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THE FIRST ISOMORPHISM THEOREM OF IMPLICATIVE SEMIGROUP WITH APARTNESS

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Abstract. Implicative semigroups with apartness have been introduced in 2016 by this author who then analyzed them in several papers. Here the author continues his research on such semigroups dealing with co-congruences. In addition, he constructs two quotient structures of these semigroups, one of which has no counterpart in the classical theory of implicative semigroups. Additionally, one form of the First Isomorphism Theorem for this type of semigroups is shown, which can be seen as an extension of corresponding theorem in the classical Semigroup Theory.

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1. Introduction

The notion of implicative semigroup was introduced by Chan and Shum [7]. Ordered ideals and filters play an important role in the theory of implicative semilattices. For this reason Chan and Shum [7] established some elementary properties and constructed a quotient structure of implicative semigroups via ordered filters, whereas Jun and Kim [10].

In the setting of Bishop's constructive mathematics, the notion of implicative semigroups with tight apartness was introduced in [14], following the ideas of above mentioned authors, and some fundamental characterizations of these semigroups were given. Then the author continued his research on implicative semigroups with apartness using co-order relations instead of partial order. In particular strongly extensional homomorphisms between implicative semigroups with apartness are discussed in [15]; co-filters and co-ideals in such semigroups were considered in [17, 18, 19]. An interested reader can find in paper [16] more information on the relations of co-quasiorder and co-order and their applications in algebraic structures having sets with apartness as carriers. This article continues the previous studies of the author focusing on co-congruences in implicative semigroups with apartness. In addition, two quotient structures of these semigroups are constructed, one of which has no counterpart in the classical theory of implicative semigroups. Additionally, one form of the First Isomorphism Theorem for this type of semigroups is shown, which can be seen as an extension of corresponding theorem in the classical Semigroup Theory.

2. Preliminaries

In this section, we recall from [6, 8, 9, 14, 15, 17, 18, 19] some concepts and processes necessary in the sequel of this paper and the reader is referred to [14, 15, 16, 17, 18, 19] for undefined notions and notations. This investigation is in Bishop's constructive algebra in the sense of papers [8, 9, 6, 16] and books [1, 2, 3, 4, 5, 11] and Chapter 8: Algebra of [20].

2.1. Set with apartness

Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation "$\neq$"). A diversity relation "$\neq$" satisfying conditions

$$\neg(x \neq x),\, x \neq y \Longrightarrow y \neq x,\, x \neq y \wedge y = z \Longrightarrow x \neq z$$

is called apartness. In this paper, we assume that the apartness is tight, i.e. it satisfies the following

$$(\forall x, y \in S)(\neg(x \neq y) \Longrightarrow x = y).$$

A subset X of S is called a strongly extensional subset of S if and only if $(\forall x \in X)(\forall y \in S)(x \neq y \vee y \in S)$. Let X, Y be subsets of S. According with Bridge and Vita definition (see for instance [5]), we say that X is set-set apartned from Y (denoted $X \bowtie Y$) if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$.

We set $x \lhd Y$ and $x \neq y$, instead of $\{x\} \bowtie Y$ and $\{x\} \bowtie \{y\}$ respectively. With $X^{\lhd} = \{x \in S : x \lhd X\}$ we denote the apartness complement of X. We say that a function $f : (S, =, \neq) \longrightarrow (T, =, \neq)$ is strongly extensional (an se-mapping, for short) if and only if ....

$$(\forall a, b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b).$$

f is an embedding if

$$(\forall a, b \in S)(a \neq b \implies f(a) \neq f(b))$$

holds.

We say that a relation $\alpha \subseteq S \times S$ S is a co-order relation on the semigroup S, if it fulfills the following properties

$(\forall x, y \in S)((x, y) \in \alpha \implies x \neq y)$ (consistency)

$(\forall x, y, z \in S)((x, z) \in \alpha \implies ((x, y) \in \alpha \vee (y, z) \in \alpha))$ (co-transitivity)

$(\forall x, y \in S)(x \neq y \implies ((x, y) \in \alpha \vee (y, z) \in \alpha))$ (linearity) and

compatibility with the product of S in the following sense

$(\forall x, y, z \in S)(((xz, yz) \in \alpha \vee (zx, zy) \in \alpha) \implies (x, y) \in \alpha)$.

