Acta Universitatis Apulensis No. 65/2021
ISSN: 1582-5329 pp. 1-12
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.65.01

CERTAIN SUBCLASS OF MEROMORPHIC FUNCTIONS INVOLVING BESSEL OPERATOR

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Abstract. In this paper, we introduce and study a new subclass of meromorphically uniformly convex functions with positive coefficients defined by a bessel operator and obtain coefficient estimates, growth and distortion theorem, radius of convexity,integral transforms, convex linear combinations, convolution properties and $\delta-$neighborhoods for the defined class.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Uniformly convex, Uniformly starlike ,Coefficient estimates.

1. Introduction

Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m, \tag{1}$$

which are regular in domain

$$U = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}.$$

Also, let Σ_p denote the subclass of Σ of functions of the form (1) with $a_m \geq 0$.

A function $f \in \Sigma$ is said to be meromorphically starlike of order α if it satisfies

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha$$

for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in U$. Further, a function $f \in \Sigma$ is said to be meromorphically convex of order α if it satisfies

$$Re\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for some $\alpha (0 \leq \alpha < 1)$ and for all $z \in U$.

Some subclasses of Σ were described and investigated by Pommerenke [5], Miller [3], Mogra et al. [4] and see also [7, 8].

We recollect here the generalized Bessel function of first kind of order γ (see [1]), denoted by

$$w(z) = \sum_{m=0}^{\infty} \frac{(-c)^m}{m!\Gamma(\gamma + m + \frac{b+1}{2})} \left(\frac{z}{2}\right)^{2m+\gamma} \; (z \in U)$$

(where Γ stands for the Gamma Euler function), which is the particular solution of the second-order linear homogeneous differential equation (see, for details, [9])

$$z^2 w''(z) + bzw'(z) + [cz^2 - \gamma^2 + (1-b)\gamma]w(z) = 0$$

where $c, \gamma, b \in \mathbb{C}$. We introduce the function φ defined, in terms of the generalized Bessel function w by

$$\varphi(z) = 2^{\gamma}\Gamma\left(\gamma + \frac{b+1}{2}\right) z^{-\left(1+\frac{\gamma}{2}\right)} w(\sqrt{z}).$$

By using the well-known Pochhammer symbol $(x)_\mu$ defined, for $x \in \mathbb{C}$ and in terms of the Euler gamma function, by

$$(x)_\mu = \frac{\Gamma(x+\mu)}{\Gamma(x)} = \begin{cases} 1, & (\mu = 0); \\ x(x+1)(x+2)\cdots(x+m-1), & (\mu = m \in \mathbb{N} = \{1, 2, 3 \cdots\}). \end{cases}$$

We obtain the following series representation for the function $\varphi(z)$

$$\varphi(z) = \frac{1}{z} + \sum_{m=0}^{\infty} \frac{(-c)^{m+1}}{4^{m+1}(m+1)!(\tau)_{m+1}} z^m \left(\tau = \gamma + \frac{b+1}{2} \notin \mathbb{Z}_0^- = \{0, -1, -2, \cdots\}\right).$$

For functions $f \in \Sigma$ given by (1) and $g \in \Sigma$ given by

$$g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m,$$

we define the Hadamard product of f and g by

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m.$$

Corresponding to the function φ define the Bessel operator ${S_\tau}^c$ by the following Hadamard product

$$\begin{aligned} {S_\tau}^c f(z) = (\varphi * f)(z) &= \frac{1}{z} + \sum_{m=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^{m+1} a_m}{(m+1)!(\tau)_{m+1}} z^m \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} \phi(m,\tau,c,)a_m z^m, \end{aligned} \tag{2}$$

where $\phi(m,\tau,c,) = \frac{\left(\frac{-c}{4}\right)^m}{(m)!(\tau)_m}$.

It easy to verify from the definition (2) that

$$z[S^c_{\tau+1} f(z)] = \tau {S_\tau}^c f(z) - (\tau+1) S^c_{\tau+1} f(z). \tag{3}$$

Motivated by Venkateswarlu et al. [7, 8], now we define a new subclass $\Sigma_p(\tau,c,\alpha,\beta)$ of Σ_p.

Definition 1. *For $-1 \le \alpha < 1$ and $\beta \ge 1$, we let $\Sigma_p(\tau,c,\alpha,\beta)$ be the subclass of Σ_p consisting of the form (1) and satisfying the analytic criterion*

$$-Re\left\{\frac{z({S_\tau}^c f(z))'}{{S_\tau}^c f(z)} + \alpha\right\} > \beta \left|\frac{z({S_\tau}^c f(z))'}{{S_\tau}^c f(z)} + 1\right|, \tag{4}$$

${S_\tau}^c f(z)$ is given by (2)

The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, integral operators and $\delta-$neighbourhoods for the class $\Sigma_p(\tau,c,\alpha,\beta)$.

