\sigma(x_k, x) \to 0$ if, and only if, $\sigma(x, x_k) \to 0$, the four topologies are continuous.

The following examples have suggested us to consider general metric spaces ([9], [19]).

1.12. If (M, F) (resp. (M, g)) is a Finsler (smooth) manifold (resp. a Riemann manifold), the function $\sigma : M \times M \to \mathbf{R}^+$ defined in a chart (U, Φ) by

$$\tilde{\sigma}(\xi, \eta) = F(\xi, \eta - \xi) \qquad \text{or} \qquad \tilde{\sigma}(\xi, \eta) = \left[\sum_{h,k} g_{h,k}(\xi)(\eta_h - \xi_h)(\eta_k - \xi_k) \right]^{1/2},$$

induces on M the generalized distance $\sigma(x,y) = \tilde{\sigma}(\Phi(x),\Phi(y))$, which satisfies neither symmetry nor the triangle inequality. Thus (M,σ) becomes a general metric space, hence a topological space, because σ is continuous (nay smooth); moreover the previous topologies coincide.

1.13. The spaces $(S,\mathcal{T}_h)$ in general are not metric, however the following statement holds:

1.14. Theorem. *Let σ and ρ be two generalized distances on S. A necessary and sufficient condition in order that, for $h = L, R, w, s$, the topology $\mathcal{T}_h(\sigma)$ coincides with $\mathcal{T}(\rho)$ is that*

$$x_k \xrightarrow[\sigma,h]{} x \Leftrightarrow x_k \xrightarrow[\rho,h]{} x$$

Proof. We prove the theorem for $h = L$; in the other cases we can proceed in an analogous manner.

Let $\rho(x_k,x) \to 0$ be with $x_k \neq x$ and $\limsup_k \sigma(x_k,x) = a > 0$, then it is possible to extract a subsequence of (x_k), denoted (y_n), such that

$$\sigma(y_n,x) > 0,\ \lim_n \sigma(y_n,x) = a,\ (\lim_n \rho(y_n,x) = 0).$$

If C denotes the closure of the set $\{y_n; n \in \mathbf{N}\}$ with respect to $\mathcal{T}_L(\sigma)$, then C is not closed with respect to $\mathcal{T}_L(\rho)$, provided $x \notin C$. The statement of the theorem is obtained by interchanging the roles of σ and ρ. □

It follows easily that

1.15. Theorem. *Let σ and ρ be two generalized distances on S. A sufficient condition in order that, for $h = L, R, w, s$, the topology $\mathcal{T}_h(\sigma)$ coincides with $\mathcal{T}_h(\rho)$ is that*

$$\limsup_{x_k \xrightarrow[\sigma,L]{} x} \frac{\sigma(x_k,x)}{\rho(x_k,x)} < +\infty, \quad \limsup_{x_k \xrightarrow[\rho,L]{} x} \frac{\rho(x_k,x)}{\sigma(x_k,x)} < +\infty.$$

Analogous conditions, mutatis mutandis, hold for the topologies $\mathcal{T}_R, \mathcal{T}_w, \mathcal{T}_s$.

1.16. Two generalized distances σ and ρ are called *equivalent* if

$$x_k \xrightarrow[\rho,w]{} x,\ y_k \xrightarrow[\rho,w]{} y \ \Rightarrow \limsup_k \frac{\sigma(x_k,y_k)}{\rho(x_k,y_k)} < +\infty$$

and

$$x_k \xrightarrow[\sigma,w]{} x,\ y_k \xrightarrow[\sigma,w]{} y \ \Rightarrow \limsup_k \frac{\rho(x_k,y_k)}{\sigma(x_k,y_k)} < +\infty.$$

Naturally the previous conditions are satisfied if two real numbers a, b exist such that

$$a\sigma(x,y) \leq \rho(x,y) \leq b\sigma(x,y) \quad \forall x,y \in S$$

which is the usual condition in metric spaces.

