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The Classification of $\mathbf{S}^2\times\mathbf{R}$ Space Groups*

J. Z. Farkas

Budapest University of Technology and Economics
Institute of Mathematics, Department of Geometry
Budapest XI. Egry J. str. 1 H1521, Hungary
e-mail: farkas@math.bme.hu

Abstract. The geometrization of 3-manifolds plays an important role in various topological investigations and in the geometry as well. Thurston classified the eight simply connected 3-dimensional maximal homogeneous Riemannian geometries [7], [8]. One of these is $\mathbf{S}^2\times\mathbf{R}$, i.e. the direct product of the spherical plane $\mathbf{S}^2$ and the real line $\mathbf{R}$. Our purpose is the classification of the space groups of $\mathbf{S}^2\times\mathbf{R}$, i.e. discrete transformation groups which act on $\mathbf{S}^2\times\mathbf{R}$ with a lattice on $\mathbf{R}$ (see Section 3), analogously to that of the classical Euclidean geometry $\mathbf{E}^3$.

1. Introduction

The theory of plane and space groups goes back to the 19th century to H. Poincaré, to E. S. Fedorov, and A. Schoenflies. Schoenflies and Fedorov parallelly classified the three-dimensional Euclidean crystallographic groups. We are interested in the analogous problem in other homogeneous geometries, now in $\mathbf{S}^2\times\mathbf{R}$, but the problem is unsolved in the rest of 3-dimensional Thurston-geometries (except $\mathbf{E}^3$ and $\mathbf{S}^3$, of course; because of crystallographic applications), and does not seem to be easy.
The well known solutions of the analogous problem in the spherical plane is the following: the series of the cyclic and dihedral rotation groups, and the rotation subgroups of the five Platonic solids will be normal subgroups of index 2 in the "full" reflection groups: $C_q\times I, D_q\times I, A_4\times I, S_4\times I, A_5\times I$, and the so-called mixed groups $C_{2q}C_q, D_qC_q, D_{2q}D_q, S_4A_4$ come up, too. Here we apply Weyl's notation, which is compared with the Macbeath signatures and other denotations in the following table:

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Table 1.

	Macbeath signature	H. Weyl	Schoen-flies	Coxeter-Moser	Conway
1q	$(+, 0; [q, q]; \{\})\ q \geq 1$	C_q	C_q	$[q]^+$	q, q
2q	$(+, 0; [\ ]; \{(q, q)\})\ q \geq 2$	D_qC_q	C_{qv}	[q]	$*q, q$
3q	$(+, 0; [2, 2, q]; \{\})\ q \geq 2$	D_q	D_q	$[2,q]^+$	$2, 2, q$
4qo	$(+, 0; [\ ]; \{(2, 2, q)\})\ q \geq 3$	$D_{2q}D_q$	D_{qh}	[2,q]	$*2, 2, q$
4qe	$(+, 0; [\ ]; \{(2, 2, q)\})\ q \geq 2$	$D_q \times I$	D_{qh}	[2,q]	$*2, 2, q$
5qo	$(+, 0; [q]; \{(1)\})\ q \geq 1$	$C_{2q}C_q$	C_{qh}	$[2,q^+]$	$q*$
5qe	$(+, 0; [q]; \{(1)\})\ q \geq 2$	$C_q \times I$	C_{qh}	$[2,q^+]$	$q*$
6qo	$(+, 0; [2]; \{(q)\})\ q \geq 3$	$D_q \times I$	D_{qd}	$[2^+,2q]$	$2 * q$
6qe	$(+, 0; [2]; \{(q)\})\ q \geq 2$	$D_{2q}D_q$	D_{qd}	$[2^+,2q]$	$2 * q$
7qo	$(-, 1; [q]; \{\})\ q \geq 1$	$C_q \times I$	S_{2q}	$[2^+,2q^+]$	$q\otimes$
7qe	$(-, 1; [q]; \{\})\ q \geq 2$	$C_{2q}C_q$	S_{2q}	$[2^+,2q^+]$	$q\otimes$
8	$(+, 0; [2, 3, 3]; \{\})$	A_4	T	$[3,3]^+$	$2, 3, 3$
9	$(+, 0; [2, 3, 4]; \{\})$	S_4	O	$[3,4]^+$	$2, 3, 4$
10	$(+, 0; [2, 3, 5]; \{\})$	A_5	I	$[3,5]^+$	$2, 3, 5$
11	$(+, 0; [\ ]; \{(2, 3, 3)\})$	S_4A_4	T_d	[3,3]	$*2, 3, 3$
12	$(+, 0; [\ ]; \{(2, 3, 4)\})$	$S_4 \times I$	O_h	[3,4]	$*2, 3, 4$
13	$(+, 0; [\ ]; \{(2, 3, 5)\})$	$A_5 \times I$	I_h	[3,5]	$*2, 3, 5$
14	$(+, 0; [3]; \{(2)\})$	$A_4 \times I$	T_h	$[3^+,4]$	$3 * 2$

We see the disadvantage of H. Weyl's denotation system: it can differ for odd (o) and even (e) parameter q. But it emphasizes the essential role of the central inversion I (or antipodal map) in spherical transformations; namely, I commutes with each isometry of $\mathbf{S}^2$. In 1967–69 Macbeath completed the classification of hyperbolic crystallographic plane groups, (for short NEC groups) [4]. He considered isometries containing orientation-preserving and -reversing transformations as well in the Bolyai-Lobachevskian hyperbolic plane. His paper deals with NEC groups, but with the Macbeath-signature we can very economically characterize the Euclidean and spherical plane groups, too. We recall the signature of a plane group

$$(\pm, g; [m_1, m_2, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}) \tag{1}$$

and, with the same notations, the combinatorial measure T of the fundamental polygon:

$$T\kappa = \pi\{\sum_{l=1}^{r}(\frac{2}{m_l} - 2) + \sum_{i=1}^{k}(-2 + \sum_{j=s_1}^{s_i}(-1 + \frac{1}{n_{ij}})) + 2\chi\} \tag{2}$$

where $\chi = 2 - \alpha g$ ($\alpha = 1$ for $-$, $\alpha = 2$ for $+$ orientability) is the Euler characteristic of the surface with genus g, and κ is the Gaussian curvature of the plane which realizes the signature, and is identical with $\mathbf{S}^2$, $\mathbf{E}^2$ or $\mathbf{H}^2$, whenever $\kappa > 0$, $= 0$ or < 0, respectively. The sign $\pm$, the genus g, the proper periods m_l of rotation centres and the period-cycles $(n_{i1}, n_{i2}, \ldots, n_{is_i})$ of dihedral corners together, with a marked polygon (treated in [3], e.g.)

and with a corresponding group presentation determine a plane group up to isomorphism for $\mathbf{E}^2$ and $\mathbf{H}^2$. These are described by Macbeath in [4].
Conway's (orbifold) notation provides only nicer typographic simplifications: with handles $\bigcirc\ldots\bigcirc$ at the beginning (if occur), with cross caps $\otimes\ldots\otimes$ at the end of the symbol; * introduces a boundary component, then may come numbers for its dihedral corners. Only simple numbers before the boundary components (iff occur) denote the orders of rotational centres (cone points).
For $\mathbf{S}^2$, and in general, the equivariance is the relevant concept of equivalence(see Section 3). There are 4 series of signatures where $T > 0$, but the corresponding plane groups cannot be realized in $\mathbf{S}^2$, see e.g. in [3]. These are the bad 2-orbifolds.

2. The geometry of $\mathbf{S}^2\times\mathbf{R}$

$\mathbf{S}^2\times\mathbf{R}$ is a Seifert fibre space whose point set is (X,x), where $X \in \mathbf{S}^2$, $x \in \mathbf{R}$. (X,x) describes a fibre if $X \in \mathbf{S}^2$ is fixed and $x \in \mathbf{R}$ varies, and it is a base if X varies with fixed x. In this paper we assume the analytic and synthetic models of $\mathbf{S}^2\times\mathbf{R}$, the components will be $\mathbf{S}^2$ and $\mathbf{R}$ with the usual geometric concepts. The projective-inversive spatial model (e.g. in [2], [5]) is very clear but the sphere inversions for $\mathbf{R}$-reflections and the central similarities for $\mathbf{R}$-translations are rather cumbersome in that model.
It is well-known that Isom($\mathbf{S}^2\times\mathbf{R}$):=Isom($\mathbf{S}^2$)$\times$Isom($\mathbf{R}$), for the isometry group of $\mathbf{S}^2\times\mathbf{R}$, where Isom($\mathbf{S}^2$):=$\{A \in \mathbf{O}(3)\ :\ \mathbf{S}^2 \mapsto \mathbf{S}^2\ :\ (X,x) \mapsto (XA,x)\}$, identical with the 3-dimensional orthogonal group, Isom($\mathbf{R}$):=$\{\rho\ :\ (X,x) \mapsto (X, \pm x + r)\}$, here the minus sign $-$ provides a reflection in the point $\frac{r}{2} \in \mathbf{R}$, by the $+$ sign we get a translation, which is a composition of two reflections whereby the distance of the two reflection points is equal to $\frac{r}{2}$. (For further details see [5], [6], [7], [8].)
Now we recall the classification of the isometries of $\mathbf{S}^2\times\mathbf{R}$. These are products of at most five reflections: three of the $\mathbf{S}^2$-component and two of the $\mathbf{R}$-component. $\mathbf{S}^2_i\mathbf{R}_j$ denotes the set of isometries which are products of i spherical reflections and j $\mathbf{R}$-reflections, where $i = 0,\ldots,3,\ j = 0,\ldots,2$, respectively.