2.2. Implicative semigroups with apartness

Let $((S, =, \neq), \cdot)$ be a semigroup with apartness, we recall we recall that its binary operation '$\cdot$' has to be extensional and strongly extensional, i.e $\cdot$ is a function from $S \times S$ into S such that

$$(\forall a, b, u, v \in S)((a, b) = (u, v) \implies ab = uv),$$
$$(\forall a, b, u, v \in S)(ab \neq uv \implies (a, b) \neq (u, v)).$$

By a *negatively co-ordered* semigroup (briefly, n.a-o. semigroup) we mean a semigroup with apartness S with a co-order α such that for all $x, y, z \in S$ the following hold:

(1) $(xy)z = x(yz)$,

(2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and

(3) $(xy, x) \lhd \alpha$ and $(xy, y) \lhd \alpha$.

In such case *alpha* we will be called a negative co-order relation on S.

A n.a-o. semigroup $(S, =, \neq, \cdot, \alpha)$ is said to be *implicative* if there is an additional binary operation $\otimes : S \times S \longrightarrow S$ such that the following is true

(4) $(z, x \otimes y) \in \alpha \Longleftrightarrow (zx, y) \in \alpha$

for any elements x, y, z of S.

In addition, let us recall that the internal binary operation '$\otimes$' must satisfy the following implications:

$$(\forall a, b, u, v \in S)((a, b) = (u, v) \implies a \otimes b = u \otimes v),$$
$$(\forall a, b, u, v \in S)(a \otimes b \neq u \otimes v \implies (a, b) \neq (u, v)).$$

The operation $\otimes$ is called *implication.* From now on, an implicative n.a-o. semigroup is simply called an *implicative semigroup.*

In any implicative semigroup S there exists a special element of S, the biggest element in $(S, \alpha^{\lhd})$, which is the left neutral element in $(S, \cdot)$.

Let $S = ((S, =, \neq), \cdot, \alpha, \otimes)$ and $T = ((T, =, \neq), \cdot, \beta, \otimes)$ be two implicative semigroups and let $f : S \longrightarrow T$ be a strongly extensional mapping from S in T. As the usual procedure in the construction of a mathematical system, for mapping f we say that it is a *homomorphism* ([15]) between implicative semigroups S and T if

$$(\forall x, y \in S)(f(x \otimes y) = f(x) \otimes f(y))$$

holds. In this case, f is said to be *se-homomorphism.* If f is surjective, then f is a semigroup homomorphism, that ([15], Theorem 3.1)

$$(\forall x, y \in S)(f(xy) = f(x)f(y))$$

is valid. A surjective se-homomorphism is *se-endomorphism* and an injective se-homomorphism is *se-monomorphism.*

At the end of this subsection, we repeat the two terms that will be used. The se-mapping $f : S \longrightarrow T$ between implicative semigroups is

- isotone if the following

$$(\forall x, y \in S)((x, y) \in \alpha \implies (f(x), f(y)) \in \beta)$$

is valid;

-reverse isotone if the following

$$(\forall x, y \in S)((f(x), f(y)) \in \beta \implies (x, y) \in \alpha)$$

holds.

3. The concept of co-congruences

In this section we introduce the concept of co-congruences on implicative semigroups with apartness and show some of its basic properties.

Let $(S, =, \neq)$ be a set with apartness. A relation q on S is a co-equality relation on S if the following hold

(a) $(\forall x, y \in S)((x, y) \in q \implies x \neq y)$;

(b) $(\forall x, y \in S)((x, y) \in q \implies (y, x) \in q)$ and

(c) $(\forall x, y, z \in S)((x, z) \in q \implies ((x, y) \in q \vee (y, z) \in q))$.