2. Coefficient inequality

In this section we obtain the coefficient bounds of function $f(z)$ for the class $\Sigma_p(\tau,c,\alpha,\beta)$.

Theorem 1. *A function $f(z)$ of the form (1) is in $\Sigma_p(\tau,c,\alpha,\beta)$ if*

$$\sum_{m=1}^{\infty} \phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]\, |a_m| \;\le\; (1-\alpha)\ (-1 \le \alpha < 1 \text{ and } \beta \ge 1). \tag{5}$$

Proof. It sufficient to show that

$$\beta\left|\frac{z({S_\tau}^c f(z))'}{{S_\tau}^c f(z)} + 1\right| + Re\left\{\frac{z({S_\tau}^c f(z))'}{{S_\tau}^c f(z)} + 1\right\} \le (1-\alpha).$$

$$\text{We have } \beta\left|\frac{z(S_\tau{}^c f(z))'}{S_\tau{}^c f(z)}+1\right|+Re\left\{\frac{z(S_\tau{}^c f(z))'}{S_\tau{}^c f(z)}+1\right\}$$
$$\leq (1+\beta)\left|\frac{z(S_\tau{}^c f(z))'}{S_\tau{}^c f(z)}+1\right|$$
$$\leq \frac{(1+\beta)\sum\limits_{m=1}^{\infty}\phi(m,\tau,c,)(m+1)|a_m||z^m|}{\frac{1}{|z|}-\sum\limits_{m=1}^{\infty}\phi(m,\tau,c,)|a_m||z^m|}$$

Letting $z\to 1$ along the real axis, we obtain

$$\leq \frac{(1+\beta)\sum\limits_{m=1}^{\infty}\phi(m,\tau,c,)(m+1)|a_m|}{1-\sum\limits_{m=1}^{\infty}\phi(m,\tau,c,)|a_m|}.$$

The above expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty}\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]\ |a_m|\ \leq\ (1-\alpha).$$

Hence the theorem is completed.

Corollary 2. *Let the function $f(z)$ defined by (1) be in the class $\Sigma_p(\tau,c,\alpha,\beta)$. Then*

$$a_m \leq \frac{(1-\alpha)}{\sum\limits_{m=1}^{\infty}\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}\quad (m\geq 1). \tag{6}$$

Equality holds for the function of the form

$$f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}\ z^m. \tag{7}$$

3. Distortion Theorems

In this section we obtain distortion bounds for the class $\Sigma_p(\tau,c,\alpha,\beta)$.

Theorem 3. *Let the function $f(z)$ defined by (1) be in the class $\Sigma_p(\tau,c,\alpha,\beta)$. Then for $0<|z|=r<1$,*

$$\frac{1}{r}-\frac{(1-\alpha)}{\phi(1,\tau,c)[3+2\beta-\alpha]}\ r\leq|f(z)|\leq\frac{1}{r}+\frac{(1-\alpha)}{\phi(1,\tau,c)[3+2\beta-\alpha]}r \tag{8}$$

with equality for the function

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)}{\phi(1,\tau,c)[3+2\beta-\alpha]}\ z\ at\ \ z=r,ir. \tag{9}$$

Proof. Suppose $f(z)$ is in $\Sigma_p(\tau,c,\alpha,\beta)$. In view of Theorem 1, we have
$\phi(1,\tau,c)[3+2\beta-\alpha]\sum\limits_{m=1}^{\infty}a_m\leq\sum\limits_{m=1}^{\infty}\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]\leq(1-\alpha)$
which evidently yields $\sum\limits_{m=1}^{\infty}a_m\leq\frac{1-\alpha}{\phi(1,\tau,c)[3+2\beta-\alpha]}$.
Consequently, we obtain

$$\begin{aligned}|f(z)|=\left|\frac{1}{z}+\sum_{m=1}^{\infty}a_mz^m\right|&\leq\left|\frac{1}{z}\right|+\sum_{m=1}^{\infty}a_m|z|^m\\&\leq\frac{1}{r}+r\sum_{m=1}^{\infty}a_m\\&\leq\frac{1}{r}+\frac{1-\alpha}{\phi(1,\tau,c)[3+2\beta-\alpha]}\ r.\end{aligned}$$

$$\begin{aligned}\text{Also,}\ \ |f(z)|=\left|\frac{1}{z}+\sum_{m=1}^{\infty}a_mz^m\right|&\geq\left|\frac{1}{z}\right|-\sum_{m=1}^{\infty}a_m|z|^m\\&\geq\frac{1}{r}-r\sum_{m=1}^{\infty}a_m\\&\geq\frac{1}{r}-\frac{1-\alpha}{\phi(1,\tau,c)[3+2\beta-\alpha]}\ r.\end{aligned}$$

Hence the results (8) follow.

