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Q-ANALOGUE OF RUSCHEWEYH DERIVATIVE WITH FIXED FINITELY MANY COEFFICIENTS

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Abstract. In this paper we consider the class $S^*(\lambda, q, \alpha, c_n)$, which consisting of analytic and univalent functions with negative coefficients in the unit disk $U = \{z :| z |< 1\}$ and fixed finitely many coefficients. The aim of the present paper is to drive several interesting properties as coefficient estimates, radius of convexity and closure theorems of $f(z)$ in the class $S^*(\lambda, q, \alpha, c_n)$.

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1. Introduction

In the last 25 years the area of quantum calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. Q-calculus is equivalent to classical calculus without the notion of limits. It defines q-calculus and h-calculus where h refers to Planck's constant, while q refers to quantum. Jackson [12, 13] was the first one give some applications of q-calculus and develope q-derivative and q-integral in systematic way. Aral and Gupta [6, 8, 7] defined the q-analogue of Baskakov Durrmeyer operator which based on q-analogue of beta function. Studies on quantum groups have played an important role in defining geometrical interpretation of q-analysis. It also suggests a relation between integrable systems and q-analysis. There are other important generalizations of q-calculus of complex operators known as q-Picard and q-Gauss-Weierstrass singular integral operators which discussed in [3, 4] and [5]. The authors Kanas and Răducanu defined q-analogue of Ruscheweyh derivative operator and studied some of its properties in [14]. Recently other applications of this differential operator was studied in [16] by Mohammed and Darus and Purohit and Raina [17, 18] also have used the fractional q-calculus operators in investigating certain classes of functions which are analyticin the open disk.

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

that are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$. For two analytic functions $f(z)$ and $g(z)$ are analytic functions in U, we say f subordinate to g and written as $f \prec g$ if there is a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent function, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. We provide some notations and concepts of q-calculus used in this paper. All results defined by Jackson in [12, 13].

For function $f \in S$ and $0 < q < 1$, the q-derivative D_q of a function f is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \qquad (z \neq 0) \tag{2}$$

and $D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (2), we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1},$$

where

$$[k]_q = \frac{q^k - 1}{q - 1}. \quad 0 < q < 1.$$

Hence, $[k]_q$ can be expressed as a geometric series $\sum\limits_{i=0}^{k-1} q^i$, when $k \to \infty$ the series converges to $\frac{1}{1-q}$. As $q \to 1$, $[k]_q \to k$ and this is the bookmark of a q-analogues the limit as $q \to 1$ recovers the classical object.

For $f(z) = z^k$ we observe that

$$D_q z^k = \frac{(zq)^k - z^k}{(q-1)z} = \frac{(q^k - 1)}{(q-1)} z^{k-1} = [k]_q z^{k-1}, \quad k \in \mathbb{N}.$$

The authors in [1] defined the q-analogue of Rusheweyh operator R_q^λ by

$$R_q^\lambda f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k, \tag{3}$$

where $[k]_q!$ defined by

$$[k]_q! = \begin{cases} [k]_q[k-1]_q.....[1]_q, & k = 1, 2, ... \\ 1, & k = 0 \end{cases}.$$

All details about q-calculus used in this paper can be found in [9] and [11].

If $q \to 1$ in the definition, we have

$$\begin{aligned} \lim_{q\to 1} R_q^{\lambda} f(z) &= z + \lim_{q\to 1}\left[\sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k\right] \\ &= z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!}{(\lambda)!(k-1)!} a_k z^k = R^{\lambda} f(z), \end{aligned}$$

where $R^{\lambda} f(z)$ is Ruscheweyh derivative which was defined in [19] and has been studied in many researches as [15] and [20].

Definition 1. *For $0 \le \alpha < 1$, the function $f(z)$ given by (1) is in the class $S(\lambda, q, \alpha)$ if satisfy the inequality*

$$\operatorname{Re}\left\{\frac{zD_q(R_q^{\lambda}(f(z))}{R_q^{\lambda}(f(z))}\right\} > \alpha, \quad |z| = r < 1, \tag{4}$$

where $\lambda \in N$ and $0 < q < 1$.