3. Space groups and their equivalence

We search for any group of $\mathbf{S}^2\times\mathbf{R}$ in the form

$$\Gamma := \{(A_1 \times \rho_1),\ldots,(A_n \times \rho_n)\} \tag{3}$$

where $(A_i \times \rho_i) := A_i \times (R_i, r_i) := (g_i, r_i)$ $(:= (g_i, \tau_i)$ see later), R_i is either the identity map $1_{\mathbf{R}}$ of $\mathbf{R}$ or the point reflection $\overline{1}_{\mathbf{R}}\ :\ x \mapsto -x$; $A_i \in$ Isom($\mathbf{S}^2$), $g_i = (A_i \times R_i)$ is the linear part of a transformation.

By definition we speak about a space group Γ if the linear parts form a finite group Γ_0 called the *point group of* Γ, moreover, the translation parts to the identity of this point group Γ_0, namely to $e := 1_{\mathbf{S}^2} \times 1_{\mathbf{R}}$, are required to form a one-dimensional lattice L_Γ. Then our definition corresponds to Euclidean crystallographic space groups.

So $\{\ldots,(g_i,\tau_i),\ldots,(e,\tau)\}$ generate the space group Γ, where the g_i are the generators of the point group Γ_0, however, the presentation is not uniquely determined. The multiplication formula

$$(A_1 \times R_1, r_1) \circ (A_2 \times R_2, r_2) = (A_1A_2 \times R_1R_2, r_1R_2 + r_2) \tag{4}$$

shows that Γ_0 is a homomorphic image of Γ, therefore, Γ_0 is nothing but the factor group Γ/L_Γ.

We emphasize that in this conception the point group of a space group is finite by definition and a space group contains a one-dimensional lattice. This is a classical point of view, and in this case we have a compact fundamental domain for Γ. Then we can give the following definition of equivalence of space groups:

Definition. *Two space groups Γ_1 and Γ_2 are geometrically equivalent, called equivariant, if there exists a similarity $\Sigma := S \times \sigma$ of $\mathbf{S}^2\times\mathbf{R}$, i.e. $S \in Isom(\mathbf{S}^2)$, $\sigma \in Sim(\mathbf{R})$, such that $\Gamma_2 = \Sigma^{-1}\Gamma_1\Sigma$.*

The similarity Σ maps the lattice L_{Γ_1} onto L_{Γ_2} by σ, while S maps the spherical parts of the group Γ_1 to those of Γ_2. So Σ is a bijective correspondence between the Γ_1 and Γ_2 orbits, and it transforms the action of Γ_1 onto that of Γ_2.

This definition of geometric isomorphism of space groups is very natural. The structure of the space group remain invariant under a similarity in the $\mathbf{R}$-direction, and the spherical part is uniquely determined up to an isometry of $\mathbf{S}^2$. Thus if Γ and Γ' are equivariant groups, then the equivariance of Γ/L_Γ and $\Gamma'/L_{\Gamma'}$ are also guaranteed.

Macbeath's definition of geometric isomorphism of NEC groups is the following: Γ and Γ' are called *geometrically isomorphic* if there is a homeomorphism t of $\mathbf{H}^2$, and a group isomorphism $\varphi : \Gamma \mapsto \Gamma'$, such that

$$g \mapsto g' := g^\varphi = t^{-1}gt,\ \Gamma' = t^{-1}\Gamma t, \tag{5}$$

so the groups Γ and Γ' are conjugate in the group of all homeomorphisms of $\mathbf{H}^2$.

With this we get a general definition of equivariance for space groups at least in any classical space of any dimension: two space groups Γ and Γ' are *equivariant*, so lie in the same equivariance class, if they satisfy (5). Now our equivariant space groups of $\mathbf{S}^2\times\mathbf{R}$ are conjugate in the group of all similarities of $\mathbf{S}^2\times\mathbf{R}$, which is the proper subgroup of all homeomorphisms of $\mathbf{S}^2\times\mathbf{R}$.

Obviously, if $\Gamma' = t^{-1}\Gamma t$ as above, then there exists an isomorphism φ such that $\Gamma^\varphi = \Gamma'$. Macbeath proved the converse for NEC groups: if there exists an isomorphism $\varphi : \Gamma \mapsto \Gamma'$, then this can be realized geometrically, so there is a homeomorphism $t : \mathbf{H}^2 \to \mathbf{H}^2$ such that (5) holds. The famous analogous result in $\mathbf{E}^n$ was proved by Frobenius and Bieberbach: then an affine transform $t : \mathbf{E}^n \to \mathbf{E}^n$ involves the isomorphism φ of space groups Γ and Γ' as in (5).

We can see that this theorem cannot be extended to the space groups of $\mathbf{S}^2\times\mathbf{R}$. There are isomorphic space groups Γ and $\Gamma' = \Gamma^\varphi$ of $\mathbf{S}^2\times\mathbf{R}$ such that φ cannot geometrically be realized. This is because the centre subgroup of $\mathrm{Isom}(\mathbf{S}^2)$ contains the central inversion.

However the extension of this theorem for space groups of other Thurston spaces, thus the existence of geometric realization of group isomorphisms, seems to be still open.

Remarks. We emphasize, that equivariance by diffeomorphism may yield less classes than those by similarities; namely, for fixed point free groups [8] Γ which lead to space forms

$\mathbf{S}^2 \times \mathbf{R}/\Gamma$ by factorization onto Γ-orbits. We shall illustrate this phenomenon at the end of the last section.

4. Point groups

Analogously to the "Euclidean method", first we determine the possible point groups of the space groups:

Theorem 1. *Any point group Γ_0 to a space group Γ of $\mathbf{S}^2 \times \mathbf{R}$ belongs to one of the following three types:*

I. $G_{\mathbf{S}^2} \times 1_{\mathbf{R}}$, *where $G_{\mathbf{S}^2}$ is a finite group of spherical isometries, $1_{\mathbf{R}} : x \mapsto x$ is the identity of $\mathbf{R}$.*

II. $G_{\mathbf{S}^2} \times \langle \overline{1}_{\mathbf{R}} \rangle$, *where $\langle \overline{1}_{\mathbf{R}} \rangle = \{x \mapsto x, x \mapsto -x\}$ is the special linear group of $\mathbf{R}$.*

III. *If the spherical group $G' (= G'_{\mathbf{S}^2})$ contains a normal subgroup G of index two, then $G'G := \{G \times 1_{\mathbf{R}}\} \cup \{(G' \setminus G) \times \overline{1}_{\mathbf{R}}\}$ forms a point group, too.*

Proof. Types I and II come up, and they are not equivariant with each other. Equivariance of the spherical group components would be necessary, but then type I would be a normal subgroup in type II of index two, and this excludes the possibility of equivariance.
The groups of the type III must be compared with the groups of type II. To this we write both groups in the following form:

$$G_{\mathbf{S}^2} \times \langle \overline{1}_{\mathbf{R}} \rangle = \{G_{\mathbf{S}^2} \times 1_{\mathbf{R}}\} \cup \{G_{\mathbf{S}^2} \times \overline{1}_{\mathbf{R}}\},\ G'G := \{G \times 1_{\mathbf{R}}\} \cup \{(G' \setminus G) \times \overline{1}_{\mathbf{R}}\}$$

The equivariance of the two components would be necessary, but this is impossible.
Existence of further groups is excluded: if only $1_{\mathbf{R}}$ comes to the $\mathbf{R}$-component then we obtain type I. When the $\mathbf{R}$-component of the point group Γ_0 includes the reflection $\overline{1}_{\mathbf{R}}$, then $(A_i \times \overline{1}_{\mathbf{R}})(A_j \times \overline{1}_{\mathbf{R}}) = (A_i A_j \times 1_{\mathbf{R}})$ shows, that the elements $g_k = (A_k \times 1_{\mathbf{R}})$ of Γ_0 form a normal subgroup of index two, consequently Γ_0 lies in type II or in type III. □

With this we get 52 classes of point groups (Table 4), 14-14 lie in Type I and in Type II, and the remaining 24 in Type III as we shall see later. Some classes contain infinite series, depending on a natural number q.