Theorem 4. *Let the function $f(z)$ defined by* (1) *be in the class $\Sigma_p(\tau,c,\alpha,\beta)$. Then for $0<|z|=r<1$,*

$$\frac{1}{r^2}-\frac{1-\alpha}{\phi(1,\tau,c)[3+2\beta-\alpha]}\leq|f'(z)|\leq\frac{1}{r^2}+\frac{1-\alpha}{\phi(1,\tau,c)[3+2\beta-\alpha]}.$$

The result is sharp, the extremal function being of the form (7)

Proof. From Theorem 1, we have
$\phi(1,\tau,c)[3+2\beta-\alpha]\sum\limits_{m=1}^{\infty}ma_m\leq\sum\limits_{m=1}^{\infty}\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]\leq(1-\alpha)$

which evidently yields $\sum\limits_{m=1}^{\infty} ma_m \leq \frac{1-\alpha}{\phi(1,\tau,c)[3+2\beta-\alpha]}$.
Consequently, we obtain

$$|f'(z)| \leq \left| \frac{1}{r^2} + \sum_{m=1}^{\infty} ma_m r^{m-1} \right|$$
$$\leq \frac{1}{r^2} + \sum_{m=1}^{\infty} ma_m$$
$$\leq \frac{1}{r^2} + \frac{(1-\alpha)}{\phi(1,\tau,c)[3+2\beta-\alpha]}.$$
$$\text{Also, } |f'(z)| \geq \left| \frac{1}{r^2} - \sum_{m=1}^{\infty} ma_m r^{m-1} \right|$$
$$\geq \frac{1}{r^2} - \sum_{m=1}^{\infty} ma_m$$
$$\geq \frac{1}{r^2} + \frac{(1-\alpha)}{(\phi(1,\tau,c)[3+2\beta-\alpha]}.$$

This completes the proof.

4. Class preserving integral operators

In this section we consider the class preserving integral operator of the form (1).

Theorem 5. *Let the function $f(z)$ defined by* (1) *be in the class $\Sigma_p(\tau, c, \alpha, \beta)$. Then*

$$f(z) = \upsilon z^{-\upsilon-1} \int_0^z t^{\upsilon} f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\upsilon}{\upsilon+m+1} \; a_m z^m \quad (\upsilon > 0) \tag{10}$$

is in $\Sigma_p(\delta, \beta)$, where

$$\delta(\tau, c, \alpha, \beta, \upsilon) = \frac{(m+1)(1+\beta) + \alpha\upsilon(1-\beta) + (1-\alpha)}{(\upsilon+m+1)(1+\beta) + (1-\alpha)}. \tag{11}$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{(1-\alpha)}{\phi(1,\tau,c)[3+2\beta-\alpha]} z$.

Proof. Suppose $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} \; a_m z^m$ is in $\Sigma_p(\tau, c, \alpha, \beta)$. We have
$f(z) = \upsilon z^{-\upsilon-1} \int\limits_0^z t^{\upsilon} f(t)dt = \frac{1}{z} + \sum\limits_{m=1}^{\infty} \frac{\upsilon}{\upsilon+m+1} \; a_m z^m \quad (\upsilon > 0)$.

It is sufficient to show that

$$\sum_{m=1}^{\infty} \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\delta]}{1-\delta} \frac{\upsilon}{\upsilon+m+1} a_m \leq 1. \tag{12}$$

Since $f(z)$ is in $\Sigma_p(\tau, c, \alpha, \beta)$, we have

$$\sum_{m=1}^{\infty} \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{1-\alpha} |a_m| \leq 1. \tag{13}$$

Thus (12) will be satisfied if
$\sum_{m=1}^{\infty} \frac{[(1+\beta)(m+1)+1-\delta]}{1-\delta} \frac{\upsilon}{\upsilon+m+1} \leq \sum_{m=1}^{\infty} \frac{[(1+\beta)(m+1)+1-\alpha]}{1-\alpha}$.
Solving for δ, we obtain

$$\delta \leq \frac{(1+\beta)(m+1)+\alpha\upsilon(1-\beta)+(1-\alpha)}{(\upsilon+m+1)(1-\beta)+(1-\alpha)} = G(m). \tag{14}$$

A simple computation will show that $G(m)$ is increasing and $G(m) \geq G(1)$. Using this, the result follows.