From Theorem 1.15 we have

1.17. Theorem. *If σ and ρ are equivalent, then $\mathcal{T}_h(\sigma) = \mathcal{T}_h(\rho)$ for $h = L, R, w, s$.*

2. A remarkable example

2.1. Let $\tilde{S}$ be the set of the Lebesgue measurable subsets of $\mathbf{R}^n$ and, for all $A, B \in \tilde{S}$, define

$$\sigma_r(A,B) = \left(r \int_{A\Delta B} [\mathrm{dist}(x, \partial B)]^{r-1} dx \right)^{1/r} \qquad (r \geq 1)$$

(where $A\Delta B$ is the symmetric difference of A and B).

Clearly, $(\tilde{S}, \sigma_r)$ is a general metric space. When we identify two sets A and B such that $|A\Delta B| = 0$, then σ_1 is the *Nikodým distance*, while σ_r $(r > 1)$ is not a distance in the usual sense, namely $\sigma_r(A,B) \neq \sigma_r(B,A)$.

In order to avoid pathological behavior, it is convenient to restrict $\tilde{S}$ to more meaningful subsets,

$$S = \{X \subset \mathbf{R}^n;\ X \ \text{ convex and bounded}\}$$

or

$$K = \{X \subset \mathbf{R}^n;\ X \ \text{ a convex body}\}.$$

Now, for all $A, B \in S$, with the above identification,

$$\sigma_r(A,B) = 0 \Rightarrow A = B.$$

In [21] the following statements are proved:

2.2. Theorem. *Let $A, B \in S$ with $|A| \neq 0$, $|B| \neq 0$. If (A_k), (B_k) are sequences in S and $(A_k) \to A, (B_k) \to B$ in the topology of σ_1, then*

$$\sigma_r(A_k, B_k) \to \sigma_r(A,B).$$

2.3. Theorem. *Let (A_k) be a sequence in S and $A \in S$. Then*

$$\sigma_r(A_k, A) \to 0 \Leftrightarrow \sigma_1(A_k, A) \to 0,$$

hence

$$\sigma_r(A_k, A) \to 0 \Leftrightarrow \sigma_r(A, A_k) \to 0.$$

Hence the generalized distance σ_r is continuous and the four topologies $\mathcal{T}_h(\sigma_r)$ are equal. Moreover, by Theorem 2.3, these topologies coincide with the one induced by σ_1, i.e. the Nikodým topology, which is the topology induced also by the Hausdorff distance ([16]).

3. Asymptotically equal distances

3.1. Let σ and ϱ be two generalized distances. We say that σ is *asymptotically equal* to ϱ at $x \in S$ if, and only if,

$$x_k \underset{\sigma}{\to} x,\ y_k \underset{\sigma}{\to} x \Rightarrow \lim_k \frac{\sigma(x_k, y_k)}{\varrho(x_k, y_k)} = 1.$$

If σ is asymptotically equal to ϱ at all points $x \in S$, then we write $\sigma \sim \varrho$. In general $\sigma \sim \varrho$ does not imply $\varrho \sim \sigma$, as is shown by the following example.

3.2. Example. Let $S = \mathbf{R}$ and

$$\varrho(x, y) = |\sin \sigma(x, y)|,$$

where σ might be a distance in the usual sense, in particular $\sigma(x, y) = |x - y|$. Now $\sigma \sim \varrho$, but if $\bar{x}, \bar{y} \in S$ are two points s.t. $\sigma(\bar{x}, \bar{y}) = m\pi$ $(m \in \mathbf{N} \setminus \{0\})$ then $\varrho(\bar{x}, \bar{y}) = 0$, hence ϱ is not asymptotically equal to σ.

We remark that, for example, the σ-closure of (x_k), where $x_k = 1/k$ is $\{x_k; k \in \mathbf{N}\} \cup \{0\}$, while the ρ-closure is $\{x; x = m\pi, m \in \mathbf{N}\}$. □

Observe that if $\sigma \sim \varrho$ and a real number $a > 0$ exists such that

$$a\sigma(x, y) \le \varrho(x, y),$$

then $\varrho \sim \sigma$.