Remarks. The central inversion $I := \overline{1}_{\mathbf{S}^2} \in \mathbf{S}^2_3$ commutes with any isometry of $\mathbf{S}^2$, and so does $\overline{1}_{\mathbf{R}}$ with any similarity of $\mathbf{R}$, fixing zero. So the centre subgroup of Γ_0 may consist of 4 elements forming a Kleinian group.
We know that the discrete groups of $\mathbf{S}^2$ can be listed analogously as in our Theorem 1:
Type 1: groups which contain only rotations: C_q, D_q, A_4, S_4, A_5.
Type 2: the direct products of the rotation groups with the central inversion: $C_q \times I, D_q \times I, A_4 \times I, S_4 \times I, A_5 \times I$.
Type 3: the mixed groups: $C_{2q}C_q, D_qC_q, D_{2q}D_q, S_4A_4$, with Weyl's notations. E.g. $C_{2q}C_q := \{C_q\} \cup \{(C_{2q} \setminus C_q) \times I\}$. See our Table 1 in the introduction.

5. Translation parts

As previously mentioned, we require a one-dimensional lattice in the $\mathbf{R}$-direction, and assume the point group Γ_0 to be finite. For the possible translation parts belonging to the elements of the point group Γ_0 of any $\mathbf{S}^2\times\mathbf{R}$ space group Γ, (analogously to the Euclidean space groups) we have to solve the so-called Frobenius congruences (mod L_Γ), where L_Γ is a fixed one-dimensional lattice in $\mathbf{R}$. For this we give the generators and relations of Γ_0 in the sense of a minimal presentation which can be obtained from the Macbeath-signature. The translation parts to the identity: $e := 1_{\mathbf{S}^2}\times 1_{\mathbf{R}}$ of Γ_0 form just the lattice $L_\Gamma := \{k\tau,\ k\in\mathbf{Z}\}$ generated by a minimal translation τ. The possible translation parts to the generators of Γ_0 will be determined from the multiplication formula (4), consequently from the defining relations of the point group Γ_0. Thus, we obtain the Frobenius congruences to be solved for the translation parts of the generators of Γ_0. Then we select these possible solutions (mod 1) into equivariance classes by the definition in Section 3. At the end we get the equivariance classes Γ of $\mathbf{S}^2\times\mathbf{R}$ space groups.

6. The three types of space groups

In each type of point groups we shall discuss in details one example in the finite cases 8–14 and that in the infinite series 1–7 as well, the others can similarly be discussed. We shall give the complete list of all non-equivariant space groups at the next section in Table 2.

6.1. Type I

In this case the point group Γ_0 determines a spherical group, characterized by the Macbeath signature. The generators of Γ_0 will be denoted by $g_1, g_2, \ldots \in \mathrm{Isom}(\mathbf{S}^2)$, and $(g_1,\tau_1), (g_2,\tau_2)$, $\ldots$ with (e,τ) generate the space group Γ, where τ_i denotes a translation corresponding to g_i, and τ generates the lattice L_Γ. For a fundamental domain of any $\mathbf{S}^2\times\mathbf{R}$ space group we can combine a fundamental domain of the spherical group with a part of the real line segment τ (of unit length), but the fundamental domain is not uniquely determined. The cases **1q.I** and **12.I** are presented in this subsection.

- **1q.I.** $(+,0;[\,q,q];\{\})\times 1_{\mathbf{R}}$ (see Table 1 of the introduction)
  The point group
  $$\Gamma_0 := (g_1 - g_1^q)$$
  is generated by the rotation $g_1 \in \mathbf{S}^2_2$ (see Section 2).
  $(g_1,\tau_1)^q = (g_1^q, q\tau_1) = (e, q\tau_1) \equiv 0 \pmod{\tau}$. We can choose $\tau = 1$, because for different values of τ we get the same equivariance classes of space groups. So the solutions are: 1. (0); 2. $(\frac{k}{q})$ where $k := 1,\ldots,\lfloor\frac{q}{2}\rfloor$ (the lower integer part of $\frac{q}{2}$). This is because for every $k = \frac{q-l}{q}$ where $l \le \lfloor\frac{q}{2}\rfloor$ with $\Sigma := (\overline{1}_{\mathbf{R}}, 0)$ (Definition in Section 3) we get $(\overline{1}_{\mathbf{R}},0)(g_1,\frac{q-l}{q})(\overline{1}_{\mathbf{R}},0) = (\overline{g}_1,\frac{q-l}{q})(\overline{1}_{\mathbf{R}},0) = (g_1,-\frac{q-l}{q}) \equiv (g_1,\frac{l}{q}) \pmod 1$.
  The geometric presentation of $\mathbf{\Gamma_{1q.I.2}}$ depends on k and q, of course.
  We remark that iff the greatest common divisor(g.c.d.) $(k,q)=1$, then Γ will be fixed point free and the factor space $\mathbf{S}^2\times\mathbf{R}/\Gamma$ will be a compact orientable manifold (space form). Otherwise $(k,q)(\neq q)$ provides the order of both rotation axes of (g_1,τ_1).

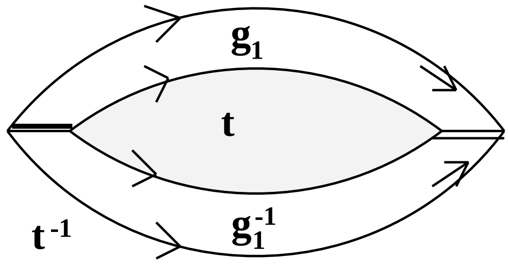

Figure 1: *The fundamental domain (by Schlegel diagram) for the presentation* $\mathbf{\Gamma_{1q.I.1}} = (\mathbf{g_1}, \mathbf{t} - \mathbf{g_1^q}, \mathbf{g_1 t g_1^{-1} t^{-1}})$. *Here the translation* $\tau = \mathbf{t}: \ f_{\mathbf{t}^{-1}} \mapsto f_{\mathbf{t}}$ *and the rotation* $g_1 = \mathbf{g_1}: \ f_{\mathbf{g_1^{-1}}} \mapsto f_{\mathbf{g_1}}$ *are the generators. Later on, the face symbol* f *will be omitted, as in the figures.*

- **12.I.** $(+, 0; [\ ]; \{(2, 3, 4)\}) \times 1_{\mathbf{R}}$

$$\Gamma_0 := (g_1, g_2, g_3 - g_1^2, g_2^2, g_3^2, (g_1 g_2)^2, (g_1 g_3)^3, (g_2 g_3)^4)$$

is the full symmetry group of the usual cube surface, generated by the three reflections: $g_i \in \mathbf{S}_1^2 \ i = 1, 2, 3$.
From the congruence relations we obtain the conditions:
$0 \equiv 2\tau_1 \equiv 2\tau_2 \equiv 2\tau_3 \equiv 2(\tau_1 + \tau_2) \equiv 3(\tau_1 + \tau_3) \equiv 4(\tau_2 + \tau_3) \pmod 1$
and the solutions: 1. $(\tau_1, \tau_2, \tau_3) \equiv (0, 0, 0)$; 2. $(0, \frac{1}{2}, 0)$; 3. $(\frac{1}{2}, 0, \frac{1}{2})$; 4.$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and they are not equivariant with each other.

6.2. Type II

Now the point group $\Gamma_0 := G_{\mathbf{S}^2} \times \langle \overline{1}_{\mathbf{R}} \rangle$ is a direct product, the generators of $G_{\mathbf{S}^2}$ will be denoted by $g_1, g_2, \ldots$, and $\overline{g}$ denotes the $\mathbf{R}$-reflection $\overline{1}_{\mathbf{R}}$. We will discuss the cases **4q.II** and **8.II** in details, and give the complete list of the groups in the next section. For a fundamental domain of a space group – as previously – we can combine a fundamental domain of the spherical group with a part of a real line segment $\frac{\tau}{2}$. Now, in this case we have a point group of double order, so it is clear that the “volume” of the fundamental domain will be the half of the corresponding previous one.

- **4q.II.**$(+, 0; [], \{(2, 2, q)\}) \times \overline{1}_{\mathbf{R}} =: \overline{(+, 0; [], \{(2, 2, q)\})}$
  The point group

$$\Gamma_0 := (g_1, g_2, g_3, \overline{g} - g_1^2, g_2^2, g_3^2, \overline{g}^2, (g_1 g_3)^2, (g_2 g_3)^2, (g_1 g_2)^q, (g_1 \overline{g})^2, (g_2 \overline{g})^2, (g_3 \overline{g})^2)$$

generated by four reflections , three ones of the $\mathbf{S}^2$-component: $g_i \in \mathbf{S}_1^2$ and $\overline{g} \in \mathbf{R}_1$
$(g_1, \tau_1)(g_1, \tau_1) = (g_1^2, 2\tau_1) \equiv (e, 0) \pmod 1$
$(g_2, \tau_2)(g_2, \tau_2) = (g_2^2, 2\tau_2) \equiv (e, 0) \pmod 1$
$(g_3, \tau_3)(g_3, \tau_3) = (g_3^2, 2\tau_3) \equiv (e, 0) \pmod 1$
$(\overline{g}, \tau_4)(\overline{g}, \tau_4) = (\overline{g}^2, 0) = (e, 0)$

$(g_1,\tau_1)(\overline{g},\tau_4)(g_1,\tau_1)(\overline{g},\tau_4)=\cdots=(e,0)$, so we may choose $\tau_4\equiv 0,\ldots$ etc.
The solutions are: 1. $(\tau_1,\tau_2,\tau_3,\tau_4)\equiv(0,0,0,0)$; 2. $(0,0,\frac{1}{2},0)$; 3. $(\frac{1}{2},\frac{1}{2},0,0)$; 4. $(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$ for every q. If q is even, then 5. $(0,\frac{1}{2},0,0)$ 6. $(0,\frac{1}{2},\frac{1}{2},0)$ also come up. The solutions: 5'. $(\frac{1}{2},0,0,0)$ and 6'. $(\frac{1}{2},0,\frac{1}{2},0)$ are equivariant to 5. and 6., respectively with $\Sigma:=S\times\sigma$ in the Definition (Section 3), where $S:=r$ is a reflection in the plane that halves the (smaller) angle of the g_1,g_2 reflection planes and $\sigma:=(e,0)$. Thus $(r^{-1},0)(g_1,\frac{1}{2})(r,0)=(g_2,\frac{1}{2})$ and $(r^{-1},0)(g_2,0)(r,0)=(g_1,0)$, and $(g_3,0)$, $(g_3,\frac{1}{2})$ are (mod 1) invariant under this transformation, since the reflection plane of g_3 is orthogonal to the r-plane (Fig.2).