Definition 1. *Let $((S, =, \neq), \cdot, \alpha, \otimes)$ be an implicative semigroup with apartness. A co-equality relation q on S is a* co-congruence *on S if the following hold*

(d1) $(\forall x, y, y \in S)((xu, yu) \in q \implies (x, y) \in q)$*,*

(d2) $(\forall x, y, y \in S)((x \otimes u, y \otimes u) \in q \implies (x, y) \in q)$*,*

(e1) $(\forall x, y, v \in S)((vx, vy) \in q \implies (x, y) \in q)$ *and*

(e2) $(\forall x, y, v \in S)((v \otimes x, v \otimes y) \in q \implies (x, y) \in q)$*.*

Remark 1. *In the language of classical algebra, a co-equality relation q on an implicative semigroup S is compatible with the internal operations if those operations are cancellative with respect to q.*

Lemma 1. *The condition* (d1)$\wedge$(e1) *is equivalent to the condition*
(f) $(\forall x, y, u, v \in S)((xu, yv) \in q \implies ((x, y) \in q \vee (u, v) \in q))$.

Proof. Assume that (f) holds. Putting $v = u$ in (f) we get (d) in accordance with (a). If we put $x = v$, $y = v$, $u = x$ and $v = y$ in (f), we obtain (e) according to (a).

In the opposite direction, assume that (d1) and (e1) are valid and let $x, y, u, v \in S$ be elements such that $(x \cdot u, y \cdot v) \in q$. Then $(x \cdot u, y \cdot u) \in q$ or $(y \cdot u, y \cdot v)$ by co-transitivitu of q. Thus $(x, y) \in q$ or $(y, v) \in q$ by (d1) and (e1). So, we have proved that the condition (f) holds. □

The proof of the following lemma is analogous to the previous one and is omitted.

Lemma 2. *The condition* (d2)$\wedge$(e2) *is equivalent to the condition*
(g) $(\forall x, y, u, v \in S)((x \otimes u, y \otimes v) \in q \implies ((x, y) \in q \vee (u, v) \in q))$.

Proposition 1. *Let q be a co-congruence on an implicative semigroup with apartness S. Then the class $xq = \{t \in S : (x, t) \in \alpha\}$, generated by the element $x \in S$, is a strongly extensional subset of S.*

Proof. Let $u, v \in S$ be elements such that $u \in xq$. Then $(x, u) \in q$. Thus $(x, v) \in q$ or $(v, u) \in q$ by co-transitivity of q. Hence $v \in xq \vee u \neq v$ by consistency of q. □

We recall from ([14], Theorem 3.7) that an inhabited proper subset G of S is an ordered co-filter of S if and only if it satisfies the following conditions:

(i) $1 \lhd G$ and

(ii) $(\forall x, y \in S)(y \in G \implies (x \otimes y \in G \vee x \in G))$.

Proposition 2. *Let q be a co-congruence on an implicative semigroup with apartness S. Then the subset $C_1 = \{x \in S : (x, 1) \in q\}$ is a co-filter of S.*

Proof. First, it is obvious that $1 \lhd C_1$ is valid because of the consistency of the relation q. Let $x, y \in S$ be elements such that $xy \in C_1$. Then $(xy, 1) \in q$ by definition of C_1. Thus $(xy, 1 \cdot y) \in q$ or $(y, 1) \in q$ by co-transitivity of q and since $y = 1 \cdot y$ ([14], Corollary 3.1). From here it follows $(x, 1) \in q$ or $(y, 1) \in q$ according to (d1). So, we have $x \in C_1 \vee y \in C_1$.

Let y be an element of C_1. Then $(1, y) \in q$. Thus for each $x \in S$ we have $(1, x \otimes y) \in q$ or $(x \otimes y, y) \in q$ by co-transitivity of q. In the first case we get $x \otimes y \in C_1$, in the latter, since by [14], Corollary 3.3, we know that $y = 1 \otimes y$, we have $(x \otimes y, 1 \otimes y) \in q$ and by (d2) $(x, 1) \in q$. So, we have proven that $y \in C_1$ implies $(x \otimes y \in C_1 \vee x \in C_1)$. □

An implicative semigroup S is said to be right self-distributive if

$$(\forall x, y, z \in S)((x \otimes y) \otimes z = (x \otimes z) \otimes (y \otimes z)).$$