3.3. Theorem. *Let σ, ρ be two generalized distances on the set S. If $\sigma \sim \rho$ and $\rho \sim \sigma$, then σ and ρ induce the same topology on S.*

Proof. If x is an L-accumulation point of a set $E \subset S$, then a sequence (x_k), with $x_k \in E \setminus \{x\}$, exists such that $\sigma(x_k, x) \to 0$. By definition, one has $\rho(x_k, x) \to 0$ too (and vice versa reversing the roles of σ and ρ), also the L-accumulation points with respect to the topology induced by σ coincide with the L-accumulation points with respect to the topology induced by ρ. Analogous conclusions hold in the other cases. □

Observe that σ and ρ may induce the same topology, without being asymptotically equal (for example σ_1 and σ_r).

The *LIP* case

3.4. Let (M, δ) be a *LIP* manifold, where δ is a distance locally equivalent to a Euclidean one. If (U, Φ) is a chart at the point $x \in M, \xi = \Phi(x), v$ is a vector of $V = \Phi(U) \subset \mathbf{R}^n$, we consider the "directional derivative" of δ at the point ξ,

$$\varphi(\xi, v) = \limsup_{t \to 0^+} \frac{\delta(\Phi^{-1}(\xi), \Phi^{-1}(\xi + tv))}{t}.$$

For almost all $\xi \in V$ there exists the limit and the function $\varphi(\xi, \cdot)$ is a norm that depends on ξ and which is locally equivalent to the Euclidean norm ([7]). Then

$$\tilde{d}(\xi, \eta) = \varphi(\xi, \eta - \xi)$$

is a generalized distance on $\mathbf{R}^n$, not continuous, which satisfies neither symmetry nor the triangle inequality.

3.5. Theorem. *Let (M, δ) be a LIP manifold, where δ is a distance locally equivalent to a Euclidean one. If $\tilde{d}(\xi, \eta) = \varphi(\xi, \eta - \xi)$ is the generalized distance induced on the chart, then δ is a.e. asymptotically equal to d, where*

$$d(x, y) = \tilde{d}(\xi, \eta), \qquad x = \Phi^{-1}(\xi), \quad y = \Phi^{-1}(\eta).$$

Proof. If $\delta(\Phi^{-1}(\xi), \Phi^{-1}(\eta)) = \tilde{\delta}(\xi, \eta)$, by (3.4) one has for $\xi \neq \eta$,

$$\frac{\tilde{\delta}(\xi, \eta)}{\tilde{d}(\xi, \eta)} = \frac{\tilde{\delta}(\xi, \eta)}{\varphi(\xi, \eta - \xi)} = \frac{\tilde{\delta}(\xi, \xi + \|\eta - \xi\| \frac{\eta - \xi}{\|\eta - \xi\|})}{\varphi(\xi, \frac{\eta - \xi}{\|\eta - \xi\|}) \|\eta - \xi\|}.$$

From every sequence (η_k) such that $\eta_k \to \xi$, it is possible to extract a subsequence (denoted again η_k) such that

$$\frac{\eta_k - \xi}{\|\eta_k - \xi\|} \to v, \qquad \|v\| = 1.$$

Then, for almost all ξ, one has $\delta \sim d$ because

$$\lim_{k \to +\infty} \frac{\delta(\xi, \eta_k)}{d(\xi, \eta_k)} = \lim_{k \to +\infty} \frac{\tilde{\delta}(\xi, \eta_k)}{\tilde{d}(\xi, \eta_k)} = \frac{\varphi(\xi, v)}{\varphi(\xi, v)} = 1. \qquad \square$$

3.6. Theorem. *Let (M, δ) be a LIP manifold, where δ is a distance locally equivalent to a Euclidean one. If ρ is a distance (on M) asymptotically equal to δ, then, for almost all ξ*

$$\varphi^\delta(\xi, v) = \varphi^\rho(\xi, v)$$

where φ^δ (resp. φ^ρ) is the "directional derivative" of δ (resp. ρ).

Proof. At the points where φ^δ and φ^ρ exist and, by the definition of asymptoticity, the relation

$$\lim_{t \to 0} \frac{\delta(\Phi^{-1}(\xi), \Phi^{-1}(\xi + tv))}{t} \frac{t}{\rho(\Phi^{-1}(\xi), \Phi^{-1}(\xi + tv))} = 1.$$

holds, whence the conclusion. $\square$

It follows in particular that

3.7. Theorem. *Let M be a metric space with respect to two asymptotically equal distances δ and ρ. Moreover let A be an open subset of $\mathbf{R}^n$ and $f : A \to M$ a LIP map. If*

$E \subset f(A)$ is $\mathcal{H}^n_\delta$-measurable (where $\mathcal{H}^n_\delta$ is the Hausdorff measure induced by δ) then E is $\mathcal{H}^n_\rho$-measurable and

$$\mathcal{H}^n_\delta(E) = \mathcal{H}^n_\rho(E).$$

It is sufficient to recall a representation theorem of type "area" ([17], [11,(3.7)]).