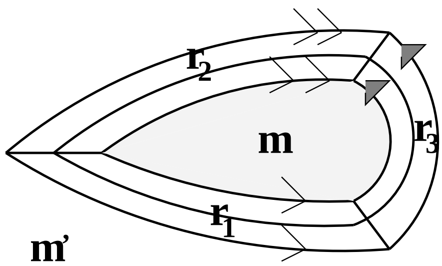

Figure 2: $\mathbf{\Gamma_{4q.II.4}=(m,m',r_1,r_2,r_3-r_1^2,r_2^2,r_3^2,(r_2r_3)^2,(r_1r_3)^2,(r_1r_2)^q,m^2,(m')^2,mr_1m'r_1,mr_2m'r_2,mr_3m'r_3)}$
Here $\mathbf{m}=(\overline{g},1)$, $\mathbf{m'}=(\overline{g},0)$, $\mathbf{r_i}=(\overline{g},0)(g_i,\frac{1}{2})$, $i=1,2,3$.

- **8.II.** $\overline{(+,0;[2,3,3];\{\})}$

$$\Gamma_0:=(g_1,g_2,\overline{g}-g_1^2,g_2^3,\overline{g}^2,(g_1g_2)^3,(g_1\overline{g}g_1\overline{g}),(g_2\overline{g}g_2^{-1}\overline{g}))$$

generated by two rotations $g_1,g_2\in\mathbf{S}_2^2$ and again $\overline{g}\in\mathbf{R}_1$.
The relations:
$0\equiv 2\tau_1\equiv 3\tau_2\equiv 3\tau_1+3\tau_2\equiv 2\tau_2$ (mod 1)
we have only one equivariance class by 1. (0,0,0) the trivial solution. This is because the congruences above, and $\tau_3\equiv 0$ to $\overline{g}$ can be achieved by changing the origin of $\mathbf{R}$.

6.3. Type III

$G'G:=\{G\times 1_{\mathbf{R}}\}\cup\{(G'\setminus G)\times\overline{1}_{\mathbf{R}}\}$
First we must find the possible pairs of spherical groups, so that the larger group contains the smaller one of index two. We have finitely many types of candidates, and this makes our work easier.
In the finite cases 8-14, we consider the fundamental domains of the spherical groups G' and G such that a domain of G' is the half part of a domain of G selected by an additional generator of G'. Then we combine this generator with the $\mathbf{R}$-reflection to obtain the group $G'G$ and its fundamental domain. The possible pairs of groups: 9-8, 11-8, 12-9, 12-11, 12-14, 13-10 and 14-8 (see Table 1).
In the series 1q-7q the parity of q plays an important role. The possible group pairs are the following:
1qe-1$\frac{q}{2}$; 2q-1q, 2qe-2$\frac{q}{2}$; 3q-1q, 3qe-3$\frac{q}{2}$; 4q-2q, 4q-3q, 4q-5q, 4qe-4$\frac{q}{2}$, 4qe-6$\frac{q}{2}$; 5q-1q, 5qe-5$\frac{q}{2}$,

5qe-7$\frac{q}{2}$; 6q-2q, 6q-3q, 6q-7q; 7q-1q. (See our Table 1 in the Introduction.) Considering the Macbeath signature and the equation (2) in the Introduction, for $\mathbf{S}^2$ it is easy to calculate that the larger group always has a half combinatorial measure as the smaller one, and in general this is a necessary condition for pairs of groups in Type III. We discuss in details the cases **5q.III** and **13.III**.

- **5q.III.a.** $(+,0;[q];\{(1)\})'(+,0;[q];\{\})$
  This point group
  $$\Gamma_0 := (g_1, \overline{g}_2 - g_1^q, \overline{g}_2^2, (g_1\overline{g}_2g_1^{-1}\overline{g}_2))$$
  is generated by the rotation $g_1 \in \mathbf{S}^2_2$ of order q and by $\overline{g}_2 \in \mathbf{S}^2_1\mathbf{R}_1$.
  The conditions for (g_1, τ_1):
  $0 \equiv q\tau_1 \equiv 2\tau_1 \pmod 1$, to $(\overline{g}_2, \tau_2)$ we may choose $\tau_2 \equiv 0 \pmod 1$.
  We get the trivial solution for every q: 1. $(\tau_1, \tau_2) \equiv (0, 0)$; and for even q 2. $(\frac{1}{2}, 0)$.

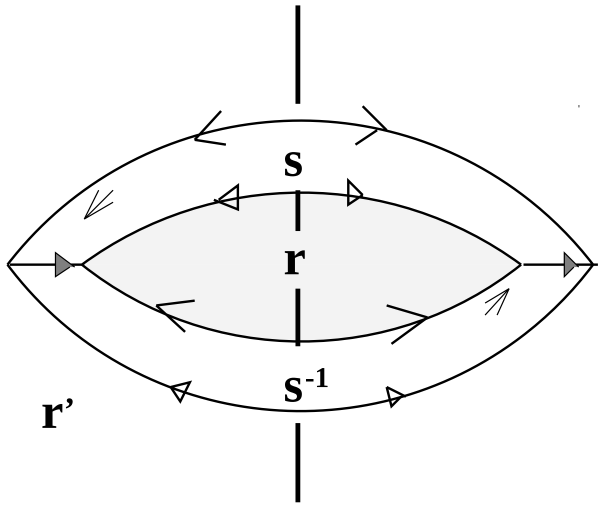

Figure 3: $\mathbf{\Gamma_{5qe.III.a.2} = (r, r', s - r^2, (r')^2, (ss)^{\frac{q}{2}}, rsr's^{-1}, rs^{-1}r's)}$
Here $s = (\overline{g}_2, 0)(g_1, \frac{1}{2})$, $s^{-1} = (g_1^{-1}, -\frac{1}{2})(\overline{g}_2, 0)$,
$r = (\overline{g}_2, , 0)$, $r' = (\overline{g}_2, 1)$.

- **5qe.III.b.** $(+,0;[q];\{(1)\})'(+,0;[\frac{q}{2}];\{(1)\})$
  The point group
  $$\Gamma_0 := (g_1, \overline{g}_2 - g_1^2, (\overline{g}_2\overline{g}_2)^{\frac{q}{2}}, (\overline{g}_2^{-1}g_1\overline{g}_2g_1))$$
  is generated by the rotatory-reflection $\overline{g}_2 \in \mathbf{S}^2_2\mathbf{R}_1$ of order q and by $g_1 \in \mathbf{S}^2_1$.
  The congruences are:
  $0 \equiv 2\tau_1 \equiv 2\tau_2 \pmod 1$.
  The solutions: 1. $(\tau_1, \tau_2) \equiv (0, 0)$; 1.' $(0, \frac{1}{2})$, 2. $(\frac{1}{2}, 0)$; 2.' $(\frac{1}{2}, \frac{1}{2})$ come up, but 1-1' and 2-2' lie in the same equivariance class by the translation $\varphi = (e, -\frac{1}{4})$: $\varphi^{-1}(\overline{g}_2, \frac{1}{2})\varphi = (\overline{g}_2, \frac{1}{4})(e, -\frac{1}{4}) = (\overline{g}_2, 0)$, and in the same way 2-2'.
- **5qe.III.c.** $(+,0;[q];\{(1)\})'(-,1;[\frac{q}{2}];\{\})$
  Our point group
  $$\Gamma_0 := (g_1, \overline{g}_2 - (g_1g_1)^{\frac{q}{2}}, \overline{g}_2^2, \overline{g}_2g_1^{-1}\overline{g}_2g_1)$$

is generated by $g_1 \in \mathbf{S}_3^2$ of order q and by $\overline{g}_2 \in \mathbf{S}_1^2\mathbf{R}_1$.
We obtain the following congruences:
$0 \equiv q\tau_1 \equiv 2\tau_1 \pmod 1$, to $(\overline{g}_2, \tau_2)$ we may choose $\tau_2 \equiv 0$.
The solutions: 1. $(\tau_1, \tau_2) \equiv (0, 0)$, 2. $(\frac{1}{2}, 0)$.