Corollary 3. *Let S be a right self-distributive semigroup with apartness. Then the co-filter C_1, constructed in the previous proposition, satisfies the following condition*

$$(\forall x, y \in S)((x, y) \in q \implies (x \otimes y \in C_1 \vee y \otimes x \in C_1)).$$

Proof. Let $x, y \in S$ be elements such that $(x, y) \in q$. Then

$$(x, (y \otimes x) \otimes x) \in q \vee ((y \otimes x) \otimes x, (x \otimes y) \otimes y) \in q \vee ((x \otimes y) \otimes y, y) \in q$$

by co-transitivity of q. Because it is

$$(y \otimes x) \otimes x = (y \otimes x) \otimes (x \otimes x) = (y \otimes x) \otimes 1 = 1$$

and

$$(x \otimes y) \otimes y = (x \otimes y) \otimes (y \otimes y) = (x \otimes y) \otimes 1 = 1$$

the second and third options lead to contradiction. The first option gives $(1 \otimes x, (y \otimes x) \otimes x) \in q \implies (1, y \otimes x) \in q$ and $y \otimes x \in C_1$ by (d2). The third option gives $((x \otimes y) \otimes y, 1 \otimes y) \in q \implies (x \otimes y, 1) \in q$ and $x \otimes y \in C_1$. □

Remark 2. *In the previous corollary, the right self-distributivity condition for an implicit semigroup S can be replaced by the requirement that the co-congruence relation q satisfies the conditions*

$$(\forall x, y \in S)(((y \otimes x) \otimes x, (x \otimes y) \otimes y) \lhd q).$$

4. Quotient structures

A relation $\sigma \subseteq S \times S$ is a co-quasiorder on S if it is consistent and co-transitive. In the following, we assume that σ is compatible with the operations in S and the following $\sigma \subseteq \alpha$ holds. In [13], Lemma 1, the author proved that the relation $q = \sigma \cup \sigma^{-1}$ is a co-congruence on S.

It is known ([16], Proposition 1.1) that the strong complement $q^{\lhd}$ of q is a congruence on S associate with the co-congruence q in the following sense $q^{\lhd} \circ q \subseteq q$ and $q \circ q^{\lhd} \subseteq q$. So, the factor-set $S/(q^{\lhd}, q) = \{[x] : x \in S\}$ can be constructed ([12], Theorem 2) with

$$[x] =_1 [y] \iff (x.y) \lhd q, \; [x] \neq_1 [y] \iff (x, y) \in q.$$

If we define the operations '$\cdot_1$' and '$\otimes_1$' and the co-order θ on $S/(q^{\lhd}, q)$ as follows

$$[x] \cdot_1 [y] =_1 [x \cdot y], \;\; [x] \otimes_1 [y] =_1 [x \otimes y], \;\; ([x],[y]) \in \theta \iff (x,y) \in \sigma,$$

the following theorem can be proved.

Theorem 4. *Let $((S, =, \neq), \cdot, \alpha, \otimes)$ be an implicative semigroup with apartness such that the co-quasiorder $\sigma \subseteq \alpha$ satisfies condition (4). If $q = \sigma \cup \sigma^{-1}$, then the system $((S/(q^{\lhd}, q), =_1, \neq_1), \cdot_1, \theta, \otimes_1)$ is an implicative semigroup with apartnes and there is a unique reverse isotone se-epimorphism $\pi : S \longrightarrow S/(q^{\lhd}, q)$.*

Proof. It can be directly verified that '$\cdot_1$' is a well-defined operation in $S/(q^{\lhd}, q)$ that satisfies the conditions (1) and (2).

Let $x, y, z, u, v \in S$ be elements such that $[x] =_1 [y]$ and $(u, v) \in q$. Then $(x, y) \lhd q$. From $(u, v) \in q$ it follows $(u, xz) \in q \vee (xz, yz) \in q \vee (yz, v) \in q$ by co-transitivity of q. Because from the second option it follows $(x, y) \in q$ by (d1), we have to have $u \neq xz \vee v \neq yz$ by consistency of q. Thus $(xz, yz) \neq (u, v) \in q$. Hence $[x] \cdot_1 [z] =_1 [xz] \neq_1 [yz] =_1 [y] \cdot_1 [z]$.