Because for the length of a curve γ constructed from the distance σ one has [6]

$$\mathcal{L}(\gamma;\sigma) = \int_a^b \varphi^\sigma(\gamma,\dot\gamma)dt$$

it follows that:

3.8. Theorem. *Let M be a metric space with respect to two asymptotically equal distances δ and ρ. If γ is a curve of M, then*

$$\mathcal{L}(\gamma;\delta) = \mathcal{L}(\gamma;\rho).$$

3.9. Example. Let (M,g) be a *LIP* Riemannian manifold embedded in $(\mathbf{R}^n,d)$, where d is the standard distance. If δ^g is the intrinsic distance induced on M by g ([6],[7]), then $\delta^g \sim d$ a.e. on M. Namely by Theorem [7,(6.2)], for almost all y

$$\lim_{x\to y} \frac{\delta^g(x,y)}{d(x,y)} = 1.$$

We recall that it is possible to have a *LIP* manifold (M,g) with $\varphi(\xi,\dot{})$ a norm, that is not derived from an inner product. Hence

3.10. Theorem. [7,(6.3)] *Given a LIP Riemannian manifold (M,g), in general it is not possible to find a number $m \in \mathbf{N}$ such that (M,g) is isometric to a LIP submanifold of $(\mathbf{R}^n, nat)$.*

4. p-Energy of a curve

4.1. As in [9], if $\gamma : [a,b] \to S$ is a (parameterized) *curve* of S, $a \le t' < t'' \le b$ and $T = \{t' = t_0 < t_1 < ... < t_{n+1} = t''\}$ is a decomposition of $[t',t'']$, we define for $p \ge 1$, $p \in \mathbf{R}$, the following functionals, called *p-energies of the curve* γ,

$$\mathcal{E}_1(\sigma,p)(\gamma;t',t'') = \sup_T \Big\{ \sum_{i=0}^n \frac{\sigma(\gamma(t_i),\gamma(t_{i+1}))^p}{(t_{i+1}-t_i)^{p-1}} \Big\};$$

$$\mathcal{E}_2(\sigma,p)(\gamma;t',t'') = \inf_T \Big\{ \sum_{i=0}^n \mathcal{E}_1(\sigma,p)(\gamma;t_i,t_{i+1}) \Big\};$$

$$\mathcal{E}_3(\sigma,p)(\gamma;t',t'') = {\int^*}_{t'}^{t''} \Big(\limsup_{h\to 0} \frac{\sigma(\gamma(t),\gamma(t+h))^p}{h^p} \Big) dt;$$

(where this latter integral is meant as a Lebesgue upper integral).

The functional $\mathcal{E}_1$ can be considered as the total p-variation of γ, with respect to the function σ. If σ is a *distance* and $p = 1$ we have the usual concept of length of a curve, for $p = 2$ we have the extension of the concept of energy to curves, that need not be smooth. In the general case,

$$\mathcal{E}_1 \geq \mathcal{E}_2 \geq \mathcal{E}_3$$

and there exist examples for which strict inequalities hold.

4.2. We say that γ satisfies the *finite energy condition* for $\mathcal{E}_h$ if some $p_0 > 1$ exists such that $\mathcal{E}_h(\sigma, p_0)(\gamma) < +\infty$.