- **13.III.**$(+, 0; [\,]; \{(2, 3, 5)\})'(+, 0; [2, 3, 5]; \{\})$
  This point group

$$\Gamma_0 := (g_1, g_2, \overline{g}_3 - g_1^2, g_2^3, \overline{g}_3^2, (g_1 g_2)^5, (g_1 \overline{g}_3)^2, (g_2 \overline{g}_3)^2)$$

  is generated by the rotations $g_1, g_2 \in \mathbf{S}_2^2$ and by $\overline{g}_3 \in \mathbf{S}_1^2\mathbf{R}_1$.
  The conditions from the relations of the point group Γ_0 are
  $0 \equiv 2\tau_1 \equiv 3\tau_2 \equiv 5\tau_1 + 5\tau_2 \pmod 1$, we may choose $\tau_3 \equiv 0$ to $(\overline{g}_3, \tau_3)$.
  We obtain only the trivial solution $(\tau_1, \tau_2, \tau_3) \equiv (0, 0, 0)$.

7. List of space groups

In this section we give the complete list of all equivariance classes of space groups of $\mathbf{S}^2 \times \mathbf{R}$ in a short form in Table 2. First we give the number of the point group and the symbol of the Macbeath signature (see Table 1), then the point group Γ_0 defined by the generators and relations, the generators are represented by Section 2, and at the end follow the symbols of the equivariance classes with the translation parts corresponding to the generators of Γ_0, here we follow a lexicographic order.

Table 2

1q.I $(+, 0; [q, q]; \{\})$, $q \geq 1$
- $\Gamma_0 = (g_1 - g_1^q)$, $g_1 \in \mathbf{S}_2^2$
- **1q.I.1**(0); **1q.I.2**$(\frac{k}{q})$ $k := 1, \ldots, \lfloor \frac{q}{2} \rfloor$ (i.e. lower integer part of $\frac{q}{2}$)

Γ is fixed point free if for g.c.d. $(k, q) = 1$, then $(\mathbf{S}^2 \times \mathbf{R})/\Gamma$ is an orientable compact manifold (space form), for $q = 1$, $k = 0$ as well.

1q.II $\overline{(+, 0; [\, q, q]; \{\})}$, $q \geq 1$
- $\Gamma_0 = (g_1, \overline{g} - g_1^q, \overline{g}^2, (g_1^{-1} \overline{g} g_1 \overline{g}))$, $g_1 \in \mathbf{S}_2^2$, $\overline{g} \in \mathbf{R}_1$
- **1q.II.1**$(0, 0)$; $\circ$ if **q** is even **1qe.II.2**$(\frac{1}{2}, 0)$

1qe.III $(+, 0; [q, q]; \{\})'(+, 0; [\frac{q}{2}, \frac{q}{2}]\{\})$, $q \geq 2$
- $\Gamma_0 = (g_1, \overline{g}_2 - g_1^{\frac{q}{2}}, \overline{g}_2 \overline{g}_2 g_1^{-1})$, $g_1 \in \mathbf{S}_2^2$, $\overline{g}_2 \in \mathbf{S}_2^2\mathbf{R}_1$
- **1qe.III.1**$(0, 0)$

2q.I $(+, 0; [\,]\{(q, q)\})$, $q \geq 2$
- $\Gamma_0 = (g_1, g_2 - g_1^2, g_2^2, (g_1 g_2)^q)$, $g_1, g_2 \in \mathbf{S}_1^2$
- **2q.I.1**$(0, 0)$; **2q.I.2**$(\frac{1}{2}, \frac{1}{2})$; $\circ$ **2qe.I.3**$(0, \frac{1}{2})$

2q.II $\overline{(+, 0; [\,]\{(q, q)\})}$, $q \geq 2$
- $\Gamma_0 = (g_1, g_2, \overline{g} - g_1^2, g_2^2, \overline{g}^2, (g_1 g_2)^q, (g_1 \overline{g})^2, (g_2 \overline{g})^2)$, $g_1, g_2 \in \mathbf{S}_1^2$, $\overline{g} \in \mathbf{R}_1$
- **2q.II.1**$(0, 0, 0)$; **2q.II.2**$(\frac{1}{2}, \frac{1}{2}, 0)$; $\circ$ **2qe.II.3**$(0, \frac{1}{2}, 0)$

2q.III.a $(+,0;[\,];\{(q,q)\})'(+,0;[q,q];\{\}),\ q\geq 2$
- $\Gamma_0=(\overline{g}_1,g_2-\overline{g}_1^2,g_2^q,(g_2\overline{g}_1)^2),\ \overline{g}_1\in\mathbf{S}_1^2\mathbf{R}_1,\ g_2\in\mathbf{S}_2^2$
- **2q.III.a.1**$(0,0)$; **2q.III.a.2**$(0,\frac{k}{q}),\ k:=1,\ldots,\lfloor\frac{q}{2}\rfloor$

2qe.III.b $(+,0;[\,]\{(q,q)\})'(+,0;[];\{(\frac{q}{2},\frac{q}{2})\}),\ q\geq 2$
- $\Gamma_0=(\overline{g}_1,g_2-\overline{g}_1^2,g_2^2,(g_2\overline{g}_1g_2\overline{g}_1)^{\frac{q}{2}}),\ \overline{g}_1\in\mathbf{S}_1^2\mathbf{R}_1,\ g_2\in\mathbf{S}_1^2$
- **2qe.III.b.1**$(0,0)$; **2qe.III.b.2**$(0,\frac{1}{2})$

3q.I $(+,0;[2,2,q];\{\}),\ q\geq 2$
- $\Gamma_0=(g_1,g_2-g_1^2,g_2^2,(g_1g_2)^q),\ g_1,g_2\in\mathbf{S}_2^2$
- **3q.I.1**$(0,0)$; **3q.I.2**$(\frac{1}{2},\frac{1}{2})$; ∘ **3qe.I.3**$(0,\frac{1}{2})$

3q.II $\overline{(+,0;[2,2,q];\{\})},\ q\geq 2$
- $\Gamma_0=(g_1,g_2,\overline{g}-g_1^2,g_2^2,\overline{g}^2,(g_1g_2)^q,(g_1\overline{g})^2,(g_2\overline{g})^2),\ g_1,g_2\in\mathbf{S}_2^2,\ \overline{g}\in\mathbf{R}_1$
- **3q.II.1**$(0,0,0)$; **3q.II.2**$(\frac{1}{2},\frac{1}{2},0)$; ∘ **3qe.II.3**$(0,\frac{1}{2},0)$

3q.III.a $(+,0;[2,2,q];\{\})'(+,0;[q,q];\{\}),\ q\geq 2$
- $\Gamma_0=(g_1,\overline{g}_2-g_1^q,\overline{g}_2^2,(g_1\overline{g}_2)^2),\ g_1\in\mathbf{S}_2^2,\ \overline{g}_2\in\mathbf{S}_2^2\mathbf{R}_1$
- **3q.III.a.1**$(0,0)$; **3q.III.a.2**$(\frac{k}{q},0),\ k:=1,\ldots,\lfloor\frac{q}{2}\rfloor$

3qe.III.b $(+,0;[2,2,q];\{\})'(+,0;[2,2,\frac{q}{2}];\{\}),\ q\geq 2$
- $\Gamma_0=(g_1,\overline{g}_2-g_1^2,\overline{g}_2^2,(g_1\overline{g}_2g_1\overline{g}_2)^{\frac{q}{2}}),\ g_1\in\mathbf{S}_2^2,\ \overline{g}_2\in\mathbf{S}_2^2\mathbf{R}_1$
- **3qe.III.b.1**$(0,0)$; **3qe.III.b.2**$(\frac{1}{2},0)$

4q.I $(+,0;[];\{(2,2,q)\}),\ q\geq 2$
- $\Gamma_0=(g_1,g_2,g_3-g_1^2,g_2^2,g_3^2-(g_1g_3)^2,(g_2g_3)^2,(g_1g_2)^q),\ g_1,g_2,g_3\in\mathbf{S}_1^2$
- **4q.I.1**$(0,0,0)$; **4q.I.2**$(0,0,\frac{1}{2})$; **4q.I.3**$(\frac{1}{2},\frac{1}{2},0)$; **4q.I.4**$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$;