Let $x, y, z, u, v \in S$ be elements such that $[x] \cdot_1 [z] \neq_1 [y] \cdot_1 [z]$. Then $[xz] \neq_1 [yz]$ and $(xz, yz) \in q$. Thus $(x, y) \in q$ by (d1). Hence $[x] \neq_1 [y]$.

We have shown that the multiplication on the right is well defined, i.e. that it is an extensive and strictly extensive total function on $S/(q^{\lhd}, q)$. Analogously, it can be shown that the multiplication on the left is also well defined.

Let us show that the multiplication in $S/(q^{\lhd}, q)$ satisfies condition (3). Let $x, y, u, v \in S$ be such $([u], [v]) \in \theta$. Then $(u, v) \in \sigma$. Thus

$$(u, xy) \in \sigma \subseteq q \vee (xy, x) \in \sigma \subseteq \alpha \vee (x, v) \in \sigma \subseteq q.$$

hence $[u] \neq_1 [x] \cdot_1 [y]$ or $[x] \neq_1 [v]$ because $(xy, x) \in \alpha$ is impossible by (3). So, $([x] \cdot_1 [y], [x]) \neq_1 ([u], [v]) \in \theta$. therefore, $([x] \cdot_1 [y], [x]) \lhd \theta$. The proof of the second part of (3) is analogous to the one of the first part.

Let us check the condition (4). Let $x, y, z \in S$ be elements such that $([z], [x] \otimes_1 [y]) \in \theta$. Then $(z, x \otimes y) \in \sigma$. Thus $(zx, y) \in \sigma$. This means $([z] \cdot_1 [y], [y]) \in \theta$.

Let us define $\pi(x) = [x]$ for any $x \in S$. It is easy to prove that π is a se-monomorphism. The reverse isotonicity remains to be verified. Suppose $([x], [y]) \in \theta$. Then $(x, y) \in \sigma \subseteq \alpha$ by definition. This means that π is a reverse isotone se-epimorphism. The uniqueness of this mapping is obvious. □

On the other hand, we can construct ([12], Theorem 3) the family $[S : q] = \{xq : x \in S\}$, where

$$xq =_2 yq \iff (x, y) \lhd q, \;\; xq \neq_2 yq \iff (x, y) \in q.$$

If we define the operations '$\cdot_2$' and '$\otimes_2$' and the co-order Θ on $[S:q]$ as follows

$$xq \cdot yq =_2 (x \cdot y)q, \;\; xq \otimes_2 yq =_2 (x \otimes y)q, \;\; (xq, yq) \in \Theta \iff (x,y) \in \sigma$$

it can be verified.

Theorem 5. *Let $((S,=,\neq),\cdot,\alpha,\otimes)$ be an implicative semigroup with apartness such that the co-quasiorder $\sigma \subseteq \alpha$ satisfies condition (4). Let $q = \sigma \cup \sigma^{-1}$. Then $(([S:q],=_2,\neq_2),\cdot_2,\Theta,\otimes_2)$ is an implicative semigroup with apartnes and there exists a unique reverse isotone se-epimorphism $\vartheta : S \longrightarrow [S:q]$.*

Proof. (i) Let us show that the operation '$\cdot_2$' is well defined.

We first show that '$\cdot_2$' is an extensive function with respect to the equality '$=_2$' in set $[S:q]$. Let $x,y,u,v,s,t \in S$ be such that $xq =_2 uq$, $qy =_2 vq$ and $(s,t) \in q$. Then $(x,u) \lhd q$, $(y,v) \lhd q$. From $(s,t) \in q$ it follows

$$(s,xy) \in q \vee (xy,xv) \in q \vee (xv,uv) \in q \vee (uv,t) \in q$$

by co-transitivity of q. Thus,

$$s \neq xy \vee (y,v) \in q \vee (x,u) \in q \vee uv \neq t$$

by consistency of q and since the second and third options are impossible because $(x,u) \lhd q$ and $(y,v) \lhd q$ hold hypothesis. This shows that $(xy,uv) \neq (s,t) \lhd q$. So $xyq =_2 uvq$. This means $xq \cdot_2 yq = uq \cdot_2 vq$ according to the definition of the operation '$\cdot_2$'.