We note that:

(i) $S(0, q, \alpha) = S_q^*(\alpha)$, $S_q^*(\alpha) = \left\{f \in S : \operatorname{Re}\left\{\frac{zD_q f(z)}{f(z)}\right\} > \alpha, z \in U\right\}$ (Aldweby and Darus in [2]);

(ii) $\lim\limits_{q\to 1} S(0, q, \alpha) = S(\alpha)$, $S(\alpha) = \left\{f \in S : \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in U\right\}$, (Srivastava, Owa and Chatterjea in [22]).

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k. \tag{5}$$

Further, we define the class $S^*(\lambda, q, \alpha)$ by

$$S^*(\lambda, q, \alpha) = S(\lambda, q, \alpha) \cap T.$$

We begain by lemma taken from [10] but by putting the harmonic part g is identically zero.

Lemma 1. *A function $f(z)$ defined by (5) is in $S^*(\lambda, q, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{([k]_q - \alpha)[k+\lambda-1]_q!}{(1-\alpha)[\lambda]_q![k-1]_q!} |a_k| \le 1, \quad 0 \le \alpha < 1. \tag{6}$$

Further if $f(z) \in S^*(\lambda, q, \alpha)$, *then*

$$|a_k| \le \frac{(1-\alpha)[\lambda]_q![k-1]_q!}{([k]_q-\alpha)[k+\lambda-1]_q!}, \tag{7}$$

with equality for the function

$$f(z) = z - \frac{(1-\alpha)[\lambda]_q![k-1]_q!}{([k]_q-\alpha)[k+\lambda-1]_q!} z^k.$$

Now we introduce a class consisting of normalized analytic univalent function of q− analogue of Ruscheweyh operator and fixed finitely many coefficients.

Let $S^*(\lambda, q, \alpha, c_n)$ denote the subclass of $S^*(\lambda, q, \alpha)$ consisting of functions of the form

$$f(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!} z^i - \sum_{k=n+1}^{\infty} |a_k| \, z^k, \tag{8}$$

where $0 \le \alpha < 1$, $0 \le c_i \le 1$ and $0 \le \sum\limits_{i=2}^{n} c_i \le 1$.

The object of this paper is to determine coefficient estimates for the class $S^*(\lambda, q, \alpha, c_n)$. Further we show that the class $S^*(\lambda, q, \alpha, c_n)$ is closed under convex linear compinations. Lastly we have obtained radius of convexity for the class $S^*(\lambda, q, \alpha, c_n)$. Various results obtained in this paper are showen to be sharp. Techniques used are similar to these of Silverman and Silvia [21].

2. Coefficient Estimates

Theorem 2. *Let the function* $f(z)$ *be defined by* (8). *Then* $f(z)$ *is in the class* $S^*(\lambda, q, \alpha, c_n)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{([k]_q-\alpha)[k+\lambda-1]_q!}{(1-\alpha)[\lambda]_q![k-1]_q!} |a_k| \le \left(1 - \sum_{i=2}^{n} c_i\right) \tag{9}$$

where $0 \le c_i \le 1$ *and* $0 \le \sum\limits_{i=2}^{n} c_i \le 1$. *The result is sharp.*

Proof. Putting

$$|a_i| = \frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!}, \qquad (i = 2, 3, 4, ..., n), \tag{10}$$

in (6), we have

$$\sum_{i=2}^{n} c_i + \sum_{k=n+1}^{\infty} \frac{([k]_q - \alpha)[k+\lambda-1]_q!}{(1-\alpha)[\lambda]_q![k-1]_q!} |a_k| \leq 1, \tag{11}$$

which implies the result. The result is sharp for the function

$$f(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q - \alpha)[i+\lambda-1]_q!} z^i - \frac{(1-\alpha)[\lambda]_q![k-1]_q! \left(1 - \sum_{i=2}^{n} c_i\right)}{([k]_q - \alpha)[k+\lambda-1]_q!} z^k, \tag{12}$$

for $k \geq n+1$. ∎

Corollary 3. *Let the function $f(z)$ defined by* (8) *be in the class $S^*(\lambda, q, \alpha, c_n)$, then*

$$|a_k| \leq \frac{(1-\alpha)[\lambda]_q![k-1]_q! \left(1 - \sum_{i=2}^{n} c_i\right)}{([k]_q - \alpha)[k+\lambda-1]_q!}, \qquad (k \geq n+1). \tag{13}$$

The result is sharp for the function $f(z)$ given by (12).