∘ **4qe.I.5**$(0,\frac{1}{2},0)$; **4qe.I.6**$(0,\frac{1}{2},\frac{1}{2})$

4q.II $\overline{(+,0;[];\{(2,2,q)\})},\ q\geq 2$
- $\Gamma_0=(g_1,g_2,g_3,\overline{g}-g_1^2,g_2^2,g_3^2,\overline{g}^2,(g_1g_3)^2,(g_2g_3)^2,(g_1g_2)^q,(g_1\overline{g})^2,(g_2,\overline{g})^2,(g_3\overline{g})^2),\ g_1,g_2,g_3\in\mathbf{S}_1^2,\ \overline{g}\in\mathbf{R}_1$
- **4q.II.1**$(0,0,0,0)$; **4q.II.2**$(0,0,\frac{1}{2},0)$; **4q.II.3**$(\frac{1}{2},\frac{1}{2},0,0)$; **4q.II.4**$(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$;

∘ **4qe.II.5**$(0,\frac{1}{2},0,0)$; **4qe.II.6**$(0,\frac{1}{2},\frac{1}{2},0)$

4q.III.a $(+,0;[];\{(2,2,q)\})'(+,0;[];\{(q,q)\}),\ q\geq 2$
- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^2,\overline{g}_3^2,(g_1g_2)^q,(g_1\overline{g}_3)^2,(g_2\overline{g}_3)^2),\ g_1,g_2\in\mathbf{S}_1^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **4q.III.a.1**$(0,0,0)$; **4q.III.a.2**$(\frac{1}{2},\frac{1}{2},0)$; ∘ **4qe.III.a.3**$(0,\frac{1}{2},0)$

4q.III.b $(+,0;[];\{(2,2,q)\})'(+,0;[2,2,q];\{\}),\ q\geq 2$
- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^2,\overline{g}_3^2,(g_1g_2)^q,(g_1\overline{g}_3)^2,(g_2\overline{g}_3)^2),\ g_1,g_2\in\mathbf{S}_2^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **4q.III.b.1**$(0,0,0)$; **4q.III.b.2**$(\frac{1}{2},\frac{1}{2},0)$; ∘ **4qe.III.b.3**$(0,\frac{1}{2},0)$

4q.III.c $(+,0;[];\{(2,2,q)\})'(+,0;[q];\{(1)\}),\ q\geq 2$
- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^q,\overline{g}_3^2,g_1g_2g_1g_2^{-1},(g_1\overline{g}_3)^2,(g_2\overline{g}_3)^2),\ g_1\in\mathbf{S}_1^2,g_2\in\mathbf{S}_2^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **4q.III.c.1**$(0,0,0)$; **4q.III.c.2**$(0,\frac{k}{q},0)$ $k:=1,\ldots,\lfloor\frac{q}{2}\rfloor$; **4q.III.c.3**$(\frac{1}{2},0,0)$;

4q.III.c.4$(\frac{1}{2},\frac{k}{q},0),\ k:=1,\ldots,\lfloor\frac{q}{2}\rfloor$

4qe.III.d $(+,0;[];\{(2,2,q)\})'(+,0;[];\{(2,2,\frac{q}{2})\}),\ q\geq 2$
- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^2,\overline{g}_3^2,(g_1g_2)^2,(g_1\overline{g}_3)^2,(g_2\overline{g}_3g_2\overline{g}_3)^{\frac{q}{2}}),\ g_1,g_2\in\mathbf{S}_1^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **4qe.III.d.1**$(0,0,0)$, **4qe.III.d.2**$(0,\frac{1}{2},0)$; **4qe.III.d.3**$(\frac{1}{2},0,0)$; **4qe.III.d.4**$(\frac{1}{2},\frac{1}{2},0)$

4qe.III.e $(+,0;[];\{(2,2,q)\})'(+,0;[2];\{(\frac{q}{2})\}),\ q\geq 2$
- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^2,\overline{g}_3^2,(g_1g_2)^{\frac{q}{2}},(g_1\overline{g}_3)^2,(g_2\overline{g}_3)^2),\ g_1\in\mathbf{S}_1^2,\ g_2\in\mathbf{S}_2^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **4qe.III.e.1**$(0,0,0)$; **4qe.III.e.2**$(\frac{1}{2},\frac{1}{2},0)$;
- ◦ if q is divisible by four **4qf.III.e.3**$(0,\frac{1}{2},0)$; **4qf.III.e.4**$(\frac{1}{2},0,0)$.

5q.I $(+,0;[q];\{(1)\}),\ q\geq 1$
- $\Gamma_0=(g_1,g_2-g_1^2,g_2^q,(g_1g_2g_1g_2^{-1}),\ g_1\in\mathbf{S}_1^2,\ g_2\in\mathbf{S}_2^2$
- **5q.I.1**$(0,0)$; **5q.I.2**$(0,\frac{k}{q}),\ k:=1,\dots,\lfloor\frac{q}{2}\rfloor$; **5q.I.3**$(\frac{1}{2},0)$; **5q.I.4**$(\frac{1}{2},\frac{k}{q}),\ k:=1,\dots,\lfloor\frac{q}{2}\rfloor$.

This Γ is fixed point free iff $(k,q)=1$, then $\mathbf{S}^2\times\mathbf{R}/\Gamma$ is a nonorientable compact manifold(space form).

5q.II $\overline{(+,0;[q];\{(1)\})},\ q\geq 1$
- $\Gamma_0=(g_1,g_2,\overline{g}-g_1^2,g_2^q,\overline{g}^2,(g_1g_2g_1g_2^{-1}),(g_1\overline{g})^2,(g_2\overline{g}g_2^{-1}\overline{g})),\ g_1\in\mathbf{S}_1^2,\ g_2\in\mathbf{S}_2^2,\ \overline{g}\in\mathbf{R}_1$
- **5q.II.1**$(0,0,0)$; **5q.II.2**$(\frac{1}{2},0,0)$; **5qe.II.3**$(0,\frac{1}{2},0)$; **5qe.II.4**$(\frac{1}{2},\frac{1}{2},0)$

5q.III.a $(+,0;[q];\{(1)\})'(+,0;[q];\{\}),\ q\geq 1$
- $\Gamma_0=(g_1,\overline{g}_2-g_1^q,\overline{g}_2^2,(g_1\overline{g}_2g_1^{-1}\overline{g}_2)),\ g_1\in\mathbf{S}_2^2,\ \overline{g}_2\in\mathbf{S}_1^2\mathbf{R}_1$
- **5q.III.a.1**$(0,0)$; ◦ **5qe.III.a.2**$(\frac{1}{2},0)$

5qe.III.b $(+,0;[q];\{(1)\})'(+,0;[\frac{q}{2}];\{(1)\}),\ q\geq 2$
- $\Gamma_0=(g_1,\overline{g}_2-g_1^2,(\overline{g}_2\overline{g}_2)^{\frac{q}{2}},(\overline{g}_2^{-1}g_1\overline{g}_2g_1)),\ g_1\in\mathbf{S}_1^2,\ \overline{g}_2\in\mathbf{S}_2^2\mathbf{R}_1$
- **5qe.III.b.1**$(0,0)$; ◦ **5qe.III.b.2**$(\frac{1}{2},0)$

5qe.III.c $(+,0;[q];\{(1)\})'(-,1;[\frac{q}{2}];\{\}),\ q\geq 2$
- $\Gamma_0=(g_1,\overline{g}_2-(g_1g_1)^{\frac{q}{2}},\overline{g}_2^2,(\overline{g}_2g_1^{-1}\overline{g}_2g_1)),\ g_1\in\mathbf{S}_3^2,\ \overline{g}_2\in\mathbf{S}_1^2\mathbf{R}_1$
- **5qe.III.c.1**$(0,0)$; **5qe.III.c.2**$(\frac{1}{2},0)$

6q.I $(+,0;[2];\{(q)\}),\ q\geq 2$
- $\Gamma_0=(g_1,g_2-g_1^2,g_2^2,(g_1g_2g_1g_2)^q),\ g_1\in\mathbf{S}_1^2,\ g_2\in\mathbf{S}_2^2$
- **6q.I.1**$(0,0)$; **6q.I.2**$(0,\frac{1}{2})$; **6q.I.3**$(\frac{1}{2},0)$; **6q.I.4**$(\frac{1}{2},\frac{1}{2})$

6q.II $\overline{(+,0;[2];\{(q)\})},\ q\geq 2$
- $\Gamma_0=(g_1,g_2,\overline{g}-g_1^2,g_2^2,\overline{g}^2,(g_1g_2g_1g_2)^q,(g_1\overline{g})^2,(g_2\overline{g})^2),\ g_1\in\mathbf{S}_1^2,\ g_2\in\mathbf{S}_2^2,\ \overline{g}\in\mathbf{R}_1$
- **6q.II.1**$(0,0,0)$; **6q.II.2**$(0,\frac{1}{2},0)$; **6q.II.3**$(\frac{1}{2},0,0)$; **6q.II.4**$(\frac{1}{2},\frac{1}{2},0)$

6q.III.a $(+,0;[2];\{(q)\})'(+,0;[];\{(q,q)\}),\ q\geq 2$
- $\Gamma_0=(g_1,\overline{g}_2-g_1^2,\overline{g}_2^2,(g_1\overline{g}_2g_1\overline{g}_2)^q),\ g_1\in\mathbf{S}_1^2,\ \overline{g}_2\in\mathbf{S}_2^2\mathbf{R}_1$
- **6q.III.a.1**$(0,0)$; **6q.III.a.2**$(\frac{1}{2},0)$