To show that '$\cdot_2$' is a strictly extensive function, we take $x,u,y,v \in S$ such that $xyq \neq_2 uvq$, ie. such that $(xy,uv) \in q$. From $(xy,uv) \in q$ immediately follows $(x,y) \in q \vee (y,v) \in q$ according to Lemma 1. Therefore, we have $xq \neq_2 uq$ or $yq \neq_2 = vq$ which shows that '$\cdot_2$' is an extensive function.

(ii) Let $x,y,z,y,v \in S$ be arbitrary elements such that $(u,v) \in q$. Then $u,x(yz)) \in q \vee (x(yz),(xy)z) \in q \vee ((xy)z,v) \in q$ by co-transitivity of q. Thus, we have $u \neq x(yz) \vee (xy)z \neq v$ by consistency of q since the second option is impossible by (1). This gives $(x(yz),(xy)z) \neq (u,v) \in q$ which means that $x(yz)q =_2 (xy)zq$ is valid. This it shown that the multiplication '$\cdot_2$' in the set $[S:q]$ satisfies the condition (1).

Let $x,y,z \in S$ be such that $(xzq,yzq) \in \Theta$. Them $(xz,yz) \in \sigma$ by definition of Θ. Thus $(x,y) \in \sigma$ by compatibility of the multiplication in S with the co-quasiorder σ. So, we have $(xq,yq) \in \Theta$. The proof for the implication $(zxq,zyq) \in \Theta \implies (zq,yq) \in \Theta$ can be shown analogously to the previous one. Thus, multiplication in [S: q] satisfies the condition (2).

To show that the multiplication in semigroup $[S : q]$ satisfies condition (3), we take the elements $x, y, u, v \in S$ such that $(uq, vq) \in \Theta$. Then, from $(u, v) \in \sigma$ it follows $(u, xy) \in \sigma \vee (xy, x) \in \sigma \subseteq \alpha \vee (x, v) \in \sigma$ by co-transitivity of σ. Since the second option is impossible by (3), we have $u \neq xy$ or $x \neq v$ by consistency of σ. This means $(xy, x) \neq (u, v) \in \sigma$. So, $(xyq, xq) \neq_2 (uq, vq) \in \Theta$. Therefore, we have $(xyq, xq) \lhd \Theta$. The proof that for the elements $x, y \in S$ holds $(xyq, yq) \lhd \Theta$ can be shown analogously to the previous one.

(iii) Let us show that the operation a is well defined and that the elements semigroup $[S : q]$ satisfy the condition (4).

Let $x, y, u, v, s, t \in S$ be such that $xq =_2 uq$, $yq =_2 vq$ and $(s, y) \in q$. Then $(x, u) \lhd q$ and $(y, v) \lhd q$. From $(s, t) \in q$ it follows

$$(s, x \otimes y) \in q \vee (x \otimes y, u \otimes y) \in q \vee (u \otimes y, u \otimes v) \in q \vee (u \otimes v, t) \in q$$

by co-transitivity of q. The second and third possibilities would lead to $(x, y) \in q$ and $y, v) \in q$ which is in contradiction with the hypothesis. So, have to have $s \neq x \otimes y$ or $u \otimes v \neq t$. This means $(x \otimes y, u \otimes v) \neq (s, y) \in q$. Therefore, $(x \otimes y)q =_2 (u \otimes v)q$.

Let us choose the elements $x, y, u, v \in S$ such that $(x \otimes y)q \neq_2 (y \otimes v)q$. Then $(x \otimes y, u \otimes v) \in q$. Hence immediately follows $(x, u) \in q$ or $(y, v) \in q$ according to Lemma 2. This means $xq \neq_2 uq$ or $yq \neq_2 vq$.

