Acta Universitatis Apulensis No. 62/2020
ISSN: 1582-5329 pp. 93-110
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.62.08

SUFFICIENCY AND DUALITY IN SET-VALUED OPTIMIZATION PROBLEMS UNDER (p, r)-ρ-(η, θ)-INVEXITY

K. Das, C. Nahak

Abstract. In this paper, we introduce a new type of generalized invexity, namely (p, r)-ρ-(η, θ)-invexity, for set-valued optimization problems. We establish the sufficient optimality conditions and duality results of Mond-Weir type (MWD) under the stated (p, r)-ρ-(η, θ)-invexity assumptions. As a special case, our results reduce to the existing ones of scalar valued optimization problems.

2010 *Mathematics Subject Classification*: 26B25, 49N15.

Keywords: convex cone, contingent derivative, set-valued map, (p, r)-ρ-(η, θ)-invexity, duality.

1. Introduction

Convex Analysis has a vital role in investigating the solutions of vector optimization problems. To relax convexity assumptions, various notions of generalized convexity have been introduced. In 1981, Hanson [7] introduced the notion of invexity. Later, many authors have studied further generalizations of invexity. One of such generalizations is (p, r)-invexity introduced by Antczak [1, 2]. He established the sufficient optimality conditions and duality results under (p, r)-invexity assumptions in nonlinear multiobjective programming problems. Recently, Mandal and Nahak [9] introduced generalized (p, r)-invexity, namely (p, r)-ρ-(η, θ)-invexity, in vector optimization. They established the sufficient optimality conditions and duality results of Mond-Weir type under (p, r)-ρ-(η, θ)-invexity assumptions.

Recently, there has been an increasing interest in the extension of vector optimization problems to set-valued optimization problems, where the objective function and functions attached to constraints are set-valued maps. It has huge applications in economics, management science, and engineering. The derivative of set-valued maps is an important tool for set-valued optimization problems. Anbin and Frankowska [3] introduced the notion of contingent derivative of set-valued maps. For single-valued map, contingent derivative coincides with Frechet derivative (Remark 15.2. in [8]). In 1987, Corley [4] established the generalized Fritz John necessary optimality conditions for the maximization of set-valued maps in terms of contingent derivative. He also proved the generalized Fritz John sufficient conditions of set-valued optimization problems where the objective function and functions attached to constraints are cone concave set-valued maps. Later, Sach and Craven [10, 11] introduced invex set-valued maps and proved duality theorems of Mond-Weir type.

In this paper, we extend the notion of (p, r)-ρ-(η, θ)-invexity from vectorial case to set-valued one. We establish that the Fritz John optimality conditions are sufficient under (p, r)-ρ-(η, θ)-invexity assumptions. We also establish the duality theorems of Mond-Weir type (MWD) of a pair of set-valued optimization problems under (p, r)-ρ-(η, θ)-invexity assumptions.

2. Definitions and Preliminaries

Let K be a nonempty subset of $\mathbb{R}^m$. Then K is said to be a cone if $\lambda y \in K$, for all $y \in K$ and $\lambda \geq 0$. Also, K is called pointed if $K \cap (-K) = \{0_{\mathbb{R}^m}\}$, solid if $\text{int}(K) \neq \emptyset$, closed if $\overline{K} = K$ and convex if $\lambda y_1 + (1 - \lambda) y_2 \in K$, for all $y_1, y_2 \in K$ and $\lambda \in [0, 1]$, where $\text{int}(K)$ and $\overline{K}$ denote the interior and closure of K, respectively and $0_{\mathbb{R}^m}$ is the zero element of $\mathbb{R}^m$. The dual cone to K is

$$K^+ = \{y^* \in \mathbb{R}^m : y^* y \geq 0, \forall y \in K\},$$

where $y^* y$ is the inner product between y^* and y.

Let $\mathbb{R}^m_+ = \{y = (y_1, ..., y_m) \in \mathbb{R}^m : y_i \geq 0, \text{ for all } i = 1, ..., m\}$. Then $\mathbb{R}^m_+$ is a solid pointed closed convex cone in $\mathbb{R}^m$. It is clear that $y^* y > 0$, for any $y^* \in \mathbb{R}^m_+ \setminus \{0_{\mathbb{R}^m}\}$ and $y \in \text{int}(\mathbb{R}^m_+)$.

With respect to $\mathbb{R}^m_+$, there are two types of cone-orderings in $\mathbb{R}^m$. For any two elements $y_1, y_2 \in \mathbb{R}^m$,

$$y_1 \leq y_2 \text{ if } y_2 - y_1 \in \mathbb{R}^m_+$$

and

$$y_1 < y_2 \text{ if } y_2 - y_1 \in \text{int}(\mathbb{R}^m_+).$$

For any nonempty subsets Y, Y' of $\mathbb{R}^m$ and $y^*, y'^* \in \mathbb{R}^m$, define

$$y^* Y + y'^* Y' = \bigcup_{\substack{y \in Y \\ y' \in Y'}} \{y^* y + y'^* y'\}.$$

The ordering of two subsets of $\mathbb{R}^m$ with respect to $\mathbb{R}^m_+$ is defined as

$$Y \geq Y' \Longleftrightarrow y \geq y', \text{ for all } y \in Y \text{ and } y' \in Y'.$$

Let $y = (y_1, ..., y_m) \in \mathbb{R}^m$. The logarithm and exponential of y are defined by componentwise

$$\log y = (\log y_1, ..., \log y_m)^T, \text{ for } y > 0 \text{ (wrt. } \mathbb{R}^m_+)$$

and

$$e^y = (e^{y_1}, ..., e^{y_m})^T, \text{ for any } y.$$

Let $\emptyset \neq Y \subseteq \mathbb{R}^m$. Define two sets $\log Y$ and e^Y as

$$\log Y = \{\log y : y \in Y\}, \text{ for } Y > 0_{\mathbb{R}^m} \text{ (wrt. } \mathbb{R}^m_+)$$

and

$$e^Y = \{e^y : y \in Y\}, \text{ for any } Y.$$

Similarly, we can define $y^{\frac{1}{p}}$ and $Y^{\frac{1}{p}}$ for nonzero real number p.
Let $2^{\mathbb{R}^m}$ be the set of all subsets of $\mathbb{R}^m$ and $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$. The effective domain, range, graph, and epigraph of the set-valued map F are defined as

$$\text{dom}(F) = \{x \in \mathbb{R}^n : F(x) \neq \emptyset\},$$

$$F(X) = \bigcup_{x \in X} F(x), \text{ for any } \emptyset \neq X \subseteq \mathbb{R}^n,$$

$$\text{gr}(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\},$$

and

$$\text{epi}(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x) + \mathbb{R}^m_+\}.$$

The following notions of minimality are mainly used in $\mathbb{R}^m$ with respect to $\mathbb{R}^m_+$.