6q.III.b $(+,0;[2];\{(q)\})'(+,0;[2,2,q];\{\}),\ q\geq 2$
- $\Gamma_0=(g_1,\overline{g}_2-g_1^2,\overline{g}_2^2,(g_1\overline{g}_2g_1\overline{g}_2)^q),\ g_1\in\mathbf{S}_2^2,\overline{g}_2\in\mathbf{S}_1^2\mathbf{R}_1$
- **6q.III.b.1**$(0,0)$; **6q.III.b.2**$(\frac{1}{2},0)$

6q.III.c $(+,0;[2];\{(q)\})'(-,1;[q];\{\}),\ q\geq 2$
- $\Gamma_0=(g_1,\overline{g}_2-(g_1g_1)^q,\overline{g}_2^2,(g_1\overline{g}_2)^2),\ g_1\in\mathbf{S}_3^2,\ \overline{g}_2\in\mathbf{S}_1^2\mathbf{R}_1$
- **6q.III.c.1**$(0,0)$; **6q.III.c.2**$(\frac{k}{2q},0),\ k:=1,\dots,\lfloor\frac{q}{2}\rfloor$

7q.I $(-,0;[q];\{\}),\ q\geq 1$
- $\Gamma_0=(g_1-(g_1g_1)^q),\ g_1\in\mathbf{S}^2_3$
- **7q.I.1**(0); **7q.I.2**$(\frac{1}{2})$; **7q.I.3**$(\frac{k}{2q}),\ q\geq 2,\ k:=1,\dots,q-1$. Γ is fixed point free iff $(k,q)=1$, e.g. $q=1,\ k=0$ and $k=1$ as well, then $\mathbf{S}^2\times\mathbf{R}/\Gamma$ is a nonorientable manifold (space form).

7q.II $\overline{(-,0;[q];\{\})},\ q\geq 1$
- $\Gamma_0=(g_1,\overline{g}-(g_1g_1)^q,\overline{g}^2,(g_1\overline{g}g_1^{-1}\overline{g}_2)),\ g_1\in\mathbf{S}^2_3,\ \overline{g}\in\mathbf{R}_1$
- **7q.II.1**$(0,0)$; **7q.II.2**$(\frac{1}{2},0)$

7q.III $(-,0;[q];\{\})'(+,0;[q];\{\}),\ q\geq 1$
- $\Gamma_0=(g_1,\overline{g}_2-g_1^q,(\overline{g}_2\overline{g}_2g_1^{-1}),\ g_1\in\mathbf{S}^2_2,\ \overline{g}_2\in\mathbf{S}^2_3\mathbf{R}_1$
- **7q.III.1**$(0,0)$. Γ is fixed point free iff $q=1$. Then $\mathbf{S}^2\times\mathbf{R}/\Gamma$ is an orientable compact manifold (space form).

8.I $(+,0;[2,3,3];\{\})$
- $\Gamma_0=(g_1,g_2-g_1^2,g_2^3,(g_1g_2)^3),\ g_1,g_2\in\mathbf{S}^2_2$
- **8.I.1**$(0,0)$; **8.I.2**$(0,\frac{1}{3})$

8.II $\overline{(+,0;[2,3,3];\{\})}$
- $\Gamma_0=(g_1,g_2,\overline{g}-g_1^2,g_2^3,(g_1g_2)^3,\overline{g}^2,(g_1\overline{g}g_1\overline{g}),(g_2\overline{g}g_2^{-1}\overline{g})),\ g_1,g_2\in\mathbf{S}^2_2,\ \overline{g}\in\mathbf{R}_1$
- **8.II.1**$(0,0,0)$

9.I $(+,0;[2,3,4];\{\})$
- $\Gamma_0=(g_1,g_2-g_1^2,g_2^3,(g_1g_2)^4),\ g_1,g_2\in\mathbf{S}^2_2$
- **9.I.1**$(0,0)$; **9.I.2**$(\frac{1}{2},0)$

9.II $\overline{(+,0;[2,3,4];\{\})}$
- $\Gamma_0=(g_1,g_2,\overline{g}-g_1^2,g_2^3,(g_1g_2)^4,\overline{g}_2,(g_1\overline{g}g_1^{-1}\overline{g}),(g_2\overline{g}g_2^{-1}\overline{g})),\ g_1,g_2\in\mathbf{S}^2_2,\ \overline{g}\in\mathbf{R}_1$
- **9.II.1**$(0,0,0)$; **9.II.2**$(\frac{1}{2},0,0)$

9.III $(+,0;[2,3,4];\{\})'(+,0;[2,3,3];\{\})$
- $\Gamma_0=(g_1,g_2,\overline{g}_3,-g_1^2,g_2^3,(g_1g_2)^3,\overline{g}_3^4,(g_1\overline{g}_3^2),(g_2\overline{g}_3)^2),\ g_1,g_2\in\mathbf{S}^2_2,\ \overline{g}_3\in\mathbf{S}^2_2\mathbf{R}_1$
- **9.III.1**$(0,0,0)$; **9.III.2**$(0,\frac{1}{3},0)$

10.I $(+,0;[2,3,5];\{\})$
- $\Gamma_0=(g_1,g_2-g_1^2,g_2^3,(g_1g_2)^5),\ g_1,g_2\in\mathbf{S}^2_2$
- **10.I.1**$(0,0)$

10.II $\overline{(+,0;[2,3,5];\{\})}$
- $\Gamma_0=(g_1,g_2,\overline{g}-g_1^2,g_2^3,(g_1g_2)^5,\overline{g}_2,(g_1\overline{g}g_1\overline{g}),(g_2\overline{g}g_2^{-1}\overline{g})),\ g_1,g_2\in\mathbf{S}^2_2,\ \overline{g}\in\mathbf{R}_1$
- **10.II.1**$(0,0,0)$

11.I $(+,0;[];\{(2,3,3)\})$
- $\Gamma_0=(g_1,g_2,g_3-g_1^2,g_2^2,g_3^2,(g_1g_2)^2,(g_1g_3)^3,(g_2g_3)^3),\ g_1,g_2,g_3\in\mathbf{S}^2_1$
- **11.I.1**$(0,0,0)$; **11.I.2**$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

11.II $\overline{(+,0;[];\{(2,3,3)\})}$
- $\Gamma_0=(g_1,g_2,g_3,\overline{g}-g_1^2,g_2^2,g_3^2,(g_1g_2)^2,(g_1g_3)^3,(g_2g_3)^3,\overline{g}^2,(g_1\overline{g})^2,(g_2\overline{g})^2,(g_3\overline{g})^2),\ g_1,g_2,g_3\in$

$\mathbf{S}_1^2,\ \overline{g}\in\mathbf{R}_1$

- **11.II.1**$(0,0,0,0)$; **11.II.2**$(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$

11.III $(+,0;[];\{(2,3,3)\})'(+,0;[2,3,3];\{\})$

- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^3,(g_1g_2)^3,\overline{g}_3^2,(g_1\overline{g}_3)^2,(g_2\overline{g}_3)^2),\ g_1,g_2\in\mathbf{S}_2^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **11.III.1**$(0,0,0)$, **11.III.2**$(0,\frac{1}{3},0)$

12.I $(+,0;[];\{(2,3,4)\})$

- $\Gamma_0=(g_1,g_2,g_3-g_1^2,g_2^2,g_3^2,(g_1g_2)^2,(g_1g_3)^3,(g_2g_3)^4),\ g_1,g_2,g_3\in\mathbf{S}_1^2$
- **12.I.1**$(0,0,0)$; **12.I.2**$(0,\frac{1}{2},0)$; **12.I.3**$(\frac{1}{2},0,\frac{1}{2})$; **12.I.4**$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

12.II $\overline{(+,0;[];\{(2,3,4)\})}$

- $\Gamma_0=(g_1,g_2,g_3,\overline{g}-g_1^2,g_2^2,g_3^2,(g_1g_2)^2,(g_1g_3)^3,(g_2g_3)^4,\overline{g}^2,(g_1\overline{g})^2,(g_2\overline{g})^2,(g_3\overline{g})^2),\ g_1,g_2,g_3\in\mathbf{S}_1^2,\ \overline{g}\in\mathbf{R}_1$
- **12.II.1**$(0,0,0,0)$; **12.II.2**$(0,\frac{1}{2},0,0)$; **12.II.3**$(\frac{1}{2},0,\frac{1}{2},0)$; **12.II.4**$(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$

12.III.a $(+,0;[];\{(2,3,4)\})'(+,0:[2,3,4];\{\})$

- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^3,(g_1g_2)^4,\overline{g}_3^2,(g_1\overline{g}_3g_1\overline{g}_3),(g_2\overline{g}_3g_2\overline{g}_3)),\ g_1,g_2\in\mathbf{S}_2^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **12.III.a.1**$(0,0,0)$; **12.III.a.2**$(\frac{1}{2},0,0,0)$