Let $x, y, z \in S$ be such $(zq, (x \otimes y)q) \in \Theta$. Then $(z, x \otimes y) \in \sigma$ by definition of Θ. Since $(z, x \otimes y) \in \sigma \iff (zx, y) \in \sigma$ according to (4), we have $((zx)q, yq) \in \Theta$. Obviously, the reverse is also true. Thus, in the semigroup $[S : q]$, the condition (4) is a valid formula.

(iv) By direct checking it can be determined that the function: $\vartheta : S \longrightarrow [S : q]$, defined by $(\forall x \in S)(\vartheta(x) = xq)$, is a unique se-surjective function. Further on, for $x, y \in S$ we have $\vartheta(x \otimes y) =_2 (x \otimes y)q =_2 xq \otimes_2 yq =_2 \vartheta(x) \otimes_2 \vartheta(y)$. Let $x, y \in S$ be such that $(\vartheta(x), \vartheta(y)) \in \Theta$. Then $(xq, yq) \in \Theta$. Thus $(x, y) \in \sigma \subseteq \alpha$ by definition of Θ. This means that ϑ is a reverse isotone se-epimorphism. $\square$

It should be emphasized here that the semigroup [S: q] has no counterpart in the classical theory of implicit semigroups. However, there is a strong link between the semigroup $S/(q^{\lhd}, q)$ and the semigroup $[S : q]$.

Theorem 6. *Let $((S, =, \neq), \cdot, \alpha, \otimes)$ be an implicative semigroup with apartness such that the co-quasiorder $\sigma \subseteq \alpha$ satisfies condition (4). Let $q = \sigma \cup \sigma^{-1}$. Then there exists a unique reverse isotone se-epimorphism $\pi : S \longrightarrow S/(q^{\lhd}, q)$, defined by $\pi(x) = [x]$, and a unique reverse isotone se-epimorphism $\vartheta : S \longrightarrow [S : q]$, defined by $\vartheta(x) = xq$, and a unique strongly extensional, embedding, injective and surjective homomorphism $h : S/(q^{\lhd}, q) \longrightarrow [S : q]$, defined by $h([x]) = xq$, such that $\vartheta = h \circ \pi$ and $\pi = h^{-1} \circ \vartheta$.*

Proof. The existence and uniqueness of the se-epimorphisms π and ϑ is proved in Theorem 4 and Theorem 5 respectively.

In what follows, let $x, y \in S$ be such that $[x] =_1 [y]$. Then

$$[x] =_1 [y] \iff xq^{\lhd} =_1 yq^{\lhd} \iff (x,y) \lhd q \iff h([x]) = xq =_2 yq = h([y]).$$

This shows that h is an injective function.

Take $x, y \in S$ such that $h([x]) \neq_2 h([y])$. Then the following holds

$$h([x]) \neq_2 h([y]) \iff xq \neq_2 yq \iff (x,y) \in q \iff [x] \neq_1 [y].$$

This proves that h is a strongly extensive and embedding function.

As the following equations

$$h([x] \cdot_1 [y]) =_2 h([xy]) =_2 (xy)q =_2 xq \cdot_2 yq =_2 h([x]) \cdot_2 h([y])$$

and

$$h([x] \otimes_1 [y]) =_2 h([xy]) =_2 (xy)q =_2 xq \otimes_2 yq =_2 h([x]) \otimes_2 h([y])$$

are obviously true, we conclude that h is a se-homomorphism. Finally, h is a unique embedding, injective and surjective se-homomorphism such that $\vartheta = h \circ \pi$ is valid. □

5. The isomorphism theorem

We will start this section with two important technical lemmas.

Lemma 7. *Let $f : S \longrightarrow T$ be a reverse isotone se-epimorphism from implicative semigroups $((S, =, \neq), \cdot, \alpha, \otimes)$ onto implicative semigroup $((T, =, \neq), \cdot, \beta, \otimes)$. Then the relation $f^{-1}(\beta) = \{(\in S \times S : (f(x), f(y)) \in \beta\}$ is a co-quasiorder on S such that $f^{-1}(\beta) \subseteq \alpha$ and $f^{-1}(\beta)$ satisfies the condition (4).*

Proof. Let $x, y \in S$ such that $(x, y) \in f^{-1}(\beta)$. Then $(f(x), f(y)) \in \beta \subseteq \neq$. Thus $x \neq y$ because f is a se-mapping. So, $f^{-1}(\beta) \subseteq \neq$.