Definition 2.1. Let Y be a nonempty subset of $\mathbb{R}^m$ and $y' \in Y$. Then y' is called a minimal point of Y if there is no $y \in Y \setminus \{y'\}$ such that $y \leq y'$ and a weakly minimal point of Y if there is no $y \in Y$ such that $y < y'$.

The sets of minimal points and weak minimal points of Y are denoted by $\min Y$ and w-min Y, respectively and characterized as

$$\min Y = \{y' \in Y : (y' - \mathbb{R}^m_+) \cap Y = \{y'\}\}$$

and

$$\text{w-min } Y = \{y' \in Y : (y' - \operatorname{int}(\mathbb{R}^m_+)) \cap Y = \emptyset\}.$$

The maximal points and weak maximal points of Y are defined in similar manners. Contingent cone is an important tool in set-valued analysis. Aubin and Frankowsa [3] characterized the contingent cone in terms of sequences.

Definition 2.2. [3] Let B be a nonempty subset of $\mathbb{R}^m$ and $y_0 \in \overline{B}$. Then the contingent cone of B at y_0 is denoted by $T(B, y_0)$ and $y \in T(B, y_0)$ if there exist sequences $\{\lambda_n\}$ with $\lambda_n \to 0^+$ and $\{y_n\}$ with $y_n \to y$ such that, $y_0 + \lambda_n y_n \in B$, for all $n \in \mathbb{N}$.

It is obvious that if $y_0 \in \operatorname{int}(B)$, then $T(B, y_0) = \mathbb{R}^m$.

Proposition 2.1. [4] $T(B, y_0)$ is a closed cone of $\mathbb{R}^m$ and $T(B, y_0) \subseteq \overline{\bigcup_{h>0} \frac{B - y_0}{h}}$. If B is a convex set, then the equality holds and $B - y_0 \subseteq T(B, y_0)$.

Let X be a nonempty subset of $\mathbb{R}^n$ and $F : X \to 2^{\mathbb{R}^m}$ be a set-valued map with $\operatorname{dom}(F) = X$ and $(x_0, y_0) \in \operatorname{gr}(F)$. Aubin and Frankowsa [3] introduced the notion of contingent derivative of set-valued maps.

Definition 2.3. [3] A set-valued function $DF(x_0, y_0) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ whose graph coincides with the contingent cone to the graph of F at (x_0, y_0), i.e.

$$\operatorname{gr}(DF(x_0, y_0)) = T(\operatorname{gr}(F), (x_0, y_0)),$$

is said to be the contingent derivative of F at (x_0, y_0).

The domain of the contingent derivative, $\operatorname{dom}(DF(x_0, y_0))$ is not necessarily the whole space $\mathbb{R}^n$. It is equal to the projection of $T(\operatorname{gr}(F), (x_0, y_0))$ onto $\mathbb{R}^n$. For a single-valued map $f : \mathbb{R}^n \to \mathbb{R}^m$ which is Frechet differentiable at x_0, from Lyusternik's Theorem [8], we have

$$T(\operatorname{gr}(f), (x_0, f(x_0))) = \operatorname{gr}(f'(x_0))$$

Therefore, the contingent derivative is the natural extension of Frechet derivative from vectorial to set-valued case.

Definition 2.4. [4] Let X be a convex set of $\mathbb{R}^n$ and $F : X \to 2^{\mathbb{R}^m}$ be a set-valued map. Then F is said to be $\mathbb{R}^m_+$-convex on X if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + \mathbb{R}^m_+.$$

Lemma 2.1. [4] Let X be a convex set of $\mathbb{R}^n$ and $F : X \to 2^{\mathbb{R}^m}$ be a $\mathbb{R}^m_+$-convex set-valued map. Then for all $x, x_0 \in X$ and $y_0 \in F(x_0)$,

$$F(x) - y_0 \subseteq D(F + \mathbb{R}^m_+)(x_0, y_0)(x - x_0),$$

where $F + \mathbb{R}^m_+$ is a set-valued map defined by

$$(F + \mathbb{R}^m_+)(x) = F(x) + \mathbb{R}^m_+, x \in X.$$

Definition 2.5. [11] Let $\emptyset \neq X \subseteq \mathbb{R}^n$, $\eta : X \times X \to \mathbb{R}^n$ be a map and $F : X \to 2^{\mathbb{R}^m}$ be a set-valued map with $(x_0, y_0) \in \mathrm{gr}(F)$. Suppose that $F + \mathbb{R}^m_+$ is contingent derivable at (x_0, y_0) with

$$\eta(X, x_0) \subseteq \mathrm{dom}(D(F + \mathbb{R}^m_+)(x_0, y_0)).$$

Then F is said to be η-invex at (x_0, y_0) if

$$F(x) - y_0 \subseteq D(F + \mathbb{R}^m_+)(x_0, y_0)(\eta(x, x_0)), \text{ for all } x \in X,$$

where $\eta(X, x_0) = \{\eta(x, x_0) : x \in X\}$.

Let X be a nonempty subset of $\mathbb{R}^n$ and $F : X \to 2^{\mathbb{R}^m}$ and $G : X \to 2^{\mathbb{R}^k}$ be two set-valued maps with $\mathrm{dom}(F) = \mathrm{dom}(G) = X$. We consider a primal problem (P).

$$\begin{aligned} \underset{x \in X}{\text{minimize}} \quad & F(x) \\ \text{subject to} \quad & G(x) \cap (-\mathbb{R}^k_+) \neq \emptyset. \end{aligned} \tag{P}$$

For special case, when $f : X \to \mathbb{R}^m$ and $g : X \to \mathbb{R}^k$ are single-valued maps, we obtain a classical single-valued primal problem as

$$\begin{aligned} \underset{x \in X}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) \leq 0_{\mathbb{R}^m}. \end{aligned}$$

Definition 2.6. A point $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ is said to be a feasible point of the problem (P) if $x_0 \in X$, $y_0 \in F(x_0)$, and $G(x_0) \cap (-\mathbb{R}^k_+) \neq \emptyset$.

Let $G^-(-\mathbb{R}^k_+) = \{x \in \mathbb{R}^n : G(x) \cap (-\mathbb{R}^k_+) \neq \emptyset\}$ and $S = X \cap G^-(-\mathbb{R}^k_+)$. Then minimizers and weak minimizers of the problem (P) are defined in the following ways.

Definition 2.7. A feasible point (x_0, y_0) of (P) is said to be a minimizer of the problem (P) if

$$y_0 \in \min F(S)$$

and a weak minimizer of the problem (P) if

$$y_0 \in \text{w-min } F(S).$$

Let F_S, G_S be the restrictions of F, G to S, respectively and $(F_S + \mathbb{R}^m_+, G_S + \mathbb{R}^k_+)$ be a set-valued map defined by

$$(F_S + \mathbb{R}^m_+, G_S + \mathbb{R}^k_+)(x) = (F_S + \mathbb{R}^m_+)(x) \times (G_S + \mathbb{R}^k_+)(x), \text{ for } x \in X.$$

Corley [4] introduced the Fritz John sufficient optimality conditions of the problem (P).

Theorem 2.1. [4] Let X be a convex set and F, G be $\mathbb{R}^m_+$-convex and $\mathbb{R}^k_+$-convex on X, respectively. Suppose that there exist $x_0 \in S$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \cap (-\mathbb{R}^k_+)$, $0_{\mathbb{R}^m} \neq y^* \in \mathbb{R}^m_+$, and $z^* \in T(\mathbb{R}^k_+, z_0)^+$ such that,

$$y^* y + z^* z \geq 0,$$

for all $(y, z) \in D(F_S + \mathbb{R}^m_+, G_S + \mathbb{R}^k_+)(x_0, y_0, z_0)(x)$ and $x \in T(S, x_0)$. Then (x_0, y_0) is a weak minimizer of the problem (P).

Definition 2.8. Let X be a nonempty subset of $\mathbb{R}^n$ and $F : X \to 2^{\mathbb{R}^m}$ be a set-valued map. Then F is called locally Lipschitz at $x_0 \in X$ if there exist a neighborhood N of x_0 and a constant r such that

$$d_H(F(x), F(x')) \leq r\|x - x'\|, \text{ for all } x, x' \in N \cap \text{dom}(F),$$

where $d_H(.,.)$ is the Hausdorff distance in $2^{\mathbb{R}^m}$.

Lemma 2.2. [11] Let either F or G be locally Lipschitz at x_0. Then, we have

$$\begin{aligned} D(F_S + \mathbb{R}^m_+, G_S + \mathbb{R}^k_+)(x_0, y_0)(.) =& D(F_S + \mathbb{R}^m_+)(x_0, y_0)(.) \\ &+ D(G_S + \mathbb{R}^k_+)(x_0, z_0)(.). \end{aligned}$$

Now, since X and $G^-(-\mathbb{R}^k_+)$ are convex sets, so S is also convex. Hence, from Proposition 2.1, we have $x - x_0 \in T(S, x_0)$, for all $x \in S$. Now, if $z^* \in \mathbb{R}^k_+$ and $z^* z_0 = 0$, then $z^* \in T(\mathbb{R}^k_+, z_0)^+$. Then, we get Fritz John sufficient optimality conditions as

$$y^* D(F_S + \mathbb{R}^m_+)(x_0, y_0)(x - x_0) + z^* D(G_S + \mathbb{R}^k_+)(x_0, z_0)(x - x_0) \geq 0, \forall x \in S$$

and

$$z^* z_0 = 0.$$

3. Optimality Conditions

Our objective is to establish the sufficient optimality conditions of the problem (P) under generalized invexity assumptions. Let X be a nonempty subset of $\mathbb{R}^n$ and $F : X \to 2^{\mathbb{R}^m}$ be a set-valued map with $\text{dom}(F) = X$ and $(x_0, y_0) \in \text{gr}(F)$. Throughout the paper, we assume that $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^n$, $\mathbf{1}' = (1, ..., 1) \in \mathbb{R}^m$, and $\mathbf{1}'' = (1, ..., 1) \in \mathbb{R}^k$. We introduce the notion of (p, r)-ρ-(η, θ)-invex set-valued maps. For $p = 0$ and $r = 0$, we have the notions of ρ-(η, θ)-invex and ρ-cone convex set-valued maps, introduced by Das and Nahak [5, 6].

Definition 3.1. Let $F + \mathbb{R}^m_+$ be contingent derivable at (x_0, y_0). Then F is said to be (p, r)-ρ-(η, θ)-invex at (x_0, y_0) if there exist vector functions $\eta, \theta : X \times X \to \mathbb{R}^n$ and $\rho \in \mathbb{R}$ with $((e^{p\eta(X,x_0)} - \mathbf{1})/p) \subset \text{dom}(D(F + \mathbb{R}^m_+)(x_0, y_0))$, for $p \neq 0$ and $\eta(X, x_0) \subset \text{dom}(D(F + \mathbb{R}^m_+)(x_0, y_0))$, for $p = 0$, such that, for all $x \in X$,

$$(e^{r(F(x)-y_0)} - \mathbf{1}')/r \subset D(F + \mathbb{R}^m_+)(x_0, y_0)((e^{p\eta(x,x_0)} - \mathbf{1})/p) + \rho\|\theta(x, x_0)\|^2\mathbf{1}' \quad \text{for } p \neq 0,\ r \neq 0,$$

$$F(x) - y_0 \subset D(F + \mathbb{R}^m_+)(x_0, y_0)((e^{p\eta(x,x_0)} - \mathbf{1})/p) + \rho\|\theta(x, x_0)\|^2\mathbf{1}' \quad \text{for } p \neq 0,\ r = 0,$$

$$(e^{r(F(x)-y_0)} - \mathbf{1}')/r \subset D(F + \mathbb{R}^m_+)(x_0, y_0)(\eta(x, x_0)) + \rho\|\theta(x, x_0)\|^2\mathbf{1}' \quad \text{for } p = 0,\ r \neq 0,$$

$$F(x) - y_0 \subset D(F + \mathbb{R}^m_+)(x_0, y_0)(\eta(x, x_0)) + \rho\|\theta(x, x_0)\|^2\mathbf{1}' \quad \text{for } p = 0,\ r = 0.$$

For a continuously differentiable single valued map $f : \mathbb{R}^n \to \mathbb{R}^m$,

$$D(f + \mathbb{R}^m_+) = \nabla f(x_0)(.) + \mathbb{R}^m_+,$$

where ∇f is the gradient of f. Therefore, for single valued case, the above notion reduces to (p, r)-ρ-(η, θ)-invexity, introduced by Mandal and Nahak [9]. We have the following example of a set-valued map which is (p, r)-ρ-(η, θ)-invex but not η-invex.

Example 3.1. Let $F : \mathbb{R} \to 2^{\mathbb{R}^2}$ be a set-valued map defined by

$$F(\lambda) = \begin{cases} \{(x, x^2) : x \geq 0\}, & \text{if } \lambda \geq 0, \\ \{(x, x^2) : -1 < x < 0\}, & \text{if } \lambda < 0. \end{cases}$$

We have

$$T(\text{gr}(F + \mathbb{R}^2_+), (0, (0, 0))) = \mathbb{R} \times \mathbb{R}^2_+.$$

Hence,

$$\text{gr}(D(F + \mathbb{R}^2_+)(0, (0, 0))) = \mathbb{R} \times \mathbb{R}^2_+.$$

Now for $-1 < x < 0$,

$$(x, x^2) \notin D(F + \mathbb{R}^2_+)(0, (0, 0))\eta(\lambda, 0) + \mathbb{R}^2_+, \text{ for any } \eta.$$

Hence, F is not η-invex map for any η.
We choose $p = 0$, $r = 1$, $\rho = -1$ and η, θ such a way that

$$\eta(\lambda, 0) \geq 0 \text{ and } \theta(\lambda, 0) = 1, \text{ for any } \lambda.$$

Now

$$e^{(x,x^2)-(0,0)} - \mathbf{1} - \rho|\theta(\lambda, 0)|^2\mathbf{1} = e^{(x,x^2)}.$$

For any $x > -1$, we have

$$e^{(x,x^2)} \in D(F + \mathbb{R}^2_+)(0, (0, 0))\eta(\lambda, 0) + \mathbb{R}^2_+.$$

Hence, F is $(0, 1)$-ρ-(η, θ)-invex map.

Theorem 3.1. (Sufficient Optimality Conditions) Let (x_0, y_0) be a feasible point of the problem (P) and $z_0 \in G(x_0) \cap (-\mathbb{R}^k_+)$. Assume that F_S is (p, r)-ρ_1-(η, θ)-invex at (x_0, y_0) and G_S is (p, r)-ρ_2-(η, θ)-invex at (x_0, z_0) with respect to same functions η and θ and $\rho_1(y^*\mathbf{1}') + \rho_2(z^*\mathbf{1}'') \geq 0$. Suppose that there exists $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $y^* \neq 0_{\mathbb{R}^m}$, such that

$$\begin{aligned}
&y^*D(F_S + \mathbb{R}^m_+)(x_0, y_0)((e^{p\eta(x,x_0)} - \mathbf{1})/p) \\
&\ + z^*D(G_S + \mathbb{R}^k_+)(x_0, z_0)((e^{p\eta(x,x_0)} - \mathbf{1})/p) \geq 0, \forall x \in S, (for\ p \neq 0), \\
&y^*D(F_S + \mathbb{R}^m_+)(x_0, y_0)\eta(x, x_0) \\
&\ + z^*D(G_S + \mathbb{R}^k_+)(x_0, z_0)\eta(x, x_0) \geq 0, \forall x \in S, (for\ p = 0),
\end{aligned} \tag{3.1}$$

and

$$z^*z_0 = 0. \tag{3.2}$$

Then (x_0, y_0) is a weak minimizer of the problem (P).

Proof. We prove the theorem by the method of contradiction in the case when $p \neq 0$. For $p = 0$, we can prove likewise.
Suppose that (x_0, y_0) is not a weak minimizer of the problem (P).
Then,

$$(y_0 - \text{int}(\mathbb{R}^m_+)) \cap F(S) \neq \emptyset.$$

Therefore, there exist $x \in S$, $y \in F(x)$ such that

$$y < y_0.$$

Hence,
$$e^y < e^{y_0} \Rightarrow \frac{1}{r}e^{ry} < \frac{1}{r}e^{ry-0} \Rightarrow (e^{r(y-y_0)} - \mathbf{1}')/r < \mathbf{0}.$$
As $y^* \neq 0_{\mathbb{R}^m}$, we have
$$y^*(e^{r(y-y_0)} - \mathbf{1}')/r < 0.$$
Since, $x \in S$, there exists an element $z \in G(x) \cap (-\mathbb{R}^k_+)$.
Let $z^* = (z_1^*, ..., z_k^*)$, $z = (z_1, ..., z_k)$, and $z_0 = (z_1{}', ..., z_k{}')$.
As $z \in -\mathbb{R}^k_+$, we have
$$z \leq \mathbf{0} \Rightarrow e^z \leq \mathbf{1}'' \Rightarrow \frac{1}{r}e^{rz} \leq \frac{1}{r}\mathbf{1}'' \Rightarrow (e^{rz} - \mathbf{1}'')/r \leq 0.$$
Now $z^* z_0 = 0$, $z_0 \in -\mathbb{R}^k_+$ and $z^* \in \mathbb{R}^k_+$.
Therefore, if $z_i{}' < 0$ for some i, where $1 \leq i \leq k$, then $z_i{}^* = 0$.
So, in this case,
$$z_i{}^*(e^{r(z_i - z_i{}')} - 1)/r = 0.$$
Again, if $z_i{}' = 0$ for some i, where $1 \leq i \leq k$, then
$$z_i{}^*(e^{r(z_i - z_i{}')} - 1)/r = z_i{}^*(e^{rz_i} - 1)/r \leq 0.$$
Combining both cases, we have
$$z^*(e^{r(z-z_0)} - \mathbf{1}'')/r \leq 0.$$
Hence, we have
$$y^*(e^{r(y-y_0)} - \mathbf{1}')/r + z^*(e^{r(z-z_0)} - \mathbf{1}'')/r < 0. \tag{3.3}$$
As F_S is $\rho_1 - (\eta, \theta)$-invex at (x_0, y_0) and G_S is $\rho_2 - (\eta, \theta)$-invex at (x_0, z_0), we have
$$(e^{r(y-y_0)} - \mathbf{1}')/r \in D(F_S + \mathbb{R}^m_+)(x_0, y_0)((e^{p\eta(x,x_0)} - \mathbf{1})/p) + \rho\|\theta(x, x_0)\|\mathbf{1}' \tag{3.4}$$
and
$$(e^{r(z-z_0)} - \mathbf{1}'')/r \in D(G_S + \mathbb{R}^k_+)(x_0, z_0)((e^{p\eta(x,x_0)} - \mathbf{1})/p) + \rho\|\theta(x, x_0)\|^2\mathbf{1}''. \tag{3.5}$$
Therefore, from (3.1), (3.4), (3.5), and the condition $\rho_1(y^*\mathbf{1}') + \rho_2(z^*\mathbf{1}'') \geq 0$, we have
$$y^*(e^{r(y-y_0)} - \mathbf{1}')/r + z^*(e^{r(z-z_0)} - \mathbf{1}'')/r \geq 0.$$
This contradicts (3.3).
Hence, (x_0, y_0) is a weak minimizer of the problem (P).

4. Mond-Weir Type Duality

In several set-valued optimization problems, evaluating the dual maximization problem is comparatively easier than solving the primal minimization problem. Sach and Craven [11] proved the duality results of Mond-Weir type under invexity assumptions. We establish the duality results under (p,r)-ρ-(η,θ)-invexity assumptions.
Let $x' \in X, y' \in F(x'), z' \in G(x')$. We assume that $F_S + \mathbb{R}^m_+$ is contingent derivable at (x', y') and $G_S + \mathbb{R}^k_+$ is contingent derivable at (x', z') with,

$$((e^{p\eta(S,x')} - \mathbf{1})/p) \subseteq \mathrm{dom}(D(F_S + \mathbb{R}^m_+)(x', y')) \cap \mathrm{dom}(D(G_S + \mathbb{R}^k_+)(x', z')), \quad for\ p \neq 0,$$

and

$$\eta(S, x') \subseteq \mathrm{dom}(D(F_S + \mathbb{R}^m_+)(x', y')) \cap \mathrm{dom}(D(G_S + \mathbb{R}^k_+)(x', z')), \quad for\ p = 0.$$

For the primal problem (P), we consider a Mond-Weir type dual problem (MWD).

$$\begin{aligned} &\text{maximize } y' && \text{(MWD)} \\ &\text{subject to } y^* D(F_S + \mathbb{R}^m_+)(x', y')((e^{p\eta(x,x')} - \mathbf{1})/p) \\ &\qquad + z^* D(G_S + \mathbb{R}^k_+)(x', z')((e^{p\eta(x,x')} - \mathbf{1})/p) \geq 0, \forall x \in S, for\ p \neq 0, \\ &\qquad y^* D(F_S + \mathbb{R}^m_+)(x', y')(\eta(x, x')) \\ &\qquad + z^*(G_S + \mathbb{R}^k_+)(x', z')(\eta(x, x')) \geq 0, \forall x \in S, for\ p = 0, \\ &\qquad z^* z' \geq 0, \\ &\qquad y^* \mathbf{1}' = 1,\ y^* \in \mathbb{R}^m_+ \setminus \{0_{\mathbb{R}^m}\},\ \text{and } z^* \in \mathbb{R}^k_+. \end{aligned}$$

For single-valued optimization, we have the Mond-Weir type dual problem considered in [9].

$$\begin{aligned} &\text{maximize } f(x') \\ &\text{subject to } y^* \nabla f(x')((e^{p\eta(x,x')} - \mathbf{1})/p) \\ &\qquad + z^* \nabla g(x')((e^{p\eta(x,x')} - \mathbf{1})/p) \geq 0, \forall x \in S, for\ p \neq 0, \\ &\qquad y^* \nabla f(x')(\eta(x, x')) + z^* \nabla g(x')(\eta(x, x')) \geq 0, \forall x \in S, for\ p = 0, \\ &\qquad z^* g(x') \geq 0, \\ &\qquad y^* \mathbf{1}' = 1,\ y^* \in \mathbb{R}^m_+ \setminus \{0_{\mathbb{R}^m}\},\ \text{and } z^* \in \mathbb{R}^k_+. \end{aligned}$$

This is the Mond-Weir type dual problem considered in [9].
Let $W_1 = \{y' : (x', y', z', y^*, z^*) \text{ is a feasible point of (MWD)}\}$.

Definition 4.1. A feasible point (x', y', z', y^*, z^*) of the problem (MWD) is said to be a weak maximizer of (MWD), if

$$(y' + \text{int}(\mathbb{R}^m_+)) \cap W_1 = \emptyset.$$

Theorem 4.1. (Weak Duality) Let (x_0, y_0) and (x', y', z', y^*, z^*) be feasible points for the problems (P) and (MWD), respectively, with $z' \in G(x') \cap (-\mathbb{R}^k_+)$. Assume that F_S is (p,r)-ρ_1-(η,θ)-invex at (x', y') and G_S is (p,r)-ρ_2-(η,θ)-invex at (x', z') with respect to same functions η, θ and $\rho_1 + \rho_2(z^*\mathbf{1}'') \geq 0$. Then, we have

$$y_0 \nless y'.$$

Proof. We prove the theorem by the method of contradiction.
Suppose that $y_0 < y'$. Hence,

$$e^{y_0} < e^{y'} \Rightarrow \frac{1}{r}e^{ry_0} < \frac{1}{r}e^{ry'} \Rightarrow (e^{r(y_0 - y')} - \mathbf{1}')/r < \mathbf{0}.$$

Since, $y^* \neq 0_{\mathbb{R}^m}$, we have

$$y^*(e^{r(y_0 - y')} - \mathbf{1}')/r < 0.$$

As $x_0 \in S$, there exists an element $z_0 \in G(x_0) \cap (-\mathbb{R}^k_+)$.
Let $z^* = (z^*_1, ..., z^*_k)$, $z_0 = (z_1, ..., z_k)$, and $z' = (z_1{}', ..., z_k{}')$.
Since, $z_0 \in -\mathbb{R}^k_+$, we have

$$z_0 \leq \mathbf{0} \Rightarrow e^{z_0} \leq \mathbf{1}'' \Rightarrow \frac{1}{r}e^{rz_0} \leq \frac{1}{r}\mathbf{1}'' \Rightarrow (e^{rz_0} - \mathbf{1}'')/r \leq 0.$$

As $z' \in -\mathbb{R}^k_+$ and $z^* \in \mathbb{R}^k_+$, we have

$$z^*z' \leq 0.$$

Again, from duality constraints, we have

$$z^*z' \geq 0.$$

Therefore,

$$z^*z' = 0.$$

Now $z^*z' = 0$, $z' \in -\mathbb{R}^k_+$ and $z^* \in \mathbb{R}^k_+$.
Consequently, if $z_i{}' < 0$ for some i, where $1 \leq i \leq k$, then $z_i{}^* = 0$.
So, in this case,

$$z_i{}^*(e^{r(z_i - z_i{}')} - 1)/r = 0.$$

Again, if $z_i{}' = 0$ for some i, where $1 \leq i \leq k$, then

$$z_i{}^*(e^{r(z_i - z_i{}')} - 1)/r = z_i{}^*(e^{rz_i} - 1)/r \leq 0.$$

Combining both cases, we have

$$z^*(e^{r(z_0 - z')} - \mathbf{1}'')/r \leq 0.$$

So, we have

$$y^*(e^{r(y_0 - y')} - \mathbf{1}')/r + z^*(e^{r(z_0 - z')} - \mathbf{1}'')/r < 0. \tag{4.6}$$

As F_S is (p, r)-ρ_1-(η, θ)-invex at (x', y') and G_S is (p, r)-ρ_2-(η, θ)-invex at (x', z'), we have

$$(e^{r(y_0 - y')} - \mathbf{1}')/r \in D(F_S + \mathbb{R}^m_+)((x', y')((e^{p\eta(x_0, x')} - \mathbf{1})/p) + \rho\|\theta(x_0, x')\|^2\mathbf{1}' \tag{4.7}$$

and

$$(e^{r(z_0 - z')} - \mathbf{1}'')/r \in D(G_S + \mathbb{R}^k_+)(x', z')((e^{p\eta(x_0, x')} - \mathbf{1})/p) + \rho\|\theta(x_0, x')\|^2\mathbf{1}''. \tag{4.8}$$

Hence, from (4.7), (4.8), and the conditions $y^*\mathbf{1}' = 1$ and $\rho_1 + \rho_2(z^*\mathbf{1}'') \geq 0$, we have

$$y^*(e^{r(y_0 - y')} - \mathbf{1}')/r + z^*(e^{r(z_0 - z')} - \mathbf{1}'')/r \geq 0.$$

This contradicts (4.6). Therefore

$$y_0 \not< y'.$$

Theorem 4.2. (Strong Duality) Let (x_0, y_0) be a weak minimizer of the problem (P). Assume that for some $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $y^*\mathbf{1}' = 1$, Eqs. (3.1) and (3.2) are satisfied for some $z_0 \in G(x_0) \cap (-\mathbb{R}^k_+)$. Then $(x_0, y_0, z_0, y^*, z^*)$ is a feasible solution for (MWD). Now if the Weak Duality Theorem 4.1 between (P) and (MWD) holds, then $(x_0, y_0, z_0, y^*, z^*)$ is a weak maximizer of (MWD).

Proof. Since the Eqs. (3.1) and (3.2) are satisfied, we have

$$\begin{aligned}
&y^*D(F_S + \mathbb{R}^m_+)(x_0, y_0)((e^{p\eta(x, x_0)} - \mathbf{1})/p) \\
&+ z^*D(G_S + \mathbb{R}^k_+)(x_0, z_0)((e^{p\eta(x, x_0)} - \mathbf{1})/p) \geq 0, \forall x \in S, for\ p \neq 0, \\
&y^*D(F_S + \mathbb{R}^m_+)(x_0, y_0)\eta(x, x_0) \\
&+ z^*D(G_S + \mathbb{R}^k_+)(x_0, z_0)\eta(x, x_0) \geq 0, \forall x \in S, for\ p = 0,
\end{aligned}$$

and

$$z^*z_0 = 0.$$

Hence, $(x_0, y_0, z_0, y^*, z^*)$ is a feasible solution for (MWD).
Next, we show that,

$$(y_0 + \text{int}(\mathbb{R}^m_+)) \cap W_1 = \emptyset.$$

We prove it by the method of contradiction.
Let $y' \in (y_0 + \text{int}(\mathbb{R}^m_+)) \cap W_1$.
Therefore,

$$y' - y_0 \in \text{int}(\mathbb{R}^m_+) \Rightarrow y_0 < y'.$$

This contradicts the Weak Duality Theorem 4.1 between (P) and (MWD).
Therefore,

$$(y_0 + \text{int}(\mathbb{R}^m_+)) \cap W_1 = \emptyset.$$

Hence, $(x_0, y_0, z_0, y^*, z^*)$ is a weak maximizer for (MWD).

Theorem 4.3. (Converse Duality) Let (x', y', z', y^*, z^*) be a weak maximizer of the problem (MWD) with $z' \in G(x') \cap (-\mathbb{R}^k_+)$. Assume that F_S is (p, r)-ρ_1-(η, θ)-invex at (x', y') and G_S is (p, r)-ρ_2-(η, θ)-invex at (x', z') with respect to same functions η and θ and $\rho_1 + \rho_2(z^*\mathbf{1}'') \geq 0$. Then (x', y') is a weak minimizer of (P).

Proof. Clearly, (x', y') is a feasible solution of the problem (P).
Let (x', y') be not a weak minimzer of the problem (P).
Then,

$$(y' - \text{int}(\mathbb{R}^m_+)) \cap F(S) \neq \emptyset.$$

So, there exist $x \in S$ and $y \in F(x)$, such that

$$y' - y \in \text{int}(\mathbb{R}^m_+) \text{ i.e., } y < y'.$$

Hence,

$$e^y < e^{y'} \Rightarrow \frac{1}{r}e^{ry} < \frac{1}{r}e^{ry'} \Rightarrow (e^{r(y-y')} - \mathbf{1}')/r < \mathbf{0}'.$$

Since, $y^* \neq 0_{\mathbb{R}^m}$, we have

$$y^*(e^{r(y-y')} - \mathbf{1}')/r < 0.$$

As $x \in S$, there exists an element $z \in G(x) \cap (-\mathbb{R}^k_+)$.
Let $z^* = (z_1^*, ..., z_k^*)$, $z = (z_1, ..., z_k)$, and $z' = ({z_1}', ..., {z_k}')$.
Since, $z \in -\mathbb{R}^k_+$, we have

$$z \leq \mathbf{0}'' \Rightarrow e^z \leq \mathbf{1}'' \Rightarrow \frac{1}{r}e^{rz} \leq \frac{1}{r}\mathbf{1}'' \Rightarrow (e^{rz} - \mathbf{1}'')/r \leq 0.$$

As $z' \in -\mathbb{R}^k_+$ and $z^* \in \mathbb{R}^k_+$, we have

$$z^*z' \leq 0.$$

Again, from duality constraints we have

$$z^* z' \geq 0.$$

Therefore,

$$z^* z' = 0.$$

Now $z^* z' = 0$, $z' \in -\mathbb{R}^k_+$ and $z^* \in \mathbb{R}^k_+$.
Therefore, if ${z_i}' < 0$ for some i, where $1 \leq i \leq k$, then ${z_i}^* = 0$.
So, in this case,

$${z_i}^*(e^{r(z_i - {z_i}')} - 1)/r = 0.$$

Again, if ${z_i}' = 0$ for some i, where $1 \leq i \leq k$, then

$${z_i}^*(e^{r(z_i - {z_i}')} - 1)/r = {z_i}^*(e^{r z_i} - 1)/r \leq 0.$$

Combining both cases, we have

$$z^*(e^{r(z-z')} - \mathbf{1}'')/r \leq 0.$$

Therefore, we have

$$y^*(e^{r(y-y')} - \mathbf{1}')/r + z^*(e^{r(z-z')} - \mathbf{1}'')/r < 0. \quad (4.9)$$

As F_S is (p,r)-ρ_1-(η,θ)-invex at (x', y') and G_S is (p,r)-ρ_2-(η,θ)-invex at (x', z'), we have

$$(e^{r(y-y')} - \mathbf{1}')/r \in D(F_S + \mathbb{R}^m_+)(x', y')\eta(x, x') + \rho\|\theta(x, x')\|^2 \mathbf{1}' \quad (4.10)$$

and

$$(e^{r(z-z')} - \mathbf{1}'')/r \in D(G_S + \mathbb{R}^p_+)(x', z')\eta(x, x') + \rho\|\theta(x, x')\|^2 \mathbf{1}''. \quad (4.11)$$

Hence, from (4.10), (4.11), and the conditions $y^* \mathbf{1}' = 1$ and $\rho_1 + \rho_2(z^* \mathbf{1}'') \geq 0$, we have

$$y^*(e^{r(y-y')} - \mathbf{1}')/r + z^*(e^{r(z-z')} - \mathbf{1}'')/r \geq 0.$$

This contradicts (4.9).
Hence, (x', y') is a weak minimizer of (P).

4.1. Wolfe Type Duality

For the primal problem (P), we consider a Wolfe type dual problem (WD).

$$\begin{aligned} \text{maximize } & y' + (z^*z')\mathbf{1}' \qquad \text{(WD)} \\ \text{subject to } & y^*D(F_S+\mathbb{R}^m_+)(x',y')((e^{p\eta(x,x')}-\mathbf{1})/p) \\ & + z^*D(G_S+\mathbb{R}^k_+)(x',z')((e^{p\eta(x,x')}-\mathbf{1})/p) \geq 0, \forall x \in S, for\ p \neq 0, \\ & y^*D(F_S+\mathbb{R}^m_+)(x',y')(\eta(x,x')) \\ & + z^*D(G_S+\mathbb{R}^k_+)(x',z')(\eta(x,x')) \geq 0, \forall x \in S, for\ p = 0, \\ & y^*\mathbf{1}' = 1,\ y^* \in \mathbb{R}^m_+ \setminus \{0_{\mathbb{R}^m}\},\ \text{and } z^* \in \mathbb{R}^k_+. \end{aligned}$$

For single-valued optimization, we have Wolfe type dual problem as

$$\begin{aligned} \text{maximize } & f(x') + (z^*g(z'))\mathbf{1}' \\ \text{subject to } & y^*\nabla f(x')((e^{p\eta(x,x')}-\mathbf{1})/p) \\ & + z^*\nabla g(x')((e^{p\eta(x,x')}-\mathbf{1})/p) \geq 0, \forall x \in S, for\ p \neq 0, \\ & y^*\nabla f(x')(\eta(x,x')) + z^*\nabla g(x')(\eta(x,x')) \geq 0, \forall x \in S, for\ p = 0, \\ & y^*\mathbf{1}' = 1,\ y^* \in \mathbb{R}^m_+ \setminus \{0_{\mathbb{R}^m}\},\ \text{and } z^* \in \mathbb{R}^k_+. \end{aligned}$$

This is the Wolfe type dual problem considered in [9].
Let $W_2 = \{y' + (z^*z')\mathbf{1}' : (x', y', z', y^*, z^*) \text{ is a feasible point of (WD)}\}$.

Definition 4.2. A feasible point (x', y', z', y^*, z^*) of the problem (WD) is said to be a weak maximizer of (WD), if

$$(y' + (z^*z')\mathbf{1}' + \text{int}(\mathbb{R}^m_+)) \cap W_2 = \emptyset.$$

We prove the duality results of Wolfe type of the problem (P). The proofs are very similar to Theorems 4.1 - - 4.3, and hence omitted.

Theorem 4.4. **(Weak Duality)** Let (x_0, y_0) and (x', y', z', y^*, z^*) be feasible points for the problems (P) and (WD) respectively. Assume that F_S is $(p,0)-\rho_1-(\eta,\theta)$-invex at (x', y') and G_S is $(p,0)-\rho_2-(\eta,\theta)$-invex at (x', z') with respect to same functions η and θ and $\rho_1 + \rho_2(z^*e') \geq 0$. Then, we have

$$y_0 \not< y' + (z^*z')\mathbf{1}'.$$

Theorem 4.5. **(Strong Duality)** Let (x_0, y_0) be a weak minimizer of the problem (P). Assume that for some $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $y^*\mathbf{1}' = 1$, Eqs. (3.1) and (3.2) are satisfied for some $z_0 \in G(x_0) \cap (-\mathbb{R}^k_+)$. Then $(x_0, y_0, z_0, y^*, z^*)$ is a feasible solution for (WD). Now if the Weak Duality Theorem 4.4 between (P) and (WD) holds, then $(x_0, y_0, z_0, y^*, z^*)$ is a weak maximizer of (WD).

Theorem 4.6. (Converse Duality) Let (x', y', z', y^*, z^*) be a weak maximizer of the problem (WD) with $z^*z' = 0$. Assume that F_S is $(p, 0) - \rho_1 - (\eta, \theta)$-invex at (x', y') and G_S is $(p, 0) - \rho_2 - (\eta, \theta)$-invex at (x', z') with respect to same functions η and θ and $\rho_1 + \rho_2(z^*\mathbf{1}'') \geq 0$. Then (x', y') is a weak minimizer of (P).

4.2. Mixed Type Duality

For the primal problem (P), we consider a mixed type dual problem (Mix D).

$$\begin{aligned} \text{maximize } & y' + (z^*z')\mathbf{1}' \qquad \text{(Mix D)} \\ \text{subject to } & y^*D(F_S + \mathbb{R}^m_+)(x', y')((e^{p\eta(x,x')} - \mathbf{1})/p) \\ & + z^*D(G_S + \mathbb{R}^k_+)(x', z')((e^{p\eta(x,x')} - \mathbf{1})/p) \geq 0, \forall x \in S, for\ p \neq 0, \\ & y^*D(F_S + \mathbb{R}^m_+)(x', y')(\eta(x, x')) \\ & + z^*D(G_S + \mathbb{R}^k_+)(x', z')(\eta(x, x')) \geq 0, \forall x \in S, for\ p = 0, \\ & z^*z' \geq 0, \\ & y^*\mathbf{1}' = 1,\ y^* \in \mathbb{R}^m_+ \setminus \{0_{\mathbb{R}^m}\},\ \text{and } z^* \in \mathbb{R}^k_+. \end{aligned}$$

For single-valued optimization, we have the mixed type dual problem as

$$\begin{aligned} \text{maximize } & f(x') + (z^*g(z'))\mathbf{1}' \\ \text{subject to } & y^*\nabla f(x')((e^{p\eta(x,x')} - \mathbf{1})/p) \\ & + z^*\nabla g(x')((e^{p\eta(x,x')} - \mathbf{1})/p) \geq 0, \forall x \in S, for\ p \neq 0, \\ & y^*\nabla f(x')(\eta(x, x')) + z^*\nabla g(x')(\eta(x, x')) \geq 0, \forall x \in S, for\ p = 0, \\ & z^*g(x') \geq 0, \\ & y^*\mathbf{1}' = 1,\ y^* \in \mathbb{R}^m_+ \setminus \{0_{\mathbb{R}^m}\},\ \text{and } z^* \in \mathbb{R}^k_+. \end{aligned}$$

This is the mixed type dual problem considered in [9].
Let $W_3 = \{y' + (z^*z')\mathbf{1}' : (x', y', z', y^*, z^*) \text{ is a feasible point of (Mix D)}\}$.

Definition 4.3. A feasible point (x', y', z', y^*, z^*) of the problem (Mix D) is said to be a weak maximizer of (Mix D), if

$$(y' + (z^*z')\mathbf{1}' + \text{int}(\mathbb{R}^m_+)) \cap W_3 = \emptyset.$$

We prove the duality results of mixed type of the problem (P). The proofs are very similar to Theorems 4.1 - - 4.3, and hence omitted.

Theorem 4.7. (Weak Duality) Let (x_0, y_0) and (x', y', z', y^*, z^*) be feasible points for the problems (P) and (Mix D) respectively with $z' \in G(x') \cap (-\mathbb{R}^k_+)$. Assume that F_S is (p,r)-ρ_1-(η,θ)-invex at (x', y') and G_S is (p,r)-ρ_2-(η,θ)-invex at (x', z') with respect to same functions η and θ and $\rho_1 + \rho_2(z^*\mathbf{1}'') \geq 0$. Then, we have

$$y_0 \nless y' + (z^*z')\mathbf{1}'.$$

Theorem 4.8. (Strong Duality) Let (x_0, y_0) be a weak minimizer of the problem (P). Assume that for some $(y^*, z^*) \in \mathbb{R}^m_+ \times \mathbb{R}^k_+$, with $y^*\mathbf{1}' = 1$, Eqs. (3.1) and (3.2) are satisfied for some $z_0 \in G(x_0) \cap (-\mathbb{R}^k_+)$. Then $(x_0, y_0, z_0, y^*, z^*)$ is a feasible solution for (Mix D). Now if the Weak Duality Theorem 4.7 between (P) and (Mix D) holds, then $(x_0, y_0, z_0, y^*, z^*)$ is a weak maximizer of (Mix D).

Theorem 4.9. (Converse Duality) Let (x', y', z', y^*, z^*) be a weak maximizer of the problem (Mix D) with $z' \in G(x') \cap (-\mathbb{R}^p_+)$. Assume that F_S is (p,r)-ρ_1-(η,θ)-invex at (x', y') and G_S is (p,r)-ρ_2-(η,θ)-invex at (x', z') with respect to same functions η and θ and $\rho_1 + \rho_2(z^*\mathbf{1}'') \geq 0$. Then (x', y') is a weak minimizer of (P).

5. Conclusions

In this paper, we study set-valued optimization problems with (p,r)-ρ-(η,θ)-invexity assumptions. We derive the sufficient optimality conditions and study the duality results of Mond-Weir type under the stated (p,r)-ρ-(η,θ)-invexity assumptions. We also construct an example to ensure that (p,r)-ρ-(η,θ)-invexity is more general than invexity. For special case, our results reduce to the existing ones available in single-valued optimization problems.

References

[1] T. Antczak, *(p,r)-invex sets and functions*, J. Math. Anal. Appl. 263, 2 (2001), 355–379.

[2] T. Antczak, *(p,r)-invexity in multiobjective programming*, Eur. J. Oper. Res. 152, 1 (2004), 72–87.

[3] J. P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birhäuser, Boston (1990).

[4] H. W. Corley, *Optimality conditions for maximizations of set-valued functions*, J. Optim. Theory Appl. 58, 1 (1988), 1–10.

[5] K. Das, C. Nahak, *Sufficient optimality conditions and duality theorems for set-valued optimization problem under generalized cone convexity*, Rend. Circ. Mat. Palermo (1952-) 63, 3 (2014), 329–345.

[6] K. Das, C. Nahak, *Set-valued fractional programming problems under generalized cone convexity*, Opsearch 53, 1 (2016), 157–177.

[7] M. A. Hanson, *On sufficiency of the* Kuhn-Tucker *conditions*, J. Math. Anal. Appl. 80, 2 (1981), 545–550.

[8] J. Jahn, *Vector Optimization: Theory, Applications, and Extensions*, Springer-Verlag, Berlin (2004).

[9] P. Mandal, C. Nahak, $(p,r)-\rho-(\eta,\theta)$*-invexity in multiobjective programming problems*, Int. J. Optim. Theory Methods Appl. 2, 4 (2010), 273–282.

[10] P. H. Sach, B. D. Craven, *Invex multifunctions and duality*, Numer. Func. Anal. Opt. 12, 5-6 (1991), 575–591.

[11] P. H. Sach, and B. D. Craven, *Invexity in multifunction optimization*, Numer. Func. Anal. Opt. 12, 3-4 (1991), 383–394.

Koushik Das
Department of Mathematics,
Taki Government College,
Taki - 743 429, West Bengal, India
email: *koushikdas.maths@gmail.com*

Chandal Nahak
Department of Mathematics,
Indian Institute of Technology Kharagpur,
Kharagpur - 721 302, West Bengal, India
email: *cnahak@maths.iitkgp.ernet.in*