4.3. Theorem. [9] *If σ satisfies the triangle inequality (on $\gamma(I)$), then*

$$\mathcal{E}_1(\sigma, p)(\gamma) = \mathcal{E}_2(\sigma, p)(\gamma) \qquad \forall p \geq 1.$$

Moreover if γ satisfies the finite energy condition, then

$$\mathcal{E}_1(\sigma, p)(\gamma) = \mathcal{E}_2(\sigma, p)(\gamma) = \mathcal{E}_3(\sigma, p)(\gamma) = \mathcal{E}(\sigma, p)(\gamma) \qquad \forall p \geq 1.$$

If S is a *LIP* (topological) manifold M and σ a distance δ locally equivalent to the Euclidean one, then

$$\mathcal{E}(\delta, p)(\gamma; a, b) = \int_a^b \varphi(\gamma, \dot{\gamma})^p dt.$$

where φ is the "derivative" of δ (see (3.4)).

In particular, if S is a *LIP* Finslerian manifold of class C^1 and $\delta = \delta^F$ is the intrinsic distance induced by a continuous norm F, then $\varphi = F$. We recall ([7]) that if F is a generic Finslerian structure, then in general $\varphi \neq F$, but $\varrho^\varphi = \varrho^F$.

Examples

4.4. We consider Example 3.2, where $S = \mathbf{R}$ and

$$\gamma(t) = \begin{cases} \bar{x}, & [a, b] \cap \mathbf{Q}, \\ \bar{y}, & [a, b] \cap \mathbf{R} - \mathbf{Q} \end{cases}.$$

Then $\sigma(\gamma(t), \gamma(t + h)) = 0, m\pi$, while $\rho(\gamma(t), \gamma(t + h)) = 0$. One easily sees that

$$\mathcal{E}_3(\sigma, p)(\gamma) = +\infty, \qquad \mathcal{E}_3(\varrho, p)(\gamma) = 0.$$

It follows that one may have $\sigma \sim \rho$ but $\mathcal{E}_3(\sigma, p)(\gamma) \neq \mathcal{E}_3(\varrho, p)(\gamma)$.

4.5. Even if $\sigma \sim \rho$ and $\rho \sim \sigma$, this does not imply that the energies are equal. Indeed, let $S = \mathbf{R}$ and

$$\sigma(x, y) = |x - y|, \qquad \varrho(x, y) = e^{\sigma(x,y)}\sigma(x, y),$$

then $\sigma \sim \varrho$, $\varrho \sim \sigma$, but

$$\mathcal{E}_1(\varrho, p)(\gamma) = e^{p(b-a)}(b-a) > (b-a) = \mathcal{E}_1(\sigma, p)(\gamma).$$

5. The main theorems

If M is a *LIP* (topological) manifold with σ and ρ distances (locally equivalent to a Euclidean one and) asymptotically equal, then, for every curve γ of M,

$$\mathcal{E}_h(\sigma, p)(\gamma) = \mathcal{E}_h(\rho, p)(\gamma) \qquad h = 1, 2, 3;\ p \geq 1.$$

Now we shall study under what conditions the energies are equal in the case that the generalized distances are asymptotically equal on a *set* M.

5.1. Theorem. *Let σ and ρ be generalized distances and $\sigma \sim \rho$. If γ is a curve of S such that $\mathcal{E}_3(\sigma, 1)(\gamma) < +\infty$, then $\mathcal{E}_3(\rho, 1)(\gamma) < +\infty$ too and*

$$\mathcal{E}_3(\sigma, p)(\gamma) = \mathcal{E}_3(\rho, p)(\gamma) \qquad \forall p \geq 1.$$

Proof. The condition $\mathcal{E}_3(\sigma, 1)(\gamma) < +\infty$ gives, for almost all $t \in [a, b]$,

$$\limsup_{h\to 0^+} \frac{\sigma(\gamma(t), \gamma(t+h))}{h} \in \mathbf{R} \Rightarrow \lim_{h\to 0} \sigma(\gamma(t), \gamma(t+h)) = 0.$$

Because $\sigma \sim \rho$, for every sequence (h_n) (with $h_n \geq 0$) convergent to 0,

$$\lim_n \frac{\sigma(\gamma(t), \gamma(t+h_n))}{h_n} \cdot \frac{h_n}{\rho(\gamma(t), \gamma(t+h_n))} = 1$$

holds and hence, if we choose a sequence (which we again indicate (h_n)) such that

$$\lim_n \frac{\sigma(\gamma(t), \gamma(t+h_n))}{h_n} = \limsup_n \frac{\sigma(\gamma(t), \gamma(t+h_n))}{h_n} = \psi(\gamma(t)),$$

it follows that

$$\limsup_n \frac{\rho(\gamma(t), \gamma(t+h_n))}{h_n} \geq \lim_n \frac{\rho(\gamma(t), \gamma(t+h_n))}{h_n} = \psi(\gamma(t)).$$