3. Radius of Convexity

Theorem 4. *Let the function $f(z)$ defined by* (8) *be in the class $S^*(\lambda, q, \alpha, c_n)$. Then $f(z)$ is covex of order ρ $(0 \leq \rho < 1)$ in $0 < |z| < r_0$, where r_0 is the largest value for which*

$$\sum_{i=2}^{n} \frac{ic_i(i-\rho)(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q - \alpha)[i+\lambda-1]_q!} r_0^{i-1} + \frac{k(k-\rho)(1-\alpha)[\lambda]_q![k-1]_q! \left(1 - \sum_{i=2}^{n} c_i\right)}{([k]_q - \alpha)[k+\lambda-1]_q!} r_0^{k-1} \leq 1 - \rho, \tag{14}$$

for $k \geq n+1$. The result is sharp for the function $f(z)$ given by (12).

Proof. It is sufficent to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \rho, \qquad (|z| < r_0).$$

We have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum\limits_{i=2}^{n}\frac{i(i-1)c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!}|z|^{i-1}+\sum\limits_{k=n+1}^{\infty}k(k-1)|a_k||z|^{k-1}}{1-\sum\limits_{i=2}^{n}\frac{ic_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!}|z|^{i-1}-\sum\limits_{k=n+1}^{\infty}k|a_k||z|^{k-1}}.$$

Thus

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\rho$$

if

$$\sum_{i=2}^{n}\frac{ic_i(i-\rho)(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!}|z|^{i-1} + \sum_{k=n+1}^{\infty}k(k-\rho)|a_k||z|^{k-1} \leq (1-\rho). \tag{15}$$

Hence by Theorem 2 and (15) we have

$$\sum_{i=2}^{n}\frac{ic_i(i-\rho)(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!}|z|^{i-1} + \frac{k(k-\rho)(1-\alpha)[\lambda]_q![k-1]_q!\left(1-\sum\limits_{i=2}^{n}c_i\right)}{([k]_q-\alpha)[k+\lambda-1]_q!}|z|^{k-1} \leq (1-\rho). \tag{16}$$

Theorem 4 follows easily from (16). ∎

4. Closure Theorems

Theorem 5. *The class $S^*(\lambda,q,\alpha,c_n)$ is closed under convex linear compination.*

Proof. Let the function $f(z)$ be defined by (8). Define the function $h(z)$ by

$$h(z) = z-\sum_{i=2}^{n}\frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!}z^i-\sum_{k=n+1}^{\infty}|b_k|z^k. \tag{17}$$

Suppose that $f(z)$ and $h(z)$ are in the class $S^*(\lambda,q,\alpha,c_n)$, we only need to prove the function

$$H(z) = \gamma f(z)+(1-\gamma)h(z) \qquad (0\leq\lambda\leq 1) \tag{18}$$

also be in the class $S^*(\lambda, q, \alpha, c_n)$.

Since

$$H(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q - \alpha)[i+\lambda-1]_q!} z^i - \sum_{k=n+1}^{\infty} \{\gamma |a_k| + (1-\gamma) |b_k|\} z^k, \quad (19)$$

notice that

$$\sum_{k=n+1}^{\infty} \frac{([k]_q - \alpha)[k+\lambda-1]_q!}{(1-\alpha)[\lambda]_q![k-1]_q!} \{\gamma |a_k| + (1-\gamma) |b_k|\} \leq \left(1 - \sum_{i=2}^{n} c_i\right) \quad (20)$$

with the aid of Theorem 2. Hence $H(z) \in S^*(\lambda, q, \alpha, c_n)$. This clearly completes the proof of the Theorem. ∎