Let $x, y, z \in S$ be such that $(x, z) \in f^{-1}(\beta)$. Then $(f(x), f(z)) \in \beta$. Thus $(f(x), f(y)) \in \beta \vee ((f(y), f(z)) \in \beta$ by co-transitivity of β. This means $(x, y) \in f^{-1}(\beta)$ or $(y, z) \in f^{-1}(\beta)$. So, $f^{-1}(\beta)$ is a co-transitive relation on S.

By direct verification it can be verified that $f^{-1}(\beta)$ is compatible with operations in S, i.e. that $f^{-1}(\beta))$ satisfies conditions (f) and (d).

From $(x, y) \in f^{-1}(\beta)$, i.e. from $(f(x), f(y)) \in \beta$ immediately follows $(x, y) \in \alpha$ since f is a reverse isotone se-mapping. So, $f^{-1}(\beta) \subseteq \alpha$.

Let us show that $f^{-1}(\beta)$ satisfies the condition (4). Let $x, y, z \in S$ be such that $(z, x \otimes y) \in f^{-1}(\beta)$. Then $(f(z), f(x \otimes y)) \in \beta$. Thus $(f(z) \cdot f(x), f(y)) \in \beta$ by (4). So, we have $(f(zx), f(y)) \in \beta$ according to the statement (4) and the statement (4) of Theorem 3.1 in article [15]. This means $(zx, y) \in f^{-1}(\beta)$. $\square$

Lemma 8. *Let $f : S \longrightarrow T$ be a reverse isotone se-epimorphism from implicative semigroups $((S, =, \neq), \cdot, \alpha, \otimes)$ onto implicative semigroup $((T, =, \neq), \cdot, \beta, \otimes)$. Then the relation $q_f = \{(x, y) \in S \times S : f(x) \neq f(y)\}$ is a co-congruence on S such that $q_f = f^{-1}(\beta) \cup (f^{-1}(\beta))^{-1}$.*

Proof. Proof that q_f is a co-congruence on S can be obtained by direct verification so we will omit it.

As the following statements $(x, y) \in q_f$, $f(x) \neq f(y)$, $(f(x), f(y)) \in \beta \vee (f(y), f(x)) \in \beta$ by linearity of β, and $(x, y) \in f^{-1}(\beta) \vee (y, x) \in f^{-1}(\beta)$ are mutually equivalent, we conclude that $q_f = f^{-1}(\beta) \cup (f^{-1}(\beta))^{-1}$. $\square$

The following theorem can be viewed as the First Isomorphism Theorem for implicative semigroups with apartness.

Theorem 9. *Let $f : S \longrightarrow T$ be a reverse isotone se-epimorphism from implicative semigroups $((S, =, \neq), \cdot, \alpha, \otimes)$ onto implicative semigroup $((T, =, \neq), \cdot, \beta, \otimes)$. Then there exist a unique pair $f_1 : S/(q_f^{\triangleleft}, q_f) \longrightarrow T$ and $f_2 : [S : q_f] \longrightarrow T$ of embedding, injective and surjective se-homomorphisms such that*

$$f = f_1 \circ \pi = f_2 \circ \vartheta = f_2 \circ h \circ \pi.$$

Proof. The existence and uniqueness of se-mappings π, ϑ and h is proved in Theorems 4, 5 and 6. If we define mappings f_1 and f_2 as follows $f_1([x]) = f(x)$ and $f_2(xq_f) = f(x)$, direct verification it can prove that they are unique embedding se-isomorphisms. For arbitrary $x \in S$ we have

$$f_2(xq) = (f_2 \circ \vartheta)(x) = f(x) = f_1([x]) = (f_1 \circ \pi)(x)$$

so the required properties of these mappings are proved. $\square$

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