We choose (h_n) such that

$$\lim_n \frac{\rho(\gamma(t), \gamma(t+h_n))}{h_n} = \limsup_n \frac{\rho(\gamma(t), \gamma(t+h_n))}{h_n};$$

then we obtain the opposite inequality, whence the conclusion. □

5.2. Theorem. *Let σ and ρ be generalized distances and $\sigma \sim \rho$. If γ is a curve of S such that $\mathcal{E}_2(\sigma, p_0)(\gamma) < +\infty$ for some $p_0 > 1$, then $\mathcal{E}_2(\rho, p_0)(\gamma) < +\infty$ too and*

$$\mathcal{E}_2(\sigma, p)(\gamma) = \mathcal{E}_2(\rho, p)(\gamma) \qquad \forall p \geq 1.$$

Proof. First we remark that $\mathcal{E}_1(\sigma, p_0)(\gamma) < +\infty$, for some $p_0 > 1$, implies for $t < \tau$

$$\lim_{t,\tau \to t^*} \sigma(\gamma(t), \gamma(\tau)) = 0, \tag{5.3}$$

i.e. the continuity of $\sigma(\gamma(t), \gamma(\tau))$ at the point (t^*, t^*) of the diagonal; but in general the continuity of $\sigma(\gamma(t), \gamma(\tau))$, as a function of (t, τ) does not follow.

i) If $\mathcal{E}_1(\sigma, p_0)(\gamma) < +\infty$ for some $p_0 > 1$, then one proves that, $\forall \varepsilon > 0$ a δ_ε exists such that

$$(1 - \varepsilon) < \frac{\sigma(\gamma(t), \gamma(\tau))}{\tau(\gamma(t), \gamma(\tau))} < (1 + \varepsilon), \qquad 0 < \tau - t < \delta_\varepsilon.$$

Indeed, suppose ab absurdo that, $\forall n$ the points $t_n, \tau_n \in [a, b]$ exist s.t.

$$a \leq t_n < \tau_n \leq b, \ \tau_n - t_n < \frac{1}{n}, \ \left| \frac{\sigma(\gamma(t_n, \gamma(\tau_n))}{\rho(\gamma(t_n), \gamma(\tau_n))} - 1 \right| \geq \varepsilon. \tag{5.4}$$

It is possible to choose subsequences, which we again call $(t_n), (\tau_n)$, convergent to a point t^*. Then by (5.3) $\forall n$ one has $\gamma(t_n) \underset{\sigma}{\longrightarrow} \gamma(t^*), \gamma(\tau_n) \underset{\sigma}{\longrightarrow} \gamma(t^*)$, and by the assumptions $\lim_n \sigma(\gamma(t_n, \gamma(\tau_n))/\rho(\gamma(t_n), \gamma(\tau_n)) = 1$, which contradicts (5.4).

Let $\bar{T}$ be a partition of $[a, b]$ with width smaller than δ_ε. Then

$$(1 - \varepsilon)^p \frac{\rho(\gamma(t_n), \gamma(t_{n+1}))^p}{(t_{n+1} - t_n)^{p-1}} \leq \frac{\sigma(\gamma(t_n), \gamma(t_{n+1}))^p}{(t_{n+1} - t_n)^{p-1}} \leq \frac{\rho(\gamma(t_n), \gamma(t_{n+1}))^p}{(t_{n+1} - t_n)^{p-1}} (1 + \varepsilon)^p,$$

whence

$$(1 - \varepsilon)^p \sum_{i=0}^{n} \mathcal{E}_1(\rho, p)(\gamma; t_i, t_{i+1}) \leq \sum_{i=0}^{n} \mathcal{E}_1(\sigma, p)(\gamma; t_i, t_{i+1}) \leq$$

$$\leq (1 + \varepsilon)^p \sum_{i=0}^{n} \mathcal{E}_1(\rho, p)(\gamma; t_i, t_{i+1}).$$

Since

$$\inf_{T \supset \bar{T}} \Big\{ \sum_{i=0}^{n} \mathcal{E}_1(\sigma, p)(\gamma; t_i, t_{i+1}) \Big\} = \inf_{T} \Big\{ \sum_{i=0}^{n} \mathcal{E}_1(\sigma, p)(\gamma; t_i, t_{i+1}) \Big\} = \mathcal{E}_2(\sigma, p)(\gamma; t', t''),$$

by the arbitrariness of ε the assertion of the theorem follows.