5. Convex linear combinations and convolution properties

In this section we obtain sharp for $f(z)$ is meromorphically convex of order δ and necessary and sufficient condition for $f(z)$ is in the class $\Sigma_p(\tau, c, \alpha, \beta)$ and also proved that convolution is in the class.

Theorem 6. *If the function $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ is in $\Sigma_p(\tau, c, \alpha, \beta)$ then $f(z)$ is meromorphically convex of order $\delta(0 \leq \delta < 1)$ in $|z| < r = r(\tau, c, \alpha, \beta, \delta)$ where*

$$r(\tau, c, \alpha, \beta, \delta) = \inf_{m \geq 1} \left\{ \frac{(1-\delta)\phi(m,\tau,c,)[(1+\beta)(1+m)+1-\alpha]}{(1-\alpha)m(m+2-\delta)} \right\}^{\frac{1}{m+1}}.$$

The result is sharp.

Proof. Let $f(z)$ be in $\Sigma_p(\tau, c, \alpha, \beta)$. Then, by Theorem 1, we have

$$\sum_{m=1}^{\infty} \phi(m,\tau,c,)[(1+\beta)(1+m)+1-\alpha]|a_m| \leq (1-\alpha). \tag{15}$$

It is sufficient to show that $\left|2+\frac{zf''(z)}{f'(z)}\right| \leq (1-\delta)$ for $|z|<r=r(\tau,c,\alpha,\beta,\delta)$, where $r(\tau,c,\alpha,\beta,\delta)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m z^{m-1}}{\frac{-1}{z^2}+\sum\limits_{m=1}^{\infty} m a_m z^{m-1}}\right| \leq \frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m |z|^{m+1}}{1-\sum\limits_{m=1}^{\infty} m a_m |z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty} \frac{m(m+2-\delta)}{1-\delta} a_m |z|^{m+1} \leq 1. \tag{16}$$

By (15), it follows that (16) is true if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1} \leq \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{1-\alpha}|a_m| \quad (m\geq 1)$$

$$\text{or} \quad |z| \leq \left\{\frac{(1-\delta)\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}. \tag{17}$$

Setting $|z|=r(\tau,c,\alpha,\beta,\delta)$ in (17), the result follows. The result is sharp for the function

$$f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}z^m \quad (m\geq 1).$$

Theorem 7. *Let* $f_0(z)=\frac{1}{z}$ *and* $f_m(z)=\frac{1}{z}+\frac{(1-\alpha)}{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}z^m \quad (m\geq 1)$. *Then* $f(z)=\frac{1}{z}+\sum\limits_{m=1}^{\infty} a_m z^m$ *is in the class* $\Sigma_p(\tau,c,\alpha,\beta)$ *if and only if it can be expressed in the form* $f(z)=\omega_0 f_0(z)+\sum\limits_{m=1}^{\infty}\omega_m f_m(z)$, *where* $\omega_0\geq 0, \omega_m\geq 0, m\geq 1$ *and* $\omega_0+\sum\limits_{m=1}^{\infty}\omega_m=1$.

Proof. Let $f(z)=\omega_0 f_0(z)+\sum\limits_{m=1}^{\infty}\omega_m f_m(z)$ with $\omega_0\geq 0, \omega_m\geq 0, m\geq 1$ and ω_0+

$\sum\limits_{m=1}^{\infty} \omega_m = 1$. Then

$$\begin{aligned} f(z) &= \omega_0 f_0(z) + \sum_{m=1}^{\infty} \omega_m f_m(z) \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} \omega_m \frac{(1-\alpha)}{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]} z^m \end{aligned}$$

Since $\displaystyle\sum_{m=1}^{\infty} \frac{[\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{(1-\alpha)} \omega_m \frac{(1-\alpha)}{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}$

$$= \sum_{m=1}^{\infty} \omega_m = 1 - \omega_0 \leq 1.$$

By Theorem 1, $f(z)$ is in the class $\Sigma_p(\tau, c, \alpha, \beta)$.