12.III.b $(+,0;[];\{(2,3,4)\})'(+,0;[];\{(2,3,3)\})$

- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^2,(g_1g_2)^3,\overline{g}_3^2,(g_1\overline{g}_3)^2,(g_2\overline{g}_3)^4),\ g_1,g_2\in\mathbf{S}_1^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **12.III.b.1**$(0,0,0)$; **12.III.b.2**$(\frac{1}{2},\frac{1}{2},0)$

12.III.c $(+,0;[];\{(2,3,4)\})'(+,0:[3];\{(2)\})$ • $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^3,(g_1g_2^{-1}g_1g_2)^2,\overline{g}_3^2,(g_1\overline{g}_3)^2,(g_2\overline{g}_3)^2),g_1\in\mathbf{S}_1^2,g_2\in\mathbf{S}_2^2,\overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$ • **12.III.c.1**$(0,0,0)$; **12.III.c.2**$(0,\frac{1}{3},0)$; **12.III.c.3**$(\frac{1}{2},0,0)$; **12.III.c.4**$(\frac{1}{2},\frac{1}{3},0)$

13.I $(+,0;[];\{(2,3,5)\})$

- $\Gamma_0=(g_1,g_2,g_3-g_1^2,g_2^2,g_3^2,(g_1g_2)^2,(g_1g_3)^3,(g_2g_3)^5),\ g_1,g_2,g_3\in\mathbf{S}_1^2$
- **13.I.1**$(0,0,0)$; **13.I.2**$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$

13.II $\overline{(+,0;[];\{(2,3,5)\})}$

- $\Gamma_0=(g_1,g_2,g_3,\overline{g}-g_1^2,g_2^2,g_3^2,(g_1g_2)^2,(g_1g_3)^3,(g_2g_3)^5,\overline{g}^2,(g_1\overline{g})^2,(g_2\overline{g})^2,(g_3\overline{g})^2),\ g_1,g_2,g_3\in\mathbf{S}_1^2,\ \overline{g}\in\mathbf{R}_1$
- **13.II.1**$(0,0,0,0)$; **13.II.2**$(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$

13.III $(+,0;[];\{2,3,5\})'(+,0;[2,3,5];\{\})$

- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^2,g_2^3,(g_1g_2)^5,\overline{g}_3^2,(g_1\overline{g}_3)^2,(g_2\overline{g}_3)^2),\ g_1,g_2\in\mathbf{S}_2^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **13.III.1**$(0,0,0)$

14.I $(+,0;[3];\{(2)\})$

- $\Gamma_0=(g_1,g_2-g_1^2,g_2^3,(g_2g_1g_2^{-1}g_1)^2),\ g_1\in\mathbf{S}_1^2,\ g_2\in\mathbf{S}_2^2$
- **14.I.1**$(0,0)$; **14.I.2**$(0,\frac{1}{3})$; **14.I.3**$(\frac{1}{2},0)$; **14.I.4**$(\frac{1}{2},\frac{1}{3})$

14.II $\overline{(+,0;[3];\{(2)\})}$

- $\Gamma_0=(g_1,g_2,\overline{g}-g_1^2,g_2^3,(g_2g_1g_2^{-1}g_1)^2,\overline{g}^2,(g_1\overline{g})^2,(g_2\overline{g}g_2^{-1}\overline{g})),\ g_1\in\mathbf{S}_1^2,\ g_2\in\mathbf{S}_2^2,\ \overline{g}\in\mathbf{R}_1$
- **14.II.1**$(0,0,0)$; **14.II.2**$(\frac{1}{2},0,0)$

14.III $(+,0;[3];\{(2)\})'(+,0;[2,3,3];\{\})$
- $\Gamma_0=(g_1,g_2,\overline{g}_3-g_1^3,g_2^3,(g_1g_2)^2,\overline{g}_3^2,(g_1\overline{g}_3g_2\overline{g}_3),(g_1g_2\overline{g}_3)^2),\ g_1,g_2\in\mathbf{S}_2^2,\ \overline{g}_3\in\mathbf{S}_1^2\mathbf{R}_1$
- **14.III.1**$(0,0,0)$

Now we can summarize our results:

Theorem 2. *In $\mathbf{S}^2\times\mathbf{R}$ there are infinitely many equivariance classes of space groups, all listed in our Table 2 without repetition (hopely without human error). Our statistics can be found in Table 3. Table 4 lists the $\mathbf{S}^2\times\mathbf{R}$ space forms.*

Table 3. Classes of $\mathbf{S}^2\times\mathbf{R}$ point groups ★ space groups

Type	infinite series 1-7 by q	finite cases 8-14
I	1|1|1|1|1|1|1 ★ 2|3|3|6|4|4|3	1|1|1|1|1|1|1 ★ 2|2|1|2|4|2|4
II	1|1|1|1|1|1|1 ★ 2|3|3|6|4|4|2	1|1|1|1|1|1|1 ★ 1|2|1|2|4|2|2
III	1|2|2|5|3|3|1 ★ 1|2,2|2,2|3,3,4,4,4|2,2,2|2,2,2|1	0|1|0|1|3|1|1 ★ 0|2|0|2|2,2,4|1|1
sums by types	7+7+17 ★ 25+24+40	7+7+7 ★ 17+14+14
all	31 ★ 89	21 ★ 45

Table 4. Similarity classes of $\mathbf{S}^2\times\mathbf{R}$ space forms

Symbol	Condition	Orientability
1q.I.$(\frac{k}{q})$	$(k,q)=1,\ k\le\lfloor\frac{q}{2}\rfloor$	orientable
5q.I.$(\frac{1}{2},\frac{k}{q})$	$(k,q)=1,\ k\le\lfloor\frac{q}{2}\rfloor$	nonorientable
7q.I.1(0)	$q=1$	nonorientable
7q.I.2$(\frac{1}{2})$	$q=1$	nonorientable
7q.I.3$(\frac{k}{2q})$	$q\ge 2,\ (k,q)=1$	nonorientable
7q.III	$q=1$	orientable

Remarks. As we previously promised we give an example, which illustrates an other possible definition of equivalence of space groups.
The class **1q.I.2**$(\frac{k}{q})$ with $(k,q)=1$ are diffeomorphic with the class **1q.I.1**(0) for $q=1$. Now we introduce the usual coordinates in the $\mathbf{S}^2$ factor: $-\pi<\varphi\le\pi\pmod{2\pi}$; $-\frac{\pi}{2}\le\vartheta\le\frac{\pi}{2}$ (the $\varphi=\pi$ longitude and the poles $\vartheta=\frac{\pi}{2}$, $\vartheta=-\frac{\pi}{2}$ are considered obviously), and let $-\infty<y<\infty$. Thus the points of $\mathbf{S}^2\times\mathbf{R}$ will be expressed in the form: (φ,ϑ,y). Now $T(\tau)\ \tau>0\ :\ (\varphi,\vartheta,y)\mapsto(\varphi,\vartheta,y+\tau)$ generates the transformation group, which is equivariant by an $\mathbf{R}$-similarity to **1q.I.1**(0) in case $q=1$ with $\tau=1$; and the screw motion $A(\alpha,a)$; $0<\alpha<2\pi,\ 0<a\ :\ (\varphi,\vartheta,y)\mapsto(\varphi+\alpha,\vartheta,y+a)$ generates another group, in general. We define the diffeomorphism $\Phi:(\varphi,\vartheta,y)\mapsto(\varphi+\alpha y,\vartheta,ya)$. Then $\Phi^{-1}T\Phi=A$ holds, because $(\varphi,\vartheta,y)\mapsto(\varphi-\frac{\alpha}{a}y,\vartheta,\frac{y}{a})\mapsto(\varphi-\frac{\alpha}{a}y,\vartheta,\frac{y}{a}+1)\mapsto(\varphi-\frac{\alpha}{a}y+\frac{\alpha y}{a}+\alpha,\vartheta,y+a)=(\varphi+\alpha,\vartheta,y+a)$. That means, the isomorphism is geometrized by a diffeomorphism, and we obtain only 2 diffeomorphism classes of orientable space forms: the first is generated by $T(\tau=1)$ above

(with $H_1 = \mathbf{Z}$ as first homology group and fundamental group as well), the second is by $\mathbf{7q}.\mathbf{III}$ $q = 1$ (with $H_1 = \mathbf{Z}_2 \times \mathbf{Z}_2$). Thurston's statement in [8] that we have only one diffeomorphism class of nonorientable space forms, seems to be erroneous. Our case $\mathbf{7q}.\mathbf{I}.\mathbf{1}(0)$ for $q = 1$ yields the homology group $H_1 = \mathbf{Z}_2 \times \mathbf{Z}$, while $\mathbf{7q}.\mathbf{I}.\mathbf{2}(\frac{1}{2})$, $q = 1$ leads to $H_1 = \mathbf{Z}$. The other nonorientable space forms in Table 4 will be diffeomorphic to one of these two, similarly as in our last Remarks.

We with E.Molnár intend to come back to the diffeomorphism classification and to the orbifold interpretation in a forthcoming paper. A figure collection will be available for special request from the author.

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