(ii) By the definition of $\mathcal{E}_2$ and because of the assumptions, a partition of $[a, b]$ exists such that $\mathcal{E}_1(\sigma, p_0)(\gamma; t_i, t_{i+1}) < +\infty$. Then by (i)

$$\mathcal{E}_2(\sigma, p_0)(\gamma; t_i, t_{i+1}) = \mathcal{E}_2(\rho, p_0)(\gamma; t_i, t_{i+1}),$$

from which the conclusion follows provided $\mathcal{E}_2$ is an additive function. □

Remarks

5.5. The result of the Theorem 5.2 is not true for $\mathcal{E}_1$, as the Example 4.5 shows.

5.6. The condition $\mathcal{E}_2(\sigma,1)(\gamma) < +\infty$ does not imply the equality $\mathcal{E}_2(\sigma,p)(\gamma) = \mathcal{E}_2(\varrho,p)(\gamma)$ even for finite energies. For example, if

$$\gamma(t) = \begin{cases} \bar{x}, & a \le t \le c, \\ \bar{y}, & c \le t \le b, \end{cases}$$

and $\sigma(\bar{x},\bar{y}) \neq \varrho(\bar{x},\bar{y})$, we have

$$\mathcal{E}_2(\sigma,1)(\gamma) = \sigma(\bar{x},\bar{y}) \neq \varrho(\bar{x},\bar{y}) = \mathcal{E}_2(\varrho,1)(\gamma).$$

5.7. For the equality in 5.2 the condition $\mathcal{E}_3(\sigma,1)(\gamma) < +\infty$ is essential, as Example 4.4 shows.

5.8. The conditions

$$\mathcal{E}_3(\sigma,1)(\gamma) < +\infty, \qquad \mathcal{E}_2(\sigma,p_0)(\gamma) < +\infty, \quad p_0 > 1,$$

can be replaced by

$$\lim_{t_n \to t^-} \sigma(\gamma(t_n),\gamma(t)) = 0, \qquad \lim_{t_n \to t^+} \sigma(\gamma(t),\gamma(t_n)) = 0.$$

5.9. If $\mathcal{E}_2(\sigma,p)(\gamma) = +\infty$ for all $p > 1$, then the result of the Theorem 5.2 is true if

$$\frac{\rho(\gamma(t),\gamma(\tau))}{\sigma(\gamma(t),\gamma(\tau))} \ge c, \qquad \forall t,\tau \in [a,b].$$

For example, if for $t_n \to t$,

$$\limsup_n \sigma(\gamma(t),\gamma(t_n)) = l > 0,$$

then

$$\limsup_n \rho(\gamma(t),\gamma(t_n)) \ge cl > 0,$$

and hence

$$\mathcal{E}_2(\sigma,p)(\gamma) = \mathcal{E}_2(\rho,p)(\gamma) = +\infty, \qquad \forall p > 1.$$

From the remark in 5.7 it follows that:

5.10. Theorem. *Let S be a topological space and σ a continuous map. If $\sigma \sim \rho$, then, for every continuous curve γ,*

$$\mathcal{E}_h(\sigma,p)(\gamma) = \mathcal{E}_h(\rho,p)(\gamma) \qquad \forall p \ge 1, h = 2,3.$$

Remarks

5.11. For $h = 1$, the theorem is not true as shown in Example 4.5.

5.12. The Nikodým distance σ_1 and the generalized distance σ_r (introduced in Section 2) induce the same topology, but σ_1 is not asymptoyically equal to σ_r, because $\mathcal{E}_h(\sigma_1, p)(\gamma) \neq \mathcal{E}_h(\sigma_r, p)(\gamma)$ (see [9], §5).

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