Conversely suppose that the function $f(z)$ is in the class $\Sigma_p(\tau, c, \alpha, \beta)$, since

$$a_m \leq \frac{(1-\alpha)}{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]} z^m \quad (m \geq 1).$$

$$\omega_m = \sum_{m=1}^{\infty} \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{(1-\alpha)} a_m \quad \text{and} \quad \omega_0 = 1 - \sum_{m=1}^{\infty} \omega_m.$$

It follows that $f(z) = \omega_0 f_0(z) + \sum\limits_{m=1}^{\infty} \omega_m f_m(z)$.

This completes the proof of the theorem.

For the functions $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} b_m z^m$ belongs to Σ_p, we denoted by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$ and defined as

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m.$$

Theorem 8. *If the function* $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ *and* $g(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} b_m z^m$ *are in the class* $\Sigma_p(\tau, c, \alpha, \beta)$ *then* $(f * g)(z)$ *is in the class* $\Sigma_p(\tau, c, \alpha, \beta)$.

Proof. Suppose $f(z)$ and $g(z)$ are in $\Sigma_p(\tau, c, \alpha, \beta)$. By Theorem 1, we have

$$\sum_{m=1}^{\infty} \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{(1-\alpha)} a_m \leq 1$$

$$\text{and } \sum_{m=1}^{\infty} \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{(1-\alpha)} b_m \leq 1 \ .$$

Since $f(z)$ and $g(z)$ are regular are in E, so is $(f*g)(z)$. Further more

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{(1-\alpha)} a_m b_m \\
&\leq \sum_{m=1}^{\infty} \left\{ \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{(1-\alpha)} \right\}^2 a_m b_m \\
&\leq \left(\sum_{m=1}^{\infty} \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{(1-\alpha)} a_m \right) \\
&\qquad \left(\sum_{m=1}^{\infty} \frac{\phi(m,\tau,c,)[(1+\beta)(m+1)+1-\alpha]}{(1-\alpha)} b_m \right) \\
&\leq 1.
\end{aligned}$$

Hence, by Theorem 1, $(f*g)(z)$ is in the class $\Sigma_p(\tau, c, \alpha, \beta)$.

6. Neighborhoods for the class $\Sigma_p(\tau, c, \alpha, \beta, \gamma)$

In this section we define the $\delta-$neighborhood of a function $f(z)$ and establish a relation between $\delta-$neighborhood and $\Sigma_p(\tau, c, \alpha, \beta, \gamma)$ class of a function.

Definition 2. *A function $f \in \Sigma_p$ is said to in the class $\Sigma_p(\tau, c, \alpha, \beta, \gamma)$ if there exists a function $g \in \Sigma_p(\tau, c, \alpha, \beta)$ such that*

$$\left| \frac{f(z)}{g(z)} - 1 \right| < (1-\gamma) \quad (z \in E, \ 0 \leq \gamma < 1). \tag{18}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [2] and Ruschweyh [6]. We defined the $\delta-$neighborhood of a function $f \in \Sigma_p$ by

$$N_\delta(f) = \left\{ g \in \Sigma_p \ \mid \ g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m \ : \ \sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \right\}. \tag{19}$$

Theorem 9. *If $g \in \Sigma_p(\tau, c, \alpha, \beta)$ and*

$$\gamma = 1 - \frac{\delta[3+2\beta-\alpha]\phi(1,\tau,c)}{[3+2\beta-\alpha]\phi(1,\tau,c) - 1 + \alpha} \tag{20}$$

then $N_\delta(g) \subset \Sigma_p(\tau, c, \alpha, \beta, \gamma)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (19) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \tag{21}$$

which implies the coefficient of inequality $\sum\limits_{m=1}^{\infty} |a_m - b_m| \leq \delta \quad (m \in \mathbb{N})$.
Since $g \in \Sigma_p(\tau, c, \alpha, \beta)$, we have $\sum\limits_{m=1}^{\infty} b_m = \frac{1-\alpha}{[3+2\beta-\alpha]\phi(1,\tau,c)}$.
So that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum\limits_{m=1}^{\infty} |a_m - b_m|}{1 - \sum\limits_{m=1}^{\infty} b_m} \leq \frac{\delta[3+2\beta-\alpha]\phi(1,\tau,c)}{[3+2\beta-\alpha]\phi(1,\tau,c) - 1 + \alpha} = 1 - \gamma,$$

provided γ is given by (20).
Hence, by Definition 2 , $f \in \Sigma_p(\tau, c, \alpha, \beta, \gamma)$ for γ given by (20), which completes the proof of theorem.

Acknowledgments: The authors warmly thanks to the editor and referees for the careful reading of the paper and their comments.

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