Theorem 6. *Let the functions*

$$f_j(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q - \alpha)[i+\lambda-1]_q!} z^i - \sum_{k=n+1}^{\infty} |a_{k,j}| z^k \quad (21)$$

be in the class $S^*(\lambda, q, \alpha, c_n)$ for every $j = 1, 2, ..., m$. Then the function $F(z)$ defined by

$$F(z) = \sum_{j=1}^{m} d_j f_j(z) \qquad (d_j \geq 0)$$

is also in the class $S^*(\lambda, q, \alpha, c_n)$, where

$$\sum_{j=1}^{m} d_j = 1.$$

Theorem 7. *Let the function $f_j(z)$ defined by (21) be in the class $S^*(\lambda, q, \alpha, c_n)$, for each $j = 1, 2, ..., m$, then the function $\phi(z)$ defined by*

$$\phi(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q - \alpha)[i+\lambda-1]_q!} z^i - \sum_{k=n+1}^{\infty} |b_k| z^k, \quad (22)$$

also be in the class $S^*(\lambda, q, \alpha, c_n)$, where

$$b_k = \frac{1}{m} \sum_{j=1}^{m} |a_{k,j}|. \quad (23)$$

Theorem 8. *Let*

$$f_n(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!} z^i \tag{24}$$

and

$$f_k(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!} z^i - \sum_{k=n+1}^{\infty} \frac{(1-\alpha)[\lambda]_q![k-1]_q!\left(1-\sum\limits_{i=2}^{n} c_i\right)}{([k]_q-\alpha)[k+\lambda-1]_q!} z^k, \tag{25}$$

for $k \geq n+1$. Then the function $f(z)$ is in the class $S^*(\lambda, q, \alpha, c_n)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=n}^{\infty} \eta_k f_k(z), \tag{26}$$

where $\eta_k \geq 0 \quad (k \geq n)$ and

$$\sum_{k=n}^{\infty} \eta_k = 1. \tag{27}$$

Proof. We suppose that the function $f(z)$ can be expressed in the form (26). Then from (24), (25) and (27) we have

$$f(z) = z - \sum_{i=2}^{n} \frac{c_i(1-\alpha)[\lambda]_q![i-1]_q!}{([i]_q-\alpha)[i+\lambda-1]_q!} z^i - \sum_{k=n+1}^{\infty} \frac{\eta_k(1-\alpha)[\lambda]_q![k-1]_q!\left(1-\sum\limits_{i=2}^{n} c_i\right)}{([k]_q-\alpha)[k+\lambda-1]_q!} z^k.$$

Since

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} \frac{\eta_k(1-\alpha)[\lambda]_q![k-1]_q!\left(1-\sum\limits_{i=2}^{n} c_i\right)}{([k]_q-\alpha)[k+\lambda-1]_q!} . \frac{([k]_q-\alpha)[k+\lambda-1]_q!}{(1-\alpha)[\lambda]_q![k-1]_q!} \\
&= \left(1-\sum_{i=2}^{n} c_i\right) \sum_{k=n+1}^{\infty} \eta_k \\
&= \left(1-\sum_{i=2}^{n} c_i\right)(1-\eta_n) \\
&\leq \left(1-\sum_{i=2}^{n} c_i\right).
\end{aligned}$$

Then $f(z) \in S^*(\lambda, q, \alpha, c_n)$.

Conversely, assuming that $f(z)$ defined by (8) be in the class $S^*(\lambda, q, \alpha, c_n)$ which satisfies (13) for $k \geq n+1$, we obtain

$$\eta_k = \frac{([k]_q - \alpha)[k+\lambda-1]_q!}{(1-\alpha)[\lambda]_q![k-1]_q!\left(1 - \sum\limits_{i=2}^{n} c_i\right)} |a_k| \leq 1$$

and

$$\eta_n = 1 - \sum_{k=n+1}^{\infty} \lambda_k.$$

This compelets the proof of the Theorem 8. ∎

Corollary 9. *The extreme points of the class* $S^*(\lambda, q, \alpha, c_n)$ *are the functions* $f_k(z)$ $(k \geq n)$ *given by* (24) *and* (25) *in Theorem 